Circle Maps as a Simple Oscillators for Complex Behavior: I. Basics

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Abstract

The circle map and its basic properties as non-linear oscillator are discussed and related to other iterative mappings as proposed in the literature. The circle map is the simplest iterative generator for sustained periodic and chaotic sounds and is easy to interpret as a basic sine oscillator with a non-linear perturbation.

1 Introduction

Circle maps are a particularly simple yet rather general example of a mapping that exhibits many important aspects of complex dynamical behavior. A circle map is capable of demonstrating such behaviors as mode and phase-locking, period doubling and subharmonics, quasi-periodicity as well as routes to chaos via repeated period doubling or via disrutpion to quasi-periodicity (Glazier and Libchaber 1988).

Circle maps are also attractive because they have served as an important "simplest case" example of iterated dynamics in the study of these dynamics among mathematicians and physicists. They also are related to already proposed sound synthesis methods that worry about introducing functional interations or non-linearities.

The circle map is particularly suitable for the study and generation of sustained undamped sounds as the map confines the space of possible iterations exactly to functions of this nature by construction.

The purpose of this paper is to discuss the circle map and its properties and to describe how the knowledge of its parameters can be utilited for synthesis. At the same time it is an attempt to bring together existing work on iterated functions and connect it to the body of literature that discusses the dynamics of the circle map from a mathematical perspective. Then it also becomes possible to identify the place of such methods within the larger body of works of complex and symbolic dynamics as already present in literature and allows for a systematic extension of synthesis methods within these approaches.

2 Background

Non-linearities have played an ongoing important role since very early. Risset introduced waveshaping (Risset 1969). Arfib and Le Brun refined this method(Arfib 1979; Le Brun 1979). In waveshaping, a pre-existing signal would be fed through a non-linear function, hence modifying the sound. The method is able to create complex though generally only perfectly non-chaotic, periodic signals and the control is well understood.

Chaos itself became a focus of attention in the late 80s and early 90s. The use of iterated functions in computer music exploiting the rich non-linear and chaotic behavior falls into to broad categories: (1) The use of periodic pattern in the generation of music structure and (2) for direct sound synthesis purposes. Within the first category Pressing studied logistic maps (Pressing 1988). Gogins (Gogins 1991) investigated randomly switched sets of functions in his iterations. Bidlack introduced physically motivated maps of either dissipative or conservative character using Lorenz-type and Henon-Heiles type iterations(Bidlack 1992). The second category was developed by Truax (Truax 1990) and Di Scipio (Di Scipio 1990; Di Scipio 1999) motivated directly by iterated maps. DiScipio considers what he calls the sine map, an iterated sinusoid without coupling to a linear function. Rodet considered Chua's network and its time-delayed extension for sound synthesis and he also draws connections to nonlinearities in a physical context (Rodet 1993, and references therein). Dobson and Fitch considered iterated complex quadratic maps (Dobson and Fitch 1995) experimentally. Manzolli et al consider a set of two-variable interations which are variations of the so-called *standard map* which in turn is related to the circle map (Manzolli, Damiani, Tatsch, and Maia 2000). Recently Valsamakis and Miranda consider a family of two variable coupled oscillator with sine waves in the feedback loop (Valsamakis and Miranda 2005). The most widely cited reference of chaos theory in the computer music literature is (Lauterborn and Parlitz 1988). It does contain a description of the circle map but gives little interpretation or motivation of the map. Maybe for the lack of emphasis of the specific properties of the circle map, it has not been widely considered as a desirable model for interative synthesis and sequence construction in the above mentioned literature.

3 Iterated Maps from the Circle to Itself

The most general form of the circle map is

$$y_{n+1} = \phi(y_n) \tag{1}$$

where the defining property is that ϕ is a mapping from a bounded interval to a bounded interval of the same size. Typically one takes the unit interval and notates $\phi : [0, 1) \to [0, 1)$, which alternatively can be interpreted as being periodically closed. This is achieved by taking the quotient of the real numbers by the integers, repeating the reals within the interval $[0, 1)$ and we notate $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ (Milnor 2006, p. 161). Topologically this is equivalent to saying that ϕ maps points on the circle back onto a circle.

If we want to model a perfectly sinusoidal oscillator that is perturbed by some coupled non-linear function, this turns into:

$$y_{n+1} = \left(y_n + \Omega - \frac{k}{2\pi} f(y_n) \right) \mod 1 \tag{2}$$

where Ω is a constant that is the fixed angular progression of the sinusoidal oscillator, and k is the coupling strength of the non-linear perturbation $f(\cdot)$. y_0 is the starting phase. In principle, the choice of $f(\cdot)$ is very flexible and examples of discontinuous functions can be found in the literature as well as smooth cases. The canonical theoretical example is the *standard circle map*:

$$y_{n+1} = \left(y_n + \Omega - \frac{k}{2\pi} \sin(2\pi y_n) \right) \mod 1 \tag{3}$$

In order to study the long-term behavior of the iterated map $\phi(\cdot)$ we can look at the *winding number*

$$W = \lim_{n \to \infty} \frac{y_n - y_0}{n} \tag{4}$$

which measures the average angle added in the long term. If this added angle notated over the interval $[0, 1)$ is a rational number p/q with $p, q \in \mathbb{N}$ then after q interations we will have a recurrence and hence the map is periodic. Irrational winding numbers are called *quasi-periodic*.

Ω of course is essentially the frequency of the unperturbed oscillator which is calculated as $\phi_\Omega = \Omega \cdot S$ where S is the sampling rate, or time interval between two time steps for $\Omega \in [0, 0.5]$. If $\Omega > 0.5$ we get aliasing and the effective frequency decreases again, with opposite phase sign.

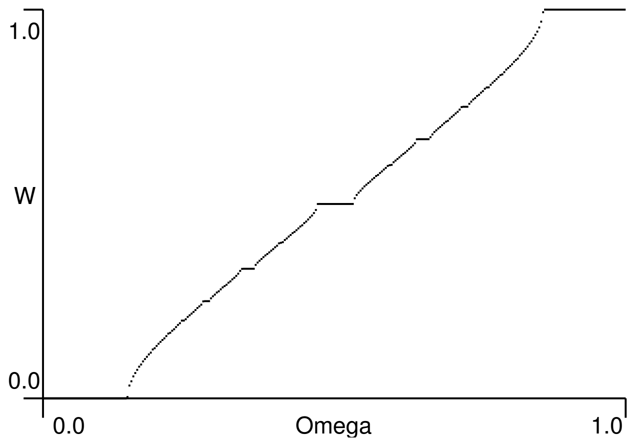

Figure 1: Devil's staircase rendered numerically for the standard circle map.

4 Perturbation of the Simple Oscillator

The parameter k defines the strength of the influence of the non-linear term is on the overall iteration. For $k << 1$ we should expect behavior that is very close to the simple sine oscillator. For $k >> 1$ we should expect that the behavior is converging to a map which is essentially just $f(y_n) = \sin(2\pi y_n)$.

For non-zero k one can observe that oscillations tend to link to rational winding numbers. As one increases Ω for a fixed k the observed frequency of the circle map will stay constant in a neighborhood of rational winding numbers and hence form successions of ascend and flat areas. This ascending line (see Figure 1) is an example of the so-called "devil's staircase" (Katok and Hasselblatt 1995; Glazier and Libchaber 1988). The width of the flat plateaus increases with increasing coupling constant k. Tracing the boundaries of the flat areas with increasing k over Ω yields curved triangular shapes, with their single-point tips at $k = 0$. These shapes are called "Arnold tongues" (Glazier and Libchaber 1988). These shapes are illustrative of the transition of the behavior of the circle map with increasing k. First off, as k increases, the area of constant frequency increases. This means that a wider range of original frequencies will lock to the particlar resulting frequency. This phenomenon is called "mode-locking". Hence mode-locking becomes stronger with increasing k. This is interesting for musical purposes, as it means that the resulting sound is generically stable in the neighborhood of frequencies corresponding to rational winding numbers. This means that small perturbations of both Ω and k do not typically perturb a locked frequency. As k is increased even more, eventually these "mode-locking plateaus" start to overlap. This is equivalent to the condition that for $k > 1$ the circle map becomes non-invertible. The circle map forms an inflection point and hence introducing hysteretic effects and eventually chaotic behavior (Glazier and Libchaber 1988). At $k = 1$ the inflection point emerges with 0 slope. This state is called *critical*. A practically relevant result about the transition through the critical case is that the choice of the non-linear function in the generic circle map (2) is not very senstive. As long as the inflection points that occur are of the same order, as for example the inflection point of the sine is cubic, the behavior stays the same (Cvitanovic, Gunaratne, and Vinson 1990, and references therein). Very roughly speaking, one can expect similar transitional behavior of period doubling and chaos even with rather widely varied non-linear functions in the circle map.

Intuitively, coupling constants $k << 1$ will lead to deformation of the oscillation, associated with additional mode-locking. At $k \approx 1$ for some frequency the behavior transitions to increasingly multi-periodic pattern, which at $k >> 1$ are chaotic. The pattern at very large k sound essentially like noise, whereas patterns those for low k are essentially periodic, with specral additions comparable to wave-shaping. Just above $k = 1$ is an interesting region with unstable periods. A main feature of the increase of k is period doubling, which will in simple cases lead to amplitude-modulation like responses (Glazier and Libchaber 1988, for another example).

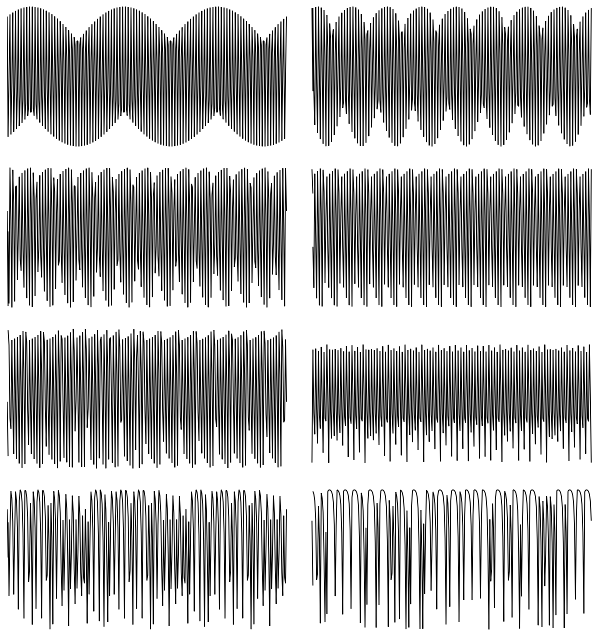

Figure 2: Wave forms for increasing coupling strength K. Ω is set to 0.33 in all cases. K is 0, 0.5, 0.8, 1.0, 1.2, 1.58, 1.8, 1.98 from left to right, top to bottom.

5 Basic Numerical Example

Some implications of these properties for wave-forms can be seen in Figure 2. The phase increase of the linear oscillator is fixed at $\Omega = 0.33$ and K is successively increased. For values K below 1 we see two effects. One is a change in frequency and the other is a deformation of the waveform. The overall waveform stays, however, perfectly periodic. As K is increased beyond 1 the pattern shows non-periodic disruptions, alongside further influence on the overall frequency. While even for K close to 2 the signal has a strong self-similar look, the pattern has deviated significantly from a periodic signal, showing irregular ad-hoc disruptions to the regular pattern.

6 Relation to Other Maps

Some maps that appeared in literature are circle maps. For example Di Scipio considered what he called the *sine map* (Di Scipio 1999):

$$y_{n+1} = \sin(2\pi r y_n) \tag{5}$$

where r is a scaling constant. This is a reduced form of the standard circle map (3) with both the linear oscillator frequency Ω removed and the linear self-increment omitted.

Manzolli and co-workers (Manzolli, Damiani, Tatsch, and Maia 2000) consider variations of the *standard map* (Glazier and Libchaber 1988; Katok and Hasselblatt 1995):

$$y_{n+1} = y_i + \Omega - \frac{k}{2\pi}\sin(2\pi y_i) + \epsilon x_i \tag{6}$$

$$x_{n+1} = \epsilon x_i - \frac{k}{2\pi}\sin(2\pi y_i) \tag{7}$$

This simplifies to the circle map if the cross-coupling constant ϵ vanishes.

Outside the computer music literature the circle map has been used in various domains. It is particularly popular in theoretical physiology where it is used to develop models of the behavior of heart rates and other cyclic body states and their coupled influence (Glass, Guevara, and Shrier 1983; Glass 2001). As a recent example McGuinness and Hong consider a piece-wise linear function for $f(\cdot)$ in (2) to model the coupling between heart rate and respiratory system (McGuinness and Hong 2004). It has also been considered for modeling physical phenomena, for example the Belousov-Zhabotinsky reaction which describes peculiar periodic or chaotic patterns as response to chemical mixing in a uniformly stirring tank (Bagley, Mayer-Kress, and Farmer 1986). They too consider piece-wise linear functions to model the observed behavior.

7 Conclusion

We discussed well-known basic properties of the circle-map and discussed their implication for sound synthesis. The circle map is a particularly interesting iterative mapping as it can be easily interpreted as a perturbed pure sinusoid, exhibiting many of the well-known properties of non-linear interations that exhibit regular and chaotic behavior.

This work is part of a larger project to place synthesis algorithms in a systematic context (Essl 2005). This too drives the desire to emphasize the circle map as an important case as it, by construction, contains both the linear and the non-linear case as extremes of a one-parameter perturbation. In addition the circle map stands in close relation to proposed methods in the literature. Its simplicity and history offers a wealth of insight into its properties.

It is planned to extend this work in two directions. One is to define the controllability of such algorithms with respect to parameter change, and the other is to study variations of the specific non-linear function within the circle map and its perceptual implications.

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