Set-theoretic study of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$

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Abstract

We give an answer to the following question: How “typical” is it for the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \qquad (\text{IVP}_f)$$

to be uniquely solvable.

Let X be a topological space. A set $A \subset X$ is called

1. *nowhere dense* if $\mathit{int}\,\mathit{cl}\,A = \varnothing$,
2. *of first category* (or *meager)* if it is a countable union of nowhere dense sets,
3. *of second category* (or *non-meager)* if it is not of first category, and
4. *residual* if $A = X \setminus M$ and M is of first category.

A set $A \subset X$ is called a F_σ*-set* respectively a G_δ*-set* if it is the countable union of closed sets respectively the countable intersection of open sets.

Theorem 1 (Baire's Theorem) *A complete metric space is of second category.*

This gives a notion for a property to be “typical” (or not) for elements of a topological space. For example, it can be shown that the set

$$D = \{f \in C[0,1] \ : \ f \text{ is differentiable for some } x \in [0,1]\}$$

is meager in $(C[0,1], || \cdot ||_\infty)$. Moreover, this also gives a (non-constructive) proof of the existence of continuous functions that are nowhere differentiable.

Let $-\infty < a < x_0 < b < \infty$ and $\Sigma = [a,b] \times \mathbb{R}$. Denote by $E = C_b(\Sigma)$ the Banach space of continuous and bounded functions with the supremum norm

$$||f||_\infty = \sup_{(x,y,)\in\Sigma} |f(x,y)|.$$

A function $\phi \in E$ is called *locally Lipschitz*[1] if for every $(x_0, y_0) \in \Sigma$ there exists a neighborhood $V(x_0, y_0)$ and a constant $0 < M = M(x_0, y_0, \phi)$ such that

$$|\phi(x, y_1) - \phi(x, y_2)| \leq M|y_1 - y_2|$$

[1] technically, only in the second coordinate

for all (x, y_1), $(x, y_2) \in V$. We denote the subspace of E of all locally Lipschitz continuous functions by $L(\Sigma)$.

Lemma 2 *$L(\Sigma)$ is dense in E.*

Proof. (Sketch) Let $\phi \in E$. Subdivide Σ into squares of side length $O(n^{-1})$ and define for $s, t \in [0,1]$ and integer k, l

$$\begin{aligned}
\phi^{\vee}_{k,l} &= \phi((1-t)x_k + tx_{k+1}, y_l),\\
\phi^{\wedge}_{k,l} &= \phi((1-t)x_k + tx_{k+1}, y_{l+1}),\\
\phi_{k,l} &= (1-s)\phi^{\vee}_{k,l} + s\phi^{\wedge}_{k,l},
\end{aligned}$$

and

$$\phi^n(x,y) = \sum_{k,l} \phi_{k,l}.$$

□

Lemma 3 *$L(\Sigma)$ is of first category.*

Proof. Indeed,

$$L(\Sigma) = \bigcup_{n \geq 1} F_n,$$

where

$$\begin{aligned}
F_n = \{ f \in L(\Sigma) \,:\, &\text{there exists } y \in \mathbb{R} \text{ with } |f(x,y) - f(x,y')| \leq n|y - y'| \\
&\text{for all } x \in [a,b] \text{ and } y' \text{ with } |y - y'| \leq n^{-1} \}.
\end{aligned}$$

The F_n are closed and nowhere dense. □

Now we define

$$\mathcal{U} = \{ f \in E \,:\, (\mathrm{IVP}_f) \text{ is uniquely solvable on } [a,b] \text{ for every } (x_0, y_0) \in \Sigma \}.$$

By the Theorem of Picard-Lindelöf and since the elements of E are bounded, we know that $L(\Sigma) \subset \mathcal{U}$ and it follows from Lemma 2 that $\mathcal{U}$ is dense in E.

Theorem 4 *$\mathcal{U}$ is a G_δ-set.*

Remark If this holds, say $\mathcal{U} = \bigcap_{n \geq 1} G_n$ with $G_n \subset E$ open, then by De Morgan's law, $E \setminus \mathcal{U} = \bigcup_{n \geq 1} E \setminus G_n$ and the sets $F_n := E \setminus G_n$ are closed. Since $\mathcal{U} \subset G_n$ and $\mathcal{U}$ is dense in E, the F_n are nowhere dense. Since E is a Banach space, it is of second category and since $E = (E \setminus \mathcal{U}) \cup \mathcal{U}$, the set $\mathcal{U}$ is of second category as well. Thus, unique solvability is "typical".

Proof. Let $\varepsilon > 0$ be rational and $q \in \mathbb{Q}$. We define

$$\begin{aligned}
P(\varepsilon, q) = \{ (f, x_0, y_0) \in E \times [a,b] \times \mathbb{R} \,:\, &\text{there exist } \phi_1, \phi_2 \in C[a,b] \text{ with} \\
&\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\,\mathrm{d}t, \quad \phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_2(t))\,\mathrm{d}t \\
&\text{for all } x \in [a,b], \text{ and } |\phi_1(q) - \phi_2(q)| \geq \varepsilon \}.
\end{aligned}$$

Claim: The sets $P(\varepsilon, q)$ are closed in $E \times [a,b] \times \mathbb{R}$. To prove this claim, assume that we have a convergent sequence

$$P(\varepsilon, q) \ni (f^n, x_0^n, y_0^n) \to (\bar{f}, \bar{x}_0, \bar{y}_0) \in E \times [a,b] \times \mathbb{R}.$$

Then there exist families of "witness functions" $\{\phi_1^n\}_{n\geq 1}$ and $\{\phi_2^n\}_{n\geq 1}$ such that

$$\phi_i^n(x) = y_0^n + \int_{x_0^n}^{x} f^n(t, \phi_i^n(t))\,\mathrm{d}t,$$

for $i = 1, 2$ and

$$|\phi_1^n(q) - \phi_2^n(q)| \geq \varepsilon.$$

One can show that the families $\{\phi_1^n\}_{n\geq 1}$ and $\{\phi_2^n\}_{n\geq 1}$ are uniformly bounded and equicontinuous. By the Arzelà-Ascoli Theorem, there exist uniformly convergent subsequences, still denoted by ϕ_1^n and ϕ_2^n, with limits $\bar{\phi}_1$ and $\bar{\phi}_2$. By the assumed convergence and the continuity of the integral, it then follows that

$$\bar{\phi}_i(x) = \bar{y}_0 + \int_{\bar{x}_0}^{x} \bar{f}(t, \bar{\phi}_i(t))\,\mathrm{d}t,$$

for $i = 1, 2$ and

$$|\bar{\phi}_1(q) - \bar{\phi}_2(q)| \geq \varepsilon.$$

Thus $(\bar{f}, \bar{x}_0, \bar{y}_0) \in P(\varepsilon, q)$ and the claim is proved. Now we define

$$P = \bigcup_{\substack{q \in \mathbb{Q} \cap [a,b] \\ \varepsilon \in \mathbb{Q}^+}} P(q, \varepsilon).$$

Then by construction, the family of non-uniquely solvable direction fields is

$$E \setminus \mathcal{U} = \pi_1(P),$$

where $\pi_1(f, x_0, y_0) = f$ is the projection onto the first coordinate. By the claim, P is a F_σ-set in $E \times [a,b] \times \mathbb{R}$.

Lemma 5 (Tube Lemma, K. Kuratowski) *Let X be a topological space, Y a compact space and $\pi_1 : X \times Y \to X$ the projection onto the first coordinate. If $A \subset X \times Y$ is closed, then so is $\pi_1(A)$.*

Lemma 6 (Generalized Kuratowski Lemma) *Let X be a topological space and Y σ-compact (i.e. $Y = \bigcup_{n\geq 1} Y_n$ with compact sets Y_n). If $A \subset X \times Y$ is a F_σ-set, then so is $\pi_1(A)$.*

Applying the Generalized Kuratowski Lemma to P gives that $E \setminus \mathcal{U}$ is a F_σ-set in E, say $E \setminus \mathcal{U} = \bigcup_{n\geq 1} F_n$, with closed sets F_n. Then $E \setminus \mathcal{U} = \bigcap_{n\geq 1} E \setminus F_n$ and so it is a G_δ-set. □