RANK GRADIENT AND THE JSJ DECOMPOSITION

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By the *rank* of a manifold M, $\operatorname{rk} M$, we will refer to the rank of its fundamental group; that is, the minimal cardinality of a generating set. Given a fixed closed manifold M, the *rank gradient* of a family of covers $\{M_n \to M\}$, each with finite degree, is defined as

$$\operatorname{rg}\{M_n\} \doteq \inf_n \frac{\operatorname{rk} M_n}{\deg\{M_n \to M\}}.$$

This was defined by M. Lackenby in [9]. Given a generating set for $\pi_1 M$ with cardinality k and a subgroup Γ of index n, the classical *Reidemeister–Schreier process* produces a generating set for Γ with cardinality $n(k-1)+1$. Thus we have:

Fact. (Reidemeister–Schreier) $\operatorname{rg}\{M_n \to M\} \leq \operatorname{rk} M - 1$.

The rank gradient of $\{M_n \to M\}$ is nonzero precisely when $\operatorname{rk} M_n$ grows linearly with the degree of the covering map. The main theorem of my talk described a family of covers for which this may be characterized.

Theorem 1. *Let M be a closed, orientable hyperbolic 3-manifold with a homomorphism $\phi\colon \pi_1 M \twoheadrightarrow \mathbb{Z}$, and for $n \in \mathbb{N}$ let $M_n \to M$ be the cover corresponding to $\phi^{-1}(n\mathbb{Z}) < \pi_1 M$. Then $\operatorname{rg}\{M_n\} = 0$ if and only if $\mathrm{PD}(\phi)$ is represented by a fiber in a fibration $M \to S^1$.*

Here $\mathrm{PD}(\phi)$ is the Poincaré dual of the cohomology class of ϕ, considered as a 1-cocycle on M. Using Stallings' fibration theorem [15], we may state Theorem 1 in a form perhaps more provocative from the standpoint of geometric group theory.

Theorem 1′. *Let M be a closed, orientable hyperbolic 3-manifold with a homomorphism $\phi\colon \pi_1 M \twoheadrightarrow \mathbb{Z}$, and for $n \in \mathbb{N}$ let $M_n \to M$ be the cover corresponding to $\phi^{-1}(n\mathbb{Z}) < \pi_1 M$. Then $\operatorname{rg}\{M_n\} = 0$ if and only if $\ker \phi$ is finitely generated.*

One direction of these theorems is obvious: if $\mathcal{S}$ is a finite generating set for $\ker \phi$, then for each $n \in \mathbb{N}$, $\pi_1 M_n$ is generated by $\mathcal{S}$ and any element mapped by ϕ to the generator of $n\mathbb{Z}$. Put another way:

Fact. If $\mathrm{PD}(\phi)$ is a fiber surface of genus g, then

$$\operatorname{rk} M_n \leq 2g + 1$$

for each $n \in \mathbb{N}$. In particular, $\operatorname{rg}\{M_n\} = 0$.

In considering Theorem 1 we are motivated by the "rank versus Heegaard genus" question about 3-manifolds:

Question. Do rank and Heegaard genus of closed hyperbolic 3-manifolds coincide?

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The *Heegaard genus* of a closed, orientable 3-manifold M, $\operatorname{Hg} M$, is the minimum genus of a separating surface S embedded in M so that if $V \subset M$ is a component of $M - S$, then $\overline{V}$ is homeomorphic to a *handlebody*, obtained from a 3-dimensional ball by attaching 1-handles. Such an S is called a *Heegaard surface* for M, and it is a classical theorem that every closed 3-manifold contains one (this follows from the existence of Morse functions, for instance).

If M has a Heegaard surface S of genus g, then the inclusion map $\overline{V} \to M$ induces a surjection at the level of π_1, where $\overline{V}$ is one of the complementary handlebodies to S. Since $\overline{V}$ has g 1-handles, we have $\operatorname{rk} M \leq g$. Therefore:

Fact. $\operatorname{rk} M \leq \operatorname{Hg} M$

One may ask a coarser version of the rank versus Heegaard genus question, in which rank and Heegaard genus are only required to behave similarly "in the large." One manifestation of this philosophy is the recent theorem of Agol (cf. [14, §9]), that all but finitely many closed hyperbolic 3-manifolds with rank 2 and injectivity radius bounded away from 0 have Heegaard genus 2. Another is the following conjecture. Below we define the *Heegaard gradient* $\operatorname{Hgr}\{M_n \to M\}$ of a family of finite-degree covers in analogy with the rank gradient, replacing rank by Heegaard genus.

Conjecture. Let M be a closed hyperbolic 3-manifold and $\{M_n \to M\}$ a family of finite-degree covers. Then $\operatorname{rg}\{M_n\} > 0$ if and only if $\operatorname{Hgr}\{M_n\} > 0$.

Theorem 1 may be considered evidence for this conjecture by comparing with the following theorem of Lackenby [10, Theorem 1.11].

Theorem (Lackenby)**.** *Let M be a finite-volume hyperbolic 3-manifold with a homomorphism $\phi\colon \pi_1 M \twoheadrightarrow \mathbb{Z}$, and for $n \in \mathbb{N}$ let $M_n \to M$ be the cover corresponding to $\phi^{-1}(n\mathbb{Z}) < \pi_1 M$. Then $\operatorname{Hgr}\{M_n\} = 0$ if and only if $\operatorname{PD}(\phi)$ is represented by a fiber in a fibration.*

Theorem 1 follows by application of two deep but well known principles. The first, known as "acylindrical accessibility," bounds the cardinality of generating sets of groups acting nicely on trees. The second, that "cylinders have bounded length," is a property of the JSJ decomposition of a manifold obtained from a hyperbolic 3-manifold by cutting along an incompressible surface. We will describe these principles and the action of $\pi_1 M$ on a tree associated to ϕ, in Section 1, then use them to prove Theorem 1. In Section 2, we will describe the JSJ decomposition and use an example to illustrate the property that cylinders have bounded length. We discuss further directions and questions in Section 3.

1. Actions on trees

Given a finitely generated group Γ acting on a tree T, an "accessibility" principle relates the combinatorics of $\Gamma \backslash T$ to the structure of Γ. *Acylindrical* accessibility, introduced by Z. Sela [13], is distinguished from other notions in that it does not require prior knowledge of the structure of vertex or edge stabilizers, but only that their action on T is "nice enough":

Definition. $\Gamma \times T \to T$ is *k-acylindrical* if no $g \in \Gamma - \{1\}$ fixes a segment of length greater than k, and *k-cylindrical* otherwise.

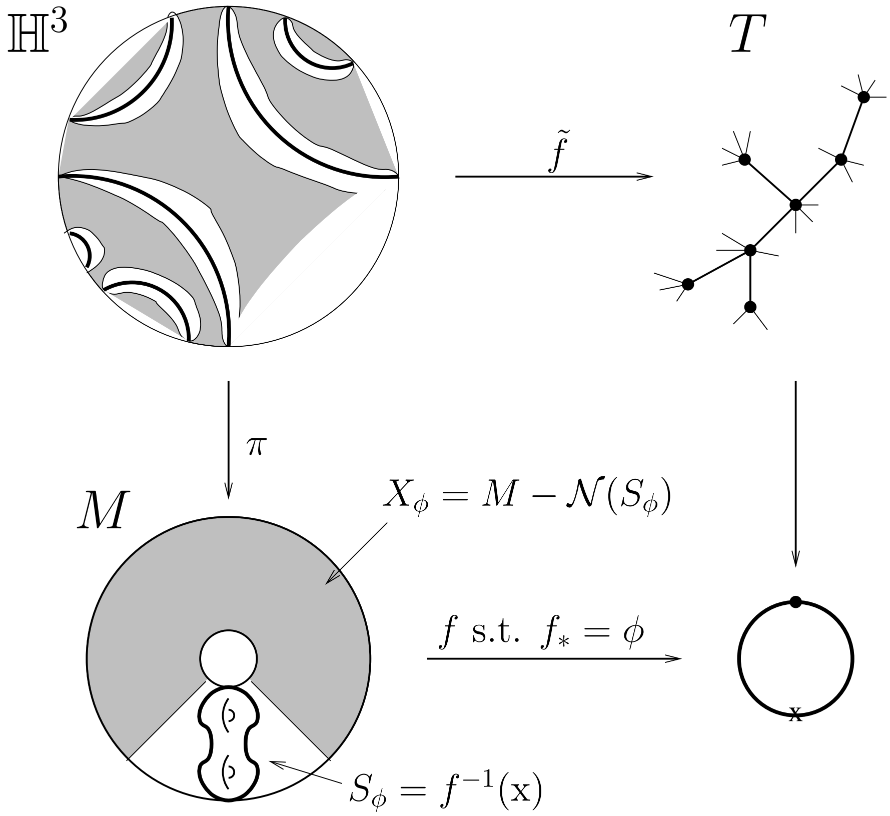

FIGURE 1. The action associated to $\phi\colon \pi_1 M \to \mathbb{Z}$.

It is a basic consequence of Bass–Serre theory that if a group Γ acts on a tree with a trivial edge stabilizer, Γ is freely decomposable or cyclic. The groups of interest here therefore act at best 1-acylindrically on trees. The acylindrical accessibility theorem that we use here, due to R. Weidmann, has constants that do not depend on the group Γ, a feature that is critical to our rank gradient computations.

Theorem ([16])**.** *Let Γ be a non-cyclic freely indecomposable finitely generated group and $\Gamma \times T \to T$ a minimal k-acylindrical action. Then $\Gamma \backslash T$ has at most $1 + 2k(\mathrm{rk}\,\Gamma - 1)$ vertices.*

We will apply this theorem to an action associated to $\phi\colon \pi_1 M \to \mathbb{Z}$, where M is a closed orientable hyperbolic manifold. Such an M is a $K(\Gamma, 1)$ space for its fundamental group Γ, since its universal cover $\mathbb{H}^3$ is contractible; thus there is a map f taking M to a graph G with a single vertex v and edge e, so that $f_* = \phi\colon \Gamma \to \pi_1 G = \mathbb{Z}$. Making this map as nice as possible, we obtain the following standard lemma.

Lemma 1 ([6], Lemma 6.5)**.** *With $f\colon M \to G$ as above, we may arrange so that $f^{-1}(\mathrm{int}(e)) = \mathcal{N}(S_\phi)$, where S_ϕ is a nonseparating π_1-injective closed surface embedded in M representing* $\mathrm{PD}(\phi)$ *and $\mathcal{N}(S_\phi)$ is an open regular neighborhood of*

S_ϕ equipped with a homeomorphism to $S_\phi \times e$ so that f factors through projection to e.

Lemma 1 motivates Figure 1 above. Since S_ϕ is π_1-injective, the inclusion $S_\phi \to M$ lifts to an embedding of universal covers $\mathbb{H}^2 \to \mathbb{H}^3$ specified by choosing the lift of a point. The preimage of S_ϕ in $\mathbb{H}^3$ is therefore a collection of properly embedded disjoint planes, each contained in a component of the preimage of $\mathcal{N}(S_\phi)$ homeomorphic to $\mathbb{H}^2 \times e$. The quotient map $f\colon M \to G$ determines a quotient $\tilde{f}\colon \mathbb{H}^3 \to T$, where T is a tree, such that $\tilde{f}$ is equivariant with respect to the action of $\Gamma = \pi_1 M$ by deck transformations. It follows that there is an action $\Gamma \times T \to T$ such that $G = \Gamma \backslash T$.

By construction, each edge stabilizer of $\Gamma \times T \to T$ is a subgroup representing $\pi_1 S$. Let $X_\phi = \overline{M - \mathcal{N}(S_\phi)}$; this is the preimage of v under f. Since S_ϕ is π_1-injective in M, so is X_ϕ and each vertex stabilizer of the action $\Gamma \times T \to T$ represents $\pi_1 X_\phi$ in Γ. An element of Γ that stabilizes a segment of length greater than 1 therefore lies in more than one conjugate of $\pi_1 S_\phi$, and so has a free homotopy through M that begins and ends on S_ϕ, passing through X_ϕ. This implies the following fact.

Fact. If $\pi_1 M \times T \to T$ is k-cylindrical, $k > 1$, then there is an immersion $(A, \partial A) \to (M, S)$, where A is an annulus, such that the interior of A has at least k components of transverse intersection with S.

Given this fact, we will define the *length* of an annulus $(A, \partial A) \to (M, S)$ immersed transverse to S to be the least number of components of transverse intersection $A \cap S$, over all homotopies sending ∂A into S. (Also see [13], Proposition 4.4 and the discussion below it.) The characteristic submanifold theory of hyperbolic 3-manifolds constrains the topology of such immersions, allowing us to prove:

Proposition 1 (Cylinders have bounded length)**.** *Suppose M is a closed, orientable hyperbolic 3-manifold and $S \subset M$ is a non-separating, embedded, orientable, π_1-injective surface, such that there is no fibration $M \to S^1$ with S as a fiber. There exists $k \in \mathbb{N}$, depending only on the topology of S, such that the length of any immersed annulus $(A, \partial A) \to (M, S)$ is bounded above by k.*

We will not prove Proposition 1 here; however, in the following section we will describe the characteristic submanifold theory and consider an example. Given the Proposition, we now prove Theorem 1.

Proof of Theorem 1. Let M be a hyperbolic manifold and $\phi\colon \pi_1 M \twoheadrightarrow \mathbb{Z}$. Let S_ϕ be a π_1-injective surface representing $\mathrm{PD}(\phi)$ supplied by Lemma 1, and suppose that S_ϕ is not a fiber in a fibration. Then taking $\Gamma = \pi_1 M$, Proposition 1 implies that there is some $k \in \mathbb{N}$ such that the action $\Gamma \times T \to T$ associated to ϕ is k-acylindrical.

For $n \in \mathbb{N}$, let $\Gamma_n = \phi^{-1}(n\mathbb{Z}) < \Gamma$ with index n. Γ_n inherits a k-acylindrical action on T from that of Γ, with quotient a graph with n vertices that is homeomorphic to a circle. (This follows from the fact that S_ϕ has n distinct homeomorphic lifts to the cover $M_n \to M$ corresponding to Γ_n, cyclically permuted by deck transformations.) Weidmann's acylindrical accessibility theorem therefore implies

$$\mathrm{rk}\, M_n \geq \frac{n-1}{2k} + 1.$$

We thus find that $\mathrm{rg}\,\{M_n\} \geq 1/2k$. □

2. Cylinders have bounded length

It is easy to see that the manifold X obtained by removing a regular neighborhood of an embedded, π_1-injective surface in a closed, orientable hyperbolic 3-manifold M is compact, irreducible, and atoroidal, with incompressible boundary. We will call such manifolds *simple*. The JSJ decomposition of a simple 3-manifold describes a "best" way to cut it apart along *essential* annuli; that is, along annuli that are properly embedded, π_1-injective, and not boundary-parallel [7], [8].

Theorem (Jaco–Shalen, Johansson)**.** *Let X be a simple 3-manifold with nonempty boundary. Up to ambient isotopy, its characteristic submanifold Σ is the unique compact submanifold of X with the following properties.*

(1) *Every component of Σ is either an I-bundle P over a surface such that $P \cap \partial X = \partial_h P$, or a Seifert fibered space S such that $S \cap \partial X$ is a saturated 2-manifold in ∂S.*
(2) *Every component of the frontier of Σ is an essential annulus or torus in X.*
(3) *No component of Σ is ambiently isotopic in X to a submanifold of another component of Σ.*
(4) *If Σ_1 is a compact submanifold of X such that (1) and (2) hold with Σ_1 in place of Σ, then Σ_1 is ambiently isotopic in X to a submanifold of Σ.*

The characteristic submanifold of X also has the *enclosing* property, that every proper, π_1-injective immersion of an annulus is properly homotopic into Σ, unless it is homotopic into the boundary. Σ may be empty, in which case X is said to be *acylindrical*, or it may be all of X, in which case X is homeomorphic to $S \times I$ for some surface S. The most interesting cases from our perspective fall between these extremes.

Example 1. Let T_1 and T_2 be punctured tori, and let V be a solid torus. For $i = 1, 2$, let $A_i = (\partial T_i) \times [0, 1]$. Orient V and the T_i, and for each i give $T_i \times [0, 1]$ the product orientation. Let $\iota_1 \colon A_1 \to \partial V$ and $\iota_2 \colon A_2 \to \partial V$ be homeomorphic embeddings with disjoint images such that for each i, ι_i reverses the boundary orientation on A_i and $\iota_i(\partial T_i \times \{1/2\})$ is a simple closed curve on ∂V that intersects the boundary of a meridian disk twice. We further require that a single component of $\overline{\partial V - \iota_i(A_i)}$ contain the images of $\partial T_1 \times \{1\}$ and $\partial T_2 \times \{1\}$.

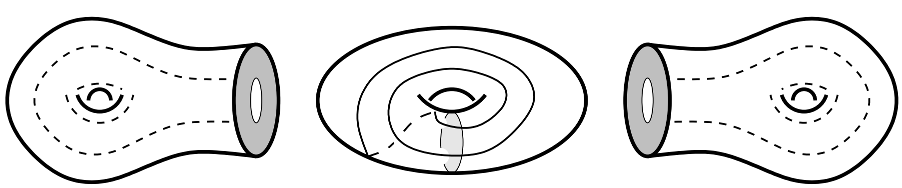

Figure 2. The components of Σ in Example 1.

The subjects of the paragraph above are pictured in Figure 2. The annuli A_1 and A_2 are shaded on the left and right, and the solid torus V is pictured in the middle with a meridian disk shaded and a simple closed curve on ∂V parallel to the images of ι_j, $j = 1, 2$, pictured. We let X be the identification space of $V \sqcup T_1 \times [0, 1] \sqcup T_2 \times [0, 1]$ by $x \sim \iota_j(x)$ for $x \in A_j$, $j = 1, 2$. Then X is a simple

3-manifold with two boundary components, and the characteristic submanifold Σ of X is homeomorphic to the disjoint union of V and the $T_j \times [0,1]$.

X has the property that $\overline{X - \Sigma}$ is the disjoint union of regular neighborhoods of A_1 and A_2 in X. Such manifolds are known as "books of I-bundles" (see [4]).

Given a surface S embedded in a hyperbolic 3-manifold M as in the statement of Proposition 1, we may regard M as obtained from $X \doteq \overline{M - \mathcal{N}(S)}$ by identifying boundary components $\partial_+ X \to \partial_- X$ by a homeomorphism ϕ. Here we fix an identification of $\mathcal{N}(S)$ with $S \times [-1,1]$, take $\partial_\pm X = S \times \{\pm 1\}$, and let $\phi\colon\ \partial_+ M \to \partial_- M$ be the map $(x,1) \mapsto (x,-1)$.

If S is not a fiber in a fibration $M \to S^1$, then the characteristic submanifold Σ is not all of X. The proof of Proposition 1 is based on the observation that in this case, we have

$$\phi(\Sigma \cap \partial_+ X) \neq \Sigma \cap \partial_- X.$$

If this were not so, then the annular components of the frontier of Σ in X would join together yielding a disjoint union of π_1-injective tori in M. But this would contradict the hyperbolicity of M.

A strengthened version of this observation, which Marc Culler has called a "vegematic argument," describes a properly shrinking sequence of subsurfaces S_k of S that carry its intersections with any annulus of length k. We will not prove Proposition 1 here, but refer the reader to antecedents in [2, §4] and [3] for applications of this general principle under somewhat different circumstances. We illustrate it below using the manifold from Example 1.

Example 2. Let X be the manifold of Example 1, and take $\partial_\pm X$ to be the component of ∂X containing $T_j \times \{\pm 1\}$, for $i = 1$ and 2. Let α, β, and γ be the curves on $\partial_+ X$ in Figure 3, and define $\phi\colon\ \partial_+ M \to \partial_- M$ to be the composition of right-handed Dehn twists $\tau_\gamma \tau_\beta \tau_\alpha$, followed by a homeomorphism that takes $T_j \times \{1\}$ to $T_j \times \{-1\}$ for each of $j = 1, 2$. Defining $M = X/\phi$ by gluing $\partial_+ X$ to $\partial_- X$ by ϕ, we claim that the action of $\pi_1 M$ on the tree T associated to $\mathrm{PD}(S)$ is 3-cylindrical but 4-acylindrical, where S is the image of ∂X under the quotient map.

We have pictured $\partial_+ M$ at the top of Figure 3, with $\Sigma \cap \partial_+ X$ unshaded. The *essential intersection* of $\phi(\Sigma \cap \partial_+ X)$ with $\Sigma \cap \partial_- X$ is the disjoint union of the annuli B_1 and B_2 pictured at the bottom of the figure. This is the union of components of $\phi(\Sigma \cap \partial_+ X) \cap \Sigma \cap \partial_- X$ that contain an essential simple closed curve. For $j = 1, 2$, there is a submanifold of Σ homeomorphic to $B_j \times [-1,1]$ such that $B_j = B_j \times \{-1\}$.

For each j, the essential intersection of $\phi(B_j \times \{1\})$ with $\Sigma \cap \partial_- M$ is empty. We thus obtain the following classification of "cylindrical" elements of $\pi_1 M \times T \to T$:

(1) Each essential curve on S determines an element of an edge stabilizer.
(2) Each essential curve in Σ bounds an immersed annulus $(A, \partial A) \to (M, S)$ with length 2.
(3) For $j = 1$ or 2, there is a simple closed curve in $\Sigma \cap \partial_+ X$, unique up to homotopy, that maps under $\tau_\gamma \tau_\beta \tau_\alpha$ to the core of B_j. Each such curve determines an annulus $(A_j, \partial A_j)$ in (M, S) with length 3.

Given an annulus with length greater than 3, its intersection with X would determine a sequence of at least 3 annuli in Σ joining components of ∂X, with boundary components exchanged by ϕ. The discussion above shows this is impossible.

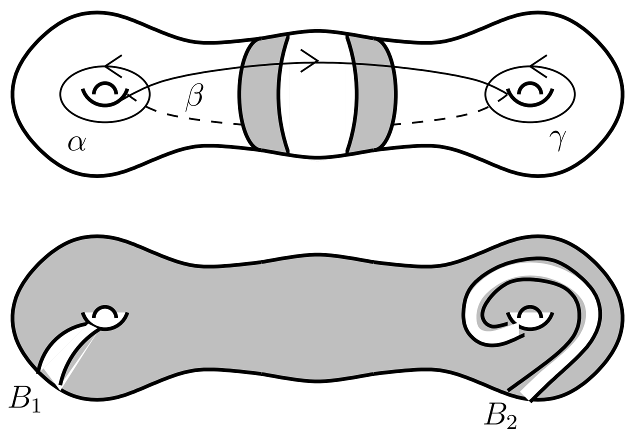

FIGURE 3. $\Sigma \cap \partial_+ X$, and the essential intersection of its image with $\Sigma \cap \partial_- X$.

3. Questions and further directions

Two directions immediately suggest themselves as avenues for further exploration of rank gradient questions. The most direct generalization seeks to expand the class of groups under consideration.

Question. Which classes of groups satisfy the conclusion of Theorem $1'$?

One could further ask whether our proof applies in a more general context. The most natural candidate for this question seems to be the class of hyperbolic groups, since these are also "atoroidal," in the sense that they do not contain free abelian groups of rank 2 or higher. Furthermore, a JSJ-type decomposition theorem that applies to these groups and has the enclosing property has been proved by Scott–Swarup [12]. Extending our proof strategy to these groups thus only requires proving a version of "cylinders have bounded length" for the class of all hyperbolic groups. A different context in which I believe Theorem 1 applies is below.

Conjecture. Theorem 1 holds for all closed 3-manifolds.

Using the prior work of Lackenby [9] and Weidmann [16], establishing this conjecture reduces to the problem of extending Theorem 1 to finite-volume (not necessarily compact) hyperbolic 3-manifolds and Seifert fibered spaces. I believe these cases can be done using *ad hoc* arguments.

A second direction for generalizing Theorem 1 lies in considering other families of covers. This direction would require different techniques than are used here. I am particularly interested in the following question.

Question. Under which circumstances does a *co-final* family of finite covers — that is, $\{M_n \to M\}$ such that $\bigcap \pi_1 M_n = \{1\} \subset \pi_1 M$ — have positive rank gradient?

This is motivated by work of Abert–Nikolov [1]. Lackenby showed that families of covers with property τ have positive Heegaard gradient [10, Theorem 1.5]. On the other hand, Abert–Nikolov described co-final towers of covers with rank gradient 0, of any hyperbolic 3-manifold that is virtually fibered [1]. In many cases, these manifolds also have co-final towers with property τ (see eg. [11]), for which the rank gradient is unknown. If the rank gradient were positive, this would give a negative answer to a question of Gaboriau [5] (see [1]); otherwise, it would provide infinitely many examples with rank unequal to Heegaard genus. Attacking this question would require new techniques.

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