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ERGODICITY AND LOSS OF CAPACITY FOR A RANDOM FAMILY OF CONCAVE MAPS

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Abstract. Random fluctuations of an environment are common in ecological and economical settings. We consider a family of concave quadratic polynomials on the unit interval that model a self-limiting growth behavior. The maps are parametrized by an independent, identically distributed random parameter. We show the existence of a unique invariant ergodic measure of the resulting random dynamical system. Moreover, there is an attenuation of the mean of the state variable compared to the constant environment with the averaged parameter.

1. **Introduction.** Random dynamical systems are a beautiful combination of deterministic and random processes and have been studied widely since the days of von Neumann and Ulam [26] and Kakutani [17]. Given are a Polish space S and a probability space $(K, \mathcal{B}, m)$ of maps $f : S \to S$ that are applied to S in an independent and identically distributed sequence. If the state space S is compact and the maps are continuous, it is straightforward to show the existence of an invariant probability measure on S using Prokhorov's Theorem (for definitions and details see Section 2). If the invariant measure is unique, it can be shown to be ergodic. Uniqueness of the invariant measure has been proved by Dubins and Freedman [15] for two situations, namely

- monotone maps of the unit interval that satisfy a certain "splitting condition", and
- maps on a complete metric space that are contractions in the logarithmic average.

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Several generalizations of these early results and further properties are discussed in the review article by Diaconis and Freedman [14] and the book by Bhattacharya and Majumdar [6]. A prominent example is the family of random logistic maps $f(x, \lambda) = \lambda x(1-x)$, where λ takes values in a subset of the interval $[0, 4]$, see for example Bhattacharya and Rao [7], and Athreya and Dai, [2]. The latter authors in [3] also provided an example for non-uniqueness of the invariant probability for random logistic maps where λ takes only two values. Their example exhibits an invariant measure supported on a two-point set and a second invariant measure supported on an invariant interval containing the two points. Further results establishing the existence of a unique invariant probability measure for randomized versions of the Beverton-Holt equation (1) are by Bezandry *et al.* [5] and Haskell and Sacker [16]. The existence of a stable unique invariant probability measure can be considered the equivalent of a globally attracting fixed point for a deterministic dynamic system. In this work we are particularly interested in the case when the uniqueness of an invariant measure holds independent of the parameter distribution m.

A second motivation for this work comes from the study of the long-term behavior of discrete population dynamics with non-constant parameters. The Beverton-Holt equation is a simple time-discrete model introduced in the late 1950s to study fish stocks subject to harvesting [4] and is given by

$$x_{n+1} = \frac{\nu K x_n}{K + (\nu - 1)x_n}, \tag{1}$$

where x_n is the population at time n, $\nu > 1$ is the inherent growth rate and $K > 0$ the carrying capacity. Several recent works deal with the Beverton-Holt equation or versions of it where the parameters K and ν vary periodically [11, 12], or randomly, obeying an independently, identically distributed process [5, 16]. In these works, it was discovered that variation in the environment described by the varying parameters is detrimental for the population, compared to the environment described by the average parameter value. This follows from Jensen's inequality for concave functions. See also the related work of Bohner and Warth [10] on the Beverton-Holt equation on time scales. We call this phenomenon a "loss of capacity", and we will show that a similar phenomenon occurs in a model of self-limiting growth proposed by Solomon to describe the evolution of financial and ecological models [8, 25]. Jensen's inequality for concave functions is known to play an important role in ecology and evolution [24].

Here we investigate a family of quadratic maps that model a self-limiting growth process. Consider a growth law with a noisy state-dependent growth parameter given by

$$x_{n+1} = \theta_n x_n$$

where θ_n is a random variable supported on $[1 - x_n, 2 - x_n]$. If x_n is normalized so that it is always between 0 and 1, the growth parameter θ_n is between 0 and 2. Informally, when x_n is small, the growth parameter is likely to be greater than one; when x_n is large, that is, close to 1, the growth parameter is likely to be less than one. In this sense, the process models "self-limiting growth". The analysis of the process is simplified by rewriting the equation using $\lambda_n = \theta_n + x_n - 1$. Then the growth law is given by

$$f_\lambda(x) = x(1 + \lambda - x) \tag{2}$$

where λ lies in some subset of $[0, 1]$ and is independent of x. This family gives rise to a Markov process on a compact state space whose long-term behavior is often difficult to characterize precisely. In the simplest cases, it can be shown that the first-passage times of such a process follow a power-law distribution [1, 9, 21], while in other cases the long-term behavior of the averages is more difficult to predict [20]. The current work contributes to the understanding of these stochastic growth laws by describing the long-run statistics for a certain class of processes.

This paper is organized as follows. In Section 2 we introduce the notation and state the definition of invariant measures for random dynamical systems. We also state some existence and uniqueness theorems for invariant probability measures. In Section 3 we focus on the family of self-limiting maps (2). Our goal is to find suitable homeomorphisms that give a universal conjugacy to a family of contractions. We also show a shift to the left of the expected average for families of doubly-concave maps. In Section 4 we provide some numerical simulations and investigate the fine structure of an invariant measure when the parameter space consists of two points. The paper ends with a discussion in Section 5.

2. Random dynamical systems.

2.1. Existence and ergodicity of invariant measures.

Let $K \subset \mathbb{R}$, let $\mathcal{B}(K)$ be the σ-algebra of Borel sets on K and m a probability measure on $\mathcal{B}(K)$. The state space S is a compact metric space equipped with the Borel σ-algebra $\mathcal{B}(S)$. We are given a map $f : S \times K \to S$ that is $(\mathcal{B}(S) \otimes \mathcal{B}(K))$-measurable and continuous for every $\lambda \in K$. We will use both notations $f(x, \lambda) = f_\lambda(x)$, whichever is convenient. The sequence space $\Omega = K^{\mathbb{N}}$ has a natural σ-algebra $\mathcal{F} = \mathcal{B}(K)^{\mathbb{N}}$ generated by cylinder sets $A_1 \times A_2 \times \dots$ with $A_i \in \mathcal{B}(K)$, of which only finitely many are proper subsets of K and all others are K. We construct a random dynamical system on the measurable product space $(S \times \Omega, \mathcal{B}(S) \otimes \mathcal{F})$ as follows. For a sequence $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ let θ denote the right shift $\theta(\lambda_1, \lambda_2, \dots) = (\lambda_2, \lambda_3, \dots)$ and $\pi_1 : \Omega \to K$ the projection onto the first coordinate $\pi_1(\lambda_1, \lambda_2, \dots) = \lambda_1$. We consider the *skew product* map $\mathcal{T} : S \times \Omega \to S \times \Omega$ given by

$$\mathcal{T}(x, \boldsymbol{\lambda}) = (f(x, \pi_1(\boldsymbol{\lambda})), \theta(\boldsymbol{\lambda}))\,.$$

For the sake of simplicity, we write $f^n(\,\cdot\,, \boldsymbol{\lambda}) : S \to S$ for the repeated application of f with the first n entries in the random sequence $\boldsymbol{\lambda}$. For an S-valued random variable X_0 independent of $\boldsymbol{\lambda}$, the sequence $X_n = f^n(X_0, \boldsymbol{\lambda})$ is a Markov process. By [18, Lemma 2.1],

$$P(x, G) = m\,\{\lambda \in K \;:\; f_\lambda(x) \in G\}$$

is a transition probability for $x \in S$ and $G \in \mathcal{B}(S)$. This includes that for fixed $G \in \mathcal{B}(S)$, the function $P(\,\cdot\,, G)$ is Borel measurable. The corresponding *transition operator* (or *Perron-Frobenius operator*) acts on bounded measurable functions $\phi \in L^\infty(S)$ by

$$P\phi(x) = \int_S \phi(y) P(x, \mathrm{d}y) = \int_K \phi(f_\lambda(x)) m(\mathrm{d}\lambda). \tag{3}$$

The adjoint operator P^* acts on Borel measures μ on S by

$$P^*\mu(G) = \int_S P(x, G)\mu(\mathrm{d}x) = \int_K \mu\left(f_\lambda^{-1}(G)\right) m(\mathrm{d}\lambda). \tag{4}$$

A probability measure μ on S is called *P^*-invariant* if $P^*\mu = \mu$. Since in this context all maps f_λ are continuous, the operator P maps $C(S)$ into itself (this is called the *Feller property*). Since S is compact, there exists at least one P^*-invariant probability measure on S [18, Lemma 2.2]. The proof of the latter statement defines for an arbitrary probability measure η on S the sequence of Cesàro means

$$\eta_n = \frac{1}{n} \sum_{k=0}^{n-1} (P^*)^k \eta$$

and uses Prokhorov's Theorem to extract a weakly convergent subsequence. This limit is then shown to be P^*-invariant.

A P^*-invariant probability measure μ is *ergodic*, if any Borel set G whose indicator function satisfies $P\chi_G = \chi_G$ is μ-trivial, that is $\mu(G) \in \{0,1\}$. This is equivalent (see [13, Theorem 5.15], [18, Corollary 2.2]) to the statement that for every $\phi \in L^\infty(S)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(\phi) = \int_S \phi(y)\mu(\mathrm{d}y).$$

2.2. **Uniqueness and stability of invariant measures.** On the space $\mathcal{P}(S)$ of all Borel probability measures, convergence in distribution is metrizable by the total variation norm, given by

$$||\mu|| = \sup_{A\in\mathcal{B}(S)} \mu(A) - \inf_{A\in\mathcal{B}(S)} \mu(A).$$

The operator P^* is called *asymptotically stable* if

$$\lim_{n\to\infty} ||(P^*)^n \mu - \mu^*|| = 0,$$

for all $\mu \in \mathcal{P}(S)$, where μ^* is the unique fixed point of P^*. Note that, if X_0 has distribution μ and the adjoint operator is asymptotically stable, then $\lim_{n\to\infty} X_n$ has distribution μ^*. Thus, if P^* is asymptotically stable, we say that the Markov process X_n is *stable in distribution*. There are several approaches to show that a process is stable in distribution. One is to show that the family of maps are themselves contractions in a certain logarithmic average. This is the content of the following theorem, where we assume that the state space is a compact metric space [6, Theorem 3.71]. Let L^r denote the (random) Lipschitz coefficient of $f^r(\cdot, \boldsymbol{\lambda})$,

$$L^r = \sup \left\{ \frac{d(f^r(x,\boldsymbol{\lambda}), f^r(y,\boldsymbol{\lambda}))}{d(x,y)} : x \neq y \right\};$$

the notation suppresses the dependence on $\boldsymbol{\lambda}$.

Proposition 1. *Assume that there exists an $r \geq 1$ such that*

$$\mathsf{E}\left[\log L^r\right] < 0,$$

where E denotes the expectation with respect to the product measure $m^{\otimes r}$. Then the Markov process X_n has a unique invariant probability and is stable in distribution.

This holds in particular, if every iterated map is a contraction. Since the total variation distance dominates the Kolmogorov distance (the uniform distance between cumulative distributions), we can take μ to be any non-atomic measure in $\mathcal{P}(S)$ and by asymptotic stability, the distribution functions of $(P^*)^n\mu$ converge uniformly to the distribution function of the invariant probability measure μ^*. Hence μ^* is also non-atomic. We use this fact in Section 4.

One can also prove stability if the Markov process is Harris irreducible and aperiodic. Assume that ν is a nonzero σ-finite measure on S. A Markov process $\{X_n\}_{n=0}^\infty$ on S with transition probability P is *Harris irreducible with reference measure* ν

(or ν-irreducible for short) if, for every $x \in S$ and every $B \in \mathcal{B}(S)$ with $\nu(B) > 0$, there is an $n \geq 1$ such that $P^n(x, B) > 0$. In other words, any $x \in S$ eventually visits every set of ν-positive measure. The following theorem shows the importance of irreducibility in characterizing long-term behavior.

Proposition 2. [6, Theorem 2.C9.3] *If a Markov process is Harris irreducible and has an invariant probability μ^* then μ^* is the unique invariant probability and*

$$\lim_{n\to\infty} \sup_{A\in\mathcal{B}(S)} \left| \frac{1}{n} \sum_{m=1}^{n} P^m(x, A) - \mu^*(A) \right| = 0.$$

By [23, Theorem 2.2] any Harris irreducible process is cyclic with period $d \geq 1$. This means that there is a sequence $\{E_0, E_1, \ldots, E_{d-1}\}$ of non-empty disjoint sets such that, for any $x \in E_i$, $P\left(x, E_{i+1}^{\complement}\right) = 0$ (where the index is taken modulo d). If $d = 1$ then the Markov process is *aperiodic*. This allows us to conclude that the process is asymptotically stable, and the convergence in Proposition 2 can be strengthened.

Proposition 3. [23, Corollary 6.7] *If a Markov process is Harris irreducible and aperiodic with invariant measure μ^* then, for any $\mu \in \mathcal{P}(S)$*

$$\lim_{n\to\infty} ||(P^*)^n \mu - \mu^*|| = 0.$$

3. **The family of self-limiting maps.**

3.1. **Uniqueness of invariant measures.** Throughout, we identify each map from equation (2) with the real-valued parameter λ. Let λ be a random variable taking values in $[0, 1]$ according to the probability measure m. Let

$$\lambda_- = \min \mathit{supp}\, m, \quad \lambda_+ = \max \mathit{supp}\, m \tag{5}$$

and $\Gamma = [\lambda_-, \lambda_+]$. Then $f_\lambda(\Gamma) \subset \Gamma$ for all $\lambda \in \Gamma$. Thus from now on the interval Γ will play the role of the state space, see Figure 1. As stated before, Γ is equipped with the σ-algebra $\mathcal{B}$ of Borel sets. An instructive example is a Bernoulli distribution with $m(\{\lambda_-\}) = p \in (0, 1)$ and $m(\{\lambda_+\}) = 1 - p$.

At first we observe that if $f'_{\lambda_+}(\lambda_-) < 1$, then all f_λ are contractions on Γ, that is $|f'_\lambda(x)| < 1$ for all $x, \lambda \in \Gamma$. By Proposition 1, the family f_λ has a unique invariant measure μ^*_m for every probability measure m with support boundaries λ_- and λ_+, and is stable in distribution. This is the case if

$$\lambda_+ < 2\lambda_-. \tag{6}$$

In view of the Theorem just cited, we say that the interval $\Gamma = [\lambda_-, \lambda_+]$ is *compressible*, if there exists a homeomorphism $h : \Gamma \to \Gamma$ such that the conjugates

$$g_\lambda = h \circ f_\lambda \circ h^{-1} \tag{7}$$

satisfy $|g'_\lambda(x)| < 1$ for all $x, \lambda \in \Gamma$. We will show that, with certain restrictions on λ_- and λ_+, this conjugacy yields a family of g_λ that are contractions.

Let $\alpha > 0$ and define a homeomorphism $h_\alpha : \Gamma \to \Gamma$ by

$$h_\alpha(x) = \lambda_- + (\lambda_+ - \lambda_-) \left(\frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^\alpha. \tag{8}$$

The inverse and the derivative are, respectively,

$$h_\alpha^{-1}(x) = h_{\alpha^{-1}}(x), \qquad h'_\alpha(x) = \alpha \left(\frac{x - \lambda_-}{\lambda_+ - \lambda_-} \right)^{\alpha-1}.$$

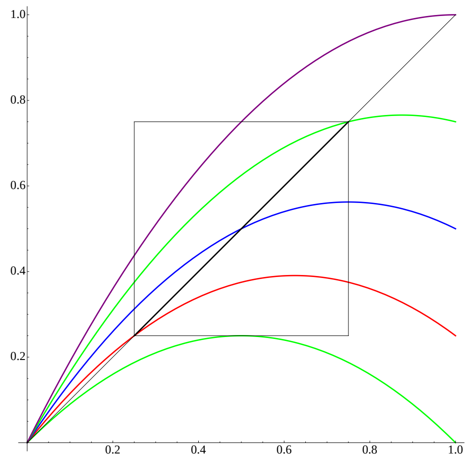

FIGURE 1. The "shark fin" family of maps from equation (2) (rotate the figure clockwise by 90°). The inset square is the example with $\lambda_- = \dfrac{1}{4}$ and $\lambda_+ = \dfrac{3}{4}$.

For every $x \in (\lambda_-, \lambda_+)$, we have that $\dfrac{x - \lambda_-}{\lambda_+ - \lambda_-} \in (0,1)$ and so

$$\lim_{\alpha \to 0+} h_\alpha(x) = \lambda_+, \quad \text{and} \quad \lim_{\alpha \to 0+} h_{\alpha^{-1}}(x) = \lambda_- \tag{9}$$

for all $x \in (\lambda_-, \lambda_+)$. We consider the family of maps obtained defined by (7) with h_α in place of h. By the chain rule,

$$\begin{aligned} g'_\lambda(x) &= h'_\alpha\left(f_\lambda\left(h_{\alpha^{-1}}(x)\right)\right) f'_\lambda\left(h_{\alpha^{-1}}(x)\right) h'_{\alpha^{-1}}(x) \\ &= \left(\frac{f_\lambda\left(h_{\alpha^{-1}}(x)\right) - \lambda_-}{\lambda_+ - \lambda_-}\right)^{\alpha-1} \left(1 + \lambda - 2h_{\alpha^{-1}}(x)\right) \left(\frac{x - \lambda_-}{\lambda_+ - \lambda_-}\right)^{\frac{1-\alpha}{\alpha}}. \end{aligned} \tag{10}$$

If $\lambda > \lambda_-$, the first term of the product is positive. It follows with (9) that

$$\lim_{\alpha \to 0+} g'_\lambda(x) = 0, \tag{11}$$

for $x < \lambda_+$, this holds for $x = \lambda_-$ as well. Moreover, for $x = \lambda_+$,

$$g'_\lambda(\lambda_+) = \left(\frac{f_\lambda(\lambda_+) - \lambda_-}{\lambda_+ - \lambda_-}\right)^{\alpha-1} (1 + \lambda - 2\lambda_+). \tag{12}$$

The first term in this product is positive, while the second can have either sign. Note that $g'_{\lambda_+}(\lambda_+) = 0$, so we can assume $\lambda < \lambda_+$. The first term is maximized by setting $\alpha = 0$ (even though that does not arise from a homeomorphism h_α). Thus we have to consider the expression

$$G(\lambda) = \frac{(\lambda_+ - \lambda_-)(1 + \lambda - 2\lambda_+)}{\lambda_+(1 + \lambda - \lambda_+) - \lambda_-}.$$

If $1 + \lambda - 2\lambda_+ > 0$, then $G(\lambda) > 0$ and we have

$$\frac{\lambda_+(1 + \lambda - 2\lambda_+) - \lambda_-(1 + \lambda - 2\lambda_+)}{\lambda_+(1 + \lambda - \lambda_+) - \lambda_-} \le \frac{\lambda_+(1 + \lambda - 2\lambda_+) - \lambda_-}{\lambda_+(1 + \lambda - \lambda_+) - \lambda_-} < 1.$$

If $1 + \lambda - 2\lambda_+ < 0$, then

$$\frac{(\lambda_+ - \lambda_-)(1 + \lambda - 2\lambda_+)}{\lambda_+(1 + \lambda - \lambda_+) - \lambda_-} \geq 1 + \lambda - 2\lambda_+ > -1,$$

if $\lambda_- > 0$ and $\lambda_+ < 1$. To summarize, we have either the convergence (11) or $|g'_\lambda(\lambda_+)| < 1$ for $\lambda > \lambda_-$. It remains to discuss the case $\lambda = \lambda_-$. We will show first that g_{λ_-} is concave.

For $\lambda = \lambda_-$ we have

$$g''_{\lambda_-}(x) = -(\lambda_+ - \lambda_-)^2 \left(\left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} \left(\lambda_- \left(\left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - \lambda_+ \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + 1 \right) \right)^{\alpha+1}$$
$$\left(\alpha - 2\alpha\lambda_+ \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + \lambda_- \left(\alpha \left(2 \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - 1 \right) + 1 \right) \tag{13}$$
$$\left(\alpha (x-\lambda_-)^2 \left(\lambda_- \left(\left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - \lambda_+ \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + 1 \right) \right)^{-3}.$$

We observe for the individual terms that, since $0 < \lambda_- \le x \le \lambda_+ < 1$ and $\lambda_- < \lambda_+$,

$$\frac{x-\lambda_-}{\lambda_+ - \lambda_-} > 0,$$
$$\lambda_- \left(\left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - \lambda_+ \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + 1 = 1 - \lambda_- + \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} (\lambda_- - \lambda_+) \ge 1 - \lambda_+ > 0, \tag{14}$$
$$\alpha - 2\alpha\lambda_+ \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} + \lambda_- \left(\alpha \left(2 \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} - 1 \right) - 1 \right) + 1 = 1 + \alpha - (1+\alpha)\lambda_- + 2\alpha \left(\frac{x-\lambda_-}{\lambda_+ - \lambda_-} \right)^{\frac{1}{\alpha}} (\lambda_- - \lambda_+)$$
$$\ge 1 + \alpha - (1+\alpha)\lambda_- + 2\alpha(\lambda_- - \lambda_+) = 1 + \alpha + (\alpha - 1)\lambda_- - 2\alpha\lambda_+ = 1 - \lambda_- + \alpha(1 + \lambda_- - 2\lambda_+ > 0.$$

Equation (13) and inequalities (14) imply that g_{λ_-} is concave and so its derivative takes its extreme values at the endpoints of the interval. To simplify the notation, we rewrite (10) as follows

$$g'_\lambda(x) = \frac{h'_\alpha\,(f_\lambda(y))\, f'_\lambda(y)}{h'_\alpha(y)},$$

where we have set $y = h_{\alpha^{-1}}(x)$. By the Rule of Bernoulli and l'Hôpital

$$\lim_{y \searrow \lambda_-} \frac{y - \lambda_-}{f_{\lambda_-}(y) - \lambda_-} = \lim_{y \searrow \lambda_-} \frac{1}{\lambda_- + 1 - 2y} = \frac{1}{1 - \lambda_-}.$$

Since the function $t \mapsto t^{1-\alpha}$ is continuous for $\alpha < 1$, as is the function h_α^{-1}, we get at the left endpoint

$$\begin{aligned} \lim_{x \searrow \lambda_-} g'_{\lambda_-}(x) &= \lim_{y \searrow \lambda_-} \left(\frac{y - \lambda_-}{f_{\lambda_-}(y) - \lambda_-} \right)^{1-\alpha} (\lambda_- + 1 - 2y) \\ &= \left(\lim_{y \searrow \lambda_-} \frac{y - \lambda_-}{f_{\lambda_-}(y) - \lambda_-} \right)^{1-\alpha} (1 - \lambda_-) \\ &= \left(\frac{1}{1 - \lambda_-} \right)^{1-\alpha} (1 - \lambda_-) = (1 - \lambda_-)^\alpha < 1. \end{aligned}$$

At the right endpoint, we require

$$g'_{\lambda_-}(\lambda_+) = (1 + \lambda_- - 2\lambda_+)(1 - \lambda_+)^{\alpha - 1} > -1.$$

This holds certainly if

$$\lambda_+ < \frac{2 + \lambda_-}{3}, \tag{15}$$

since then it is true for all $\alpha > 0$.

Theorem 3.1. *Let the pair $0 < \lambda_- < \lambda_+ < 1$ be an element of the region bounded by the inequalities (6) and (15); see Figure 2. Then the family (f_λ) admits a unique invariant probability measure μ_m^* for every probability measure m supported on $[\lambda_-, \lambda_+]$.*

Proof. Let P^* and Q^* be the transfer operators of the families (f_λ) and (g_λ), respectively. Since

$$Q^* h_\alpha \mu = \int_\Gamma \mu h_\alpha^{-1} g_\lambda^{-1}\, m(\mathrm{d}\lambda) = \int_\Gamma \mu f_\lambda^{-1} h_\alpha^{-1}\, m(\mathrm{d}\lambda) = h_\alpha P^* \mu,$$

the operators are related by the same conjugacy. Since all g_λ are contractions, it follows by Proposition 1 that the family (g_λ) admits a unique invariant probability measure ν_m^* for every probability measure m that has the same support boundaries as in equation (5). Then $\mu_m^* = h_\alpha \nu_m^*$ is the unique invariant probability measure of the family (f_λ). □

A larger region of proper subintervals $\Gamma = [\lambda_-, \lambda_+] \subset [0, 1]$ can be treated under the assumption that the measure m is supported on Γ. By the Lebesgue decomposition theorem, there exists a unique decomposition $m = m_c + m_s$ where m_c is absolutely continuous and m_s is singular with respect to the Lebesgue measure ℓ on Γ.

Theorem 3.2. *Let $0 < \lambda_- < \lambda_+ \leq 1$ and assume in addition that $\operatorname{supp} m = \Gamma$. Then there is a unique probability measure that is a fixed point of the operator P^* which also has support Γ, and the operator P^* is asymptotically stable.*

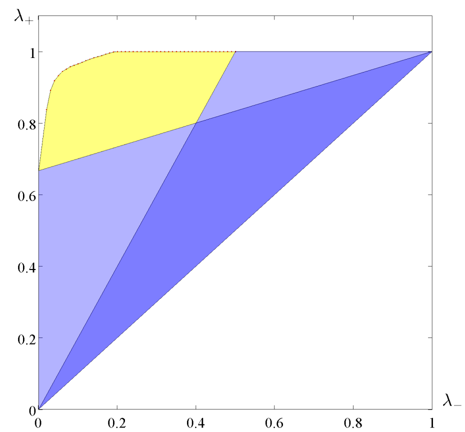

FIGURE 2. The region of interval ends (λ_-, λ_+) determined by inequalities (6) and (15) (light purple) where the invariant measure is unique independent of the probability measure m. The yellow region shows intervals that are compressible by two-parameter homeomorphisms from equation (16), as indicated numerically in Section 4.

Proof. Let E be an arbitrary open subset of Γ and $\tilde{E} \subset E$ be an open proper subset. By assumption, $m_c(\tilde{E}) = p > 0$. By the independence assumption, it follows that the event

$$H_{k,l} = \left(\lambda_k \in \tilde{E}, \dots, \lambda_{k+l-1} \in \tilde{E}\right)$$

has probability $p^l > 0$ and its complementary event $H^{\complement}_{k,l}$ has probability $1 - p^l < 1$. The events $H_{k,l}, H_{k+l,l}, H_{k+2l,l}, H_{k+3l,l}, \dots$ are independent. Therefore, for every k and l, the event $\bigcap_{i \geq 0} H^{\complement}_{k+il,l}$ has probability 0. By the Borel-Cantelli lemma, for any $x \in \Gamma$ and for a set $\Omega' \subset \Omega$ with $m^{\mathbb{N}}(\Omega') = 1$, the event $f^l(x, \boldsymbol{\lambda}) \in \tilde{E} \subset E$ occurs infinitely often for all $\boldsymbol{\lambda} \in \Omega'$. If μ is a Borel probability measure on Γ, then there exists an $l \geq 0$ such that $\left(supp\,(P^*)^l \mu\right) \cap E \neq \varnothing$. Thus the process X_n is Harris irreducible with reference measure ℓ on Γ and by Proposition 2 the invariant probability measure is unique.

To show that the Markov process is aperiodic, recall that λ is a stable fixed point for f_λ. Thus for every $x \in \Gamma$ there exists a $\delta > 0$ such that $[x - \delta, x + \delta] \subset \Gamma$ and $P(x, [x - \delta, x + \delta]) > 0$. Therefore, there can be no cycle of length $d > 1$. □

Remark 1. Theorem 3.2 holds for any family of maps such that for each point of an interval $x \in \Gamma$, there exists a map that has x as its unique globally attracting fixed point.

Remark 2. If $m = m_c$, then we can say more about the fine structure of the invariant probability measure. If $\lambda_- > 0$, then the operator P^* maps every measure to an absolutely continuous one, since the map $\lambda \mapsto f_\lambda(x)$ is injective for every $x \in \Gamma$. Hence the invariant probability measure is absolutely continuous. If $\lambda_- = 0$, then P^* maps every measure μ with $\mu(\{0\}) = 0$ to an absolutely continuous measure. Clearly the Dirac measure δ_0 is invariant. Any invariant probability measure is a

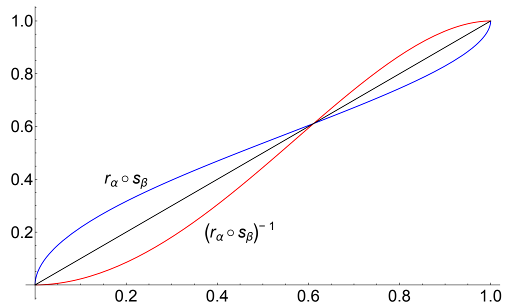

FIGURE 3. The unit interval homeomorphisms obtained from composition of $r_\alpha(x) = x^\alpha$ and $s_\beta(x) = 1-(1-x)^\beta$. Here we use $\alpha = \beta = 0.52$.

convex combination of δ_0 and an absolutely continuous measure, depending on the initial condition.

3.2. **Loss of capacity.** As stated in Section 1, it is well known that a periodically or randomly varying environment may have a deleterious effect on the state average, see for example [5, 10, 11, 12, 16]. This is the content of the following theorem.

Theorem 3.3. *Assume that $f(\,\cdot\,,\lambda)$ is strictly concave and $f(x,\,\cdot\,)$ is concave. Let μ denote the unique invariant measure and*

$$\bar\lambda = \int_K \lambda\, m(\mathrm{d}\lambda), \quad \bar x = \int_{[0,1]} x\,\mu(\mathrm{d}x).$$

Then $\bar x \le x^{\bar\lambda}$ with equality if and only if m is a Dirac measure.

Proof. Since the function $f(x,\lambda)$ is concave in either argument and strictly concave in x, applying Jensen's inequality twice gives

$$\begin{aligned}\bar x &= \int_{[0,1]} x\,\mu(\mathrm{d}x) = \int_{[0,1]} x\,P^*\mu(\mathrm{d}x) = \int_{[0,1]}\int_K f(x,\lambda)\,m(\mathrm{d}\lambda)\,\mu(\mathrm{d}x)\\ &\le \int_{[0,1]} f(x,\bar\lambda)\,\mu(\mathrm{d}x) \le f(\bar x,\bar\lambda).\end{aligned}$$

By the concavity of $f(\,\cdot\,,\bar\lambda)$, it follows that $\bar x \le x^{\bar\lambda}$. The last inequality is strict if μ is not supported in a single point, that is, if m is not supported in a single point. □

Remark 3. Theorem 3.3 holds for arbitrary random families of concave maps that admit a unique invariant measure.

4. Numerical simulations and examples.

The region of intervals that we currently know to be compressible by the one-parameter family of homeomorphisms from (8) is described by the inequalities (6) and (15). Numerical results indicate that a larger region of intervals is compressible. We define a two-parameter family of homeomorphisms of Γ by

$$h_{\alpha,\beta}(x) = \lambda_- + (\lambda_+ - \lambda_-)\left(1 - \left(\frac{\lambda_+ - x}{\lambda_+ - \lambda_-}\right)^\beta\right)^\alpha \tag{16}$$

and use these in the conjugacy (7); see Figure 3.

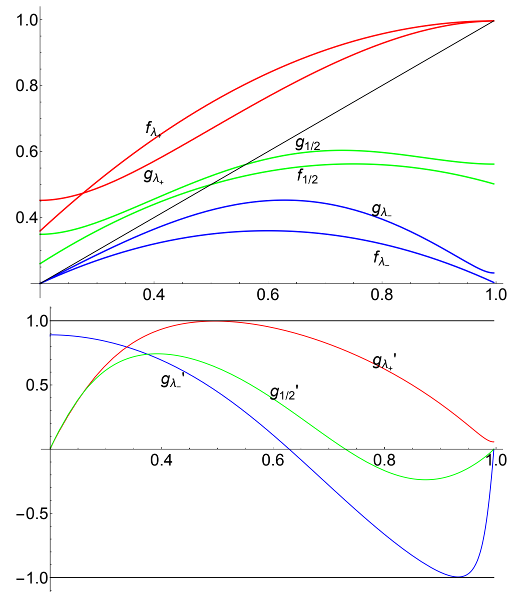

FIGURE 4. Sample members of the families of maps f_λ and g_λ *(left)* and the corresponding derivatives g'_λ *(right)*. Here $\lambda_- = 0.2$, $\lambda_+ = 0.996$ and $\alpha = \beta = 0.52$. Other derivatives g'_λ with intermediate values $\lambda \in (\lambda_-, \lambda_+)$ also stay between -1 and 1, though not always between g'_{λ_-} and g'_{λ_+}.

In Figure 4 we show an example where $\lambda_- = 0.2$, $\lambda_+ = 0.996$ and $\alpha = \beta = 0.52$; see also the yellow region in Figure 2. We observe that extremal values of the derivatives g'_λ are taken for the extremal parameter values λ_- and λ_+. We also observe that if the interval $[\lambda_-, \lambda_+]$ is compressible by the homeomorphism $h_{\alpha,\beta}$, then so is the subinterval $[\lambda_-, \tilde{\lambda}_+]$ with $\tilde{\lambda}_+ < \lambda_+$, using the same parameters α and β for the homeomorphism; see Figure 5. Values of compressing exponents for maximal intervals with varying lower boundary are listed in Table 1.

λ_-	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.22	0.24
λ_+	0.84	0.92	0.94	0.96	0.96	0.97	0.98	0.99	0.99	1.00	1.00	1.00
α	0.20	0.20	0.20	0.22	0.24	0.30	0.32	0.40	0.40	0.42	0.54	0.60
β	0.66	0.36	0.24	0.22	0.22	0.28	0.28	0.38	0.34	0.34	0.52	0.58

TABLE 1. Maximal upper interval bound λ_+ as a function of the lower interval bound λ_- and numerically computed (α,β) pairs of compressing conjugacies.

In the special case that λ is uniformly distributed on $[0, 1]$, we know from Remark 2 that the invariant probability measure has a density, provided that it does not assign a mass to the point 0. In this case the Perron-Frobenius operator can be

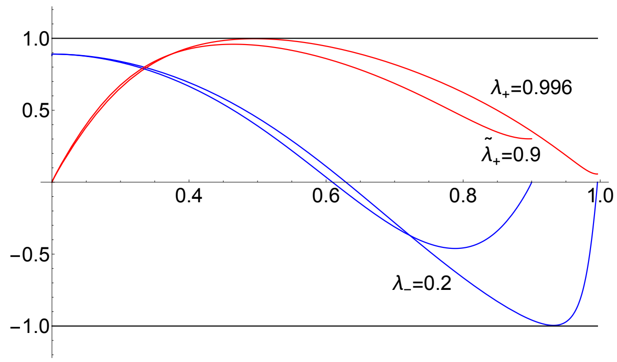

FIGURE 5. A compression of the subinterval $[\lambda_-, \tilde{\lambda}_+]$ of $[\lambda_-, \lambda_+]$ with $\tilde{\lambda}_+ < \lambda_+$ using the homeomorphism $h_{\alpha,\beta}$ with the same parameters α and β. Shown are the derivatives of the maps g_λ with the extremal parameter values. Here $\lambda_- = 0.2$, $\lambda_+ = 0.996$ and $\tilde{\lambda}_+ = 0.9$. The homeomorphism parameters are $\alpha = \beta = 0.52$.

written explicitly,

$$P\phi(x) = \begin{cases} \displaystyle\int_{1-\sqrt{1-x}}^{\frac{1-\sqrt{1-4x}}{2}} \frac{\phi(y)}{y}\,\mathrm{d}y + \int_{\frac{1+\sqrt{1-4x}}{2}}^{1} \frac{\phi(y)}{y}\,\mathrm{d}y & \text{if} \quad 0 \le x < \frac{1}{4} \\ \displaystyle\int_{1-\sqrt{1-x}}^{1} \frac{\phi(y)}{y}\,\mathrm{d}y & \text{if} \quad \frac{1}{4} \le x \le 1 \end{cases} . \tag{17}$$

We plot iterates $\phi^{n+1} = P^n\phi$, starting with the uniform density in Figure 6. The iterates converge quickly to an invariant density with a maximum at $x = \dfrac{1}{4}$, where the derivative of the density is not defined. This can also be deduced from the explicit formulation given in equation (17).

A different picture can emerge if m is supported on two points $\lambda_- < \lambda_+$ and

$$\max_{x\in\Gamma} f_{\lambda_-}(x) < \min_{x\in\Gamma} f_{\lambda_+}(x), \tag{18}$$

creating a gap on which the invariant probability measure must be zero. In this case, the invariant measure can be supported on a Cantor set. We illustrate with an example similar to the example for the logistic family given in [7]. In this example, we choose $\lambda_- = \dfrac{1}{2}$ and $\lambda_+ = \dfrac{7}{8}$. With this choice of parameter values we have $|f'_\lambda(x)| < 1$ for all x, so that both maps are strict contractions, and equation (18) holds, since $f_{\lambda_+}(\lambda_-) > \max_{x\in\Gamma} f_{\lambda_-}(x)$. By Proposition 1 and the remarks that follow, the process is stable in distribution, and the invariant measure μ^* is non-atomic. We show that the support is a closed, completely disconnected set with no isolated points.

For every word $\boldsymbol{\lambda}^{(n)} \in \{-,+\}^n$ let $I_{\boldsymbol{\lambda}^{(n)}} = f^n(\Gamma, \boldsymbol{\lambda}^{(n)})$ and let

$$J_n = \bigcup_{\boldsymbol{\lambda}^{(n)}\in\{-,+\}^n} I_{\boldsymbol{\lambda}^{(n)}}$$

where the union is taken over all possible sequences of symbols $-$ and $+$ of length n. By Equation 18, $I_- \cap I_+ = \varnothing$. Likewise, J_2 is a union of four disjoint closed intervals, since every point of $f_{\lambda_-}(I_-)$ lies to the left of every point of $f_{\lambda_-}(I_+)$. We

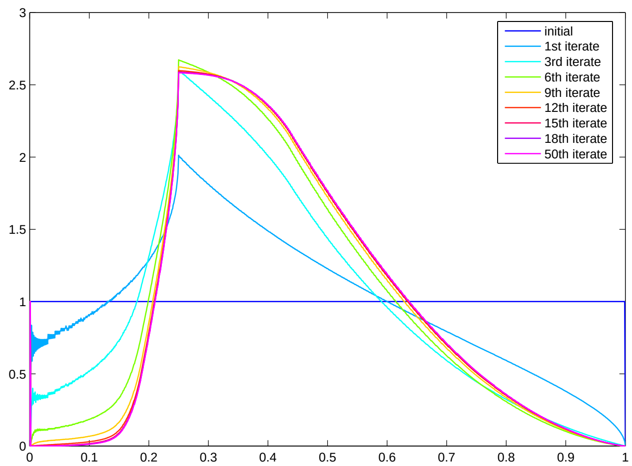

FIGURE 6. Iterates of the Perron-Frobenius operator given in equation (17) beginning with the uniform density on $[0, 1]$. The iterates converge quickly to the invariant density.

note that the restriction of either map to J_2 is injective. This follows from the fact that

$$f_{\lambda_-}\left(f_{\lambda+}\left(f_{\lambda_-}(\lambda_-)\right)\right) > f_{\lambda_-}\left(f_{\lambda_+}\left(f_{\lambda_+}(\lambda_-)\right)\right).$$

Hence all subintervals of J_3 are disjoint, see Figure 7. For $n > 3$, the injectivity of the maps restricted to each J_{n-1}, ensures that all subintervals of J_n are disjoint.

The nested sequence $\{J_n\}$ consists of closed, non-empty sets with

$$J = \bigcap_{n=1}^{\infty} J_n \supset supp\, \mu^*.$$

Since J is the closure of the endpoints of all $I_{\boldsymbol{\lambda}^{(n)}}$, it is equal to the closure of the orbits of λ_- or λ_+. Hence, $J \subset supp(\mu^*)$ and therefore $J = supp(\mu^*)$. Since both maps contract all intervals by at least a factor of $7/8 = f'_{\lambda_+}(\lambda_-)$, the lengths of the intervals $I_{\boldsymbol{\lambda}^{(n)}}$ go to zero as $n \to \infty$. Thus J does not contain any intervals, i.e. it is completely disconnected. Since μ^* is non-atomic, its support contains no isolated points, and we have shown that $supp\, \mu^*$ is a Cantor Set.

5. **Discussion.** To the best of our knowledge, there are no other general theorems that guarantee the uniqueness of an invariant probability measure for a family of interval maps than those that originated in the work by Dubins and Freedman [15] and their subsequent generalizations in [6]. Theorem 7.1 in [6] is stated for the random Lipschitz constants of compositions of maps of depth $r \geq 1$. As our maps are quadratic, compositions of depth r are polynomials of degree 2^r in x, whose coefficients themselves are polynomials in the random parameters $\lambda_1, \ldots, \lambda_r$. It is not a promising enterprise to find patterns in these polynomials and to establish that the Lipschitz constants of the compositions are all < 1 (we do not even conjecture that this is the case). Our approach via conjugacies makes it possible to extend the

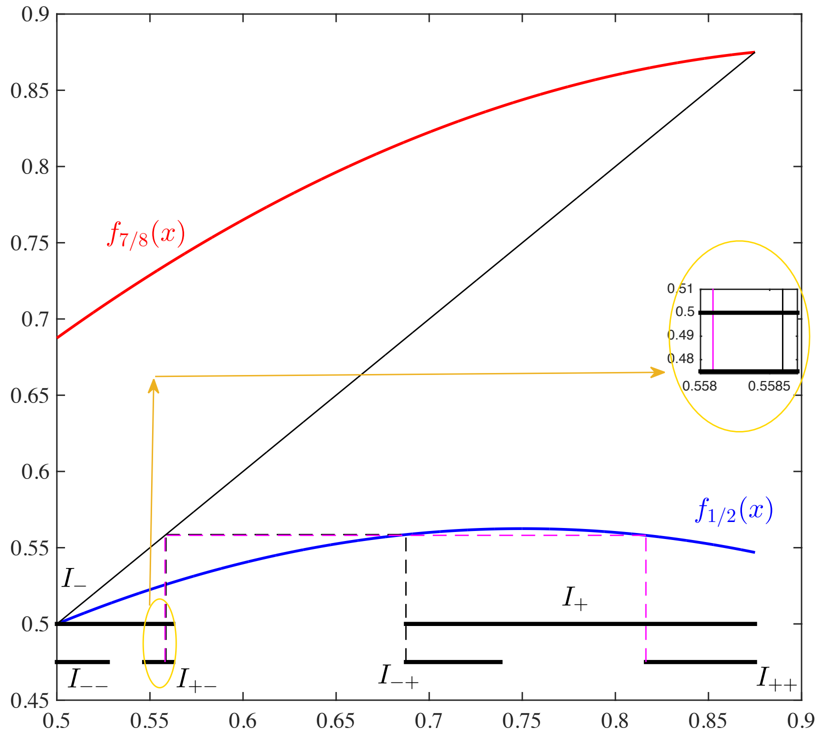

FIGURE 7. The distribution of parameter values is supported on two points: $\lambda_- = 1/2$ and $\lambda_+ = 7/8$. Both maps f_{λ_-} and f_{λ_+} are strict contractions, and the images of $\Gamma = [\lambda_-, \lambda_+]$ under f_{λ_-} and f_{λ_+} are disjoint closed intervals I_- and I_+. The inset shows that images of I_{-+} and I_{++} under f_{λ_-} are disjoint: $f_{\lambda_-}(I_{++}) = [\frac{35}{64}, \frac{269312}{482560}]$, $f_{\lambda_-}(I_{-+}) = [\frac{269555}{482560}, \frac{2047}{3640}]$. Successive iterates result in nested sequences of disjoint closed intervals whose union contains the support of the invariant measure.

applicability of the existing theorems. It seems an open question to classify function families that are conjugated to families of contractions by a single homeomorphism.

In Theorem 3.2 we have established a uniqueness result for the case that the parameter is absolutely continuous with respect to the Lebesgue measure and its density g is supported on the entire parameter space which here coincides with the state space. The papers by Bezandry *et al.* [5] and Haskell and Sacker [16] use explicit expressions for the Perron-Frobenius operator for a randomized version of the Beverton-Holt equation, acting on the space of probability densities only. The uniqueness of the invariant probability density follows from the positivity of a sufficiently large power P^n, see [19, Theorem 5.6.1]. However, even a unique invariant probability density may coexist with a second invariant probability measure supported on a Lebesgue null set, as the following example shows (similar to the example in [3]).

Let $\Gamma = [0,1]$ and select $\frac{1}{2} < b < a < 1$. We define functions

$$\begin{aligned} f_1(x) &= \begin{cases} \dfrac{bx}{a} & \text{if} \quad 0 \le x \le a \\ b - \dfrac{b(x-a)}{1-a} & \text{if} \quad a \le x \le 1 \end{cases}, \\ f_2(x) &= \begin{cases} 1 - \dfrac{bx}{1-a} & \text{if} \quad 0 \le x \le 1-a \\ 1 - b + \dfrac{b(x-1+a)}{a} & \text{if} \quad 1-a \le x \le 1 \end{cases}, \end{aligned} \tag{19}$$

see the left panel of Figure 8. Clearly, the set of end points $\{0,1\}$ is invariant under both f_1 and f_2. Thus if $m(\{\lambda = 1\}) = m(\{\lambda = 2\}) = \frac{1}{2}$ and $X_{n+1} = f_{\lambda_n}(X_n)$, then the measure $\frac{1}{2}(\delta_0 + \delta_1)$ is an invariant probability measure. For the Markov chain starting with $X_0 = x$, let

$$\mu_{x,n}(A) = \frac{1}{n} \sum_{j=0}^{n-1} P_x(X_j \in A)$$

be the sequence of *occupation measures*. Since the state space Γ is compact and the process X_n has the Feller property, any vague limit point ν of the sequence $(\mu_{x,n})_{n=1}^{\infty}$ is an invariant probability measure.

Proposition 4. *There exists a vague limit ν that assigns a positive measure to some subinterval of Γ.*

Proof. This can be shown similarly as in Foster's Theorem with the help of a Lyapunov function, see [3] and [22]. Let

$$h(x) = \begin{cases} 1 - 2x & \text{if} \quad 0 \le x < \dfrac{1}{2} \\ 2x - 1 & \text{if} \quad \dfrac{1}{2} \le x \le 1 \end{cases}.$$

Let $\kappa = \dfrac{1-a}{2b}$ and $0 < x < \kappa$, it follows that $f_2(x) > \frac{1}{2}$. Then

$$\begin{aligned} \varphi(x) := \mathsf{E}_x(h(X_1)) - h(x) &= \frac{1}{2}\left(1 - 2\frac{bx}{a}\right) + \frac{1}{2}\left(2\left(1 - \frac{bx}{1-a}\right) - 1\right) - (1-2x) \\ &= -\frac{bx}{a} - \frac{bx}{1-a} + 2x \le -\delta, \end{aligned}$$

for some $\delta > 0$ if a and b are sufficiently close to 1. By symmetry, $\varphi(1-x) = \varphi(x)$ and so the inequality holds for $1 - \kappa < x < 1$ as well. Since h is bounded, the function φ is also bounded on Γ, say $\varphi(x) \le \gamma$. Now let $J = [\kappa, 1-\kappa]$. By the linearity of the expectation and the Markov property, we have

$$\begin{aligned} \mathsf{E}_x(h(X_n)) - h(x) &= \sum_{j=1}^{n} \mathsf{E}_x(h(X_j) - h(X_{j-1})) = \sum_{j=1}^{n} \mathsf{E}_x(\varphi(X_{j-1})) \\ &= \sum_{j=1}^{n} \mathsf{E}_x(\varphi(X_{j-1}) : X_{j-1} \in J) + \sum_{j=1}^{n} \mathsf{E}_x(\varphi(X_{j-1}) : X_{j-1} \notin J). \end{aligned}$$

Dividing this by n gives

$$\frac{1}{n}(\mathsf{E}_x(h(X_n)) - h(x)) \leq \gamma\mu_{x,n}(J) - \delta\mu_{x,n}(\Gamma \setminus J) = (\gamma + \delta)\mu_{x,n}(J) - \delta.$$

Assume $\nu(J) = 0$ for all vague limits of $(\mu_{x,n})_{n=1}^{\infty}$, then $\lim_{n\to\infty} \mu_{x,n}(J) = 0$. Thus taking the limit in the previous inequality gives

$$\lim_{n\to\infty} \frac{1}{n}(\mathsf{E}_x(h(X_n)) - h(x)) \leq -\delta < 0.$$

However, the limit is indeed

$$\lim_{n\to\infty} \frac{1}{n}\mathsf{E}_x(h(X_n)) \geq 0$$

since $h(x) \geq 0$. This shows that there is a vague limit ν with $\nu(J) > 0$. □

The numerical simulation in the left panel of Figure 8 suggests that the limit measure is absolutely continuous. By the symmetry of the maps and the equal probabilities, it is symmetric about $\frac{1}{2}$. Note that in this example each map has a globally attracting fixed point. This can also happen with an absolutely continuous parameter distribution, if we extend the family by setting $f_\lambda = \lambda f_1$ for $\lambda \in [1 - \varepsilon, 1 + \varepsilon]$, with similar modifications to f_2.

Finally, there is also an interesting inverse problem that can be stated. Given two probability measures μ and ν whose supports have the same interval hull Γ and that are mutually singular, is there a family of functions $f_\lambda : \Gamma \to \Gamma$ and a probability measure m on the parameter space, such that both measures are fixed points of the operator P^*? These could be for example the Cantor measure, supported on the middle-third set, and the Lebesgue measure.

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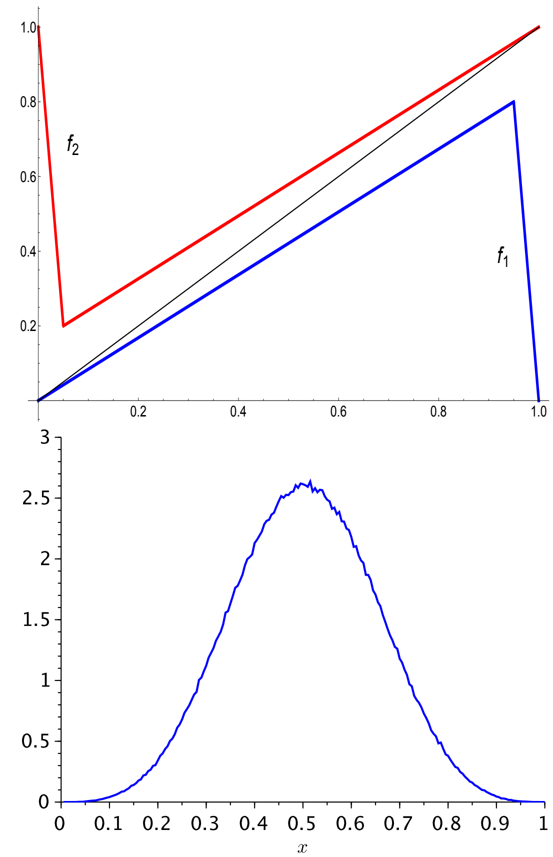

FIGURE 8. *(Left)* The functions f_1 and f_2 from equation (19) with $a = 0.95$ and $b = 0.8$. *(Right)* A numerical approximation of the invariant measure supported on $[0, 1]$ by 10^6 iterations when f_1 and f_2 are chosen with equal probability.

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