Linear Stability of Delayed Reaction-Diffusion Systems

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Collaborator

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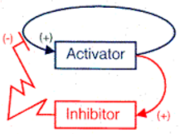
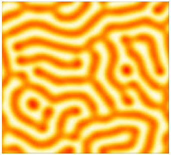
Overview of the talk

- notions of matrix stability
- delayed reaction-diffusion systems
- results

Background

- Alan Turing, 1952: diffusion can destabilize stable steady states of ODEs.
- 1970s: various reaction mechanisms proposed (Gierer-Meinhard model for example).





Background

- Steady states can also be destabilized by delays.
- In many cases the delays are off-diagonal, or “harmless” (Hofbauer & So, 2000).

How do these two phenomena interact?

Hadeler & Ruan (2007) showed that for 2-dimensional systems, stability with respect to off-diagonal delay implies stability with respect to diffusion. We extend this result to the general case.

Matrix stability

A matrix $A \in \mathbb{R}^{n \times n}$ is called

(a) *stable* if all eigenvalues of A have negative real part,

(b) *strongly stable with respect to diffusion* if $A - D$ is stable for every non-negative diagonal matrix D,

(c) *excitable with respect to diffusion* if it is stable, but not strongly stable with respect to diffusion.

Matrices that are strongly stable with respect to diffusion have been characterized up to order $n = 3$ (Cross, 1978).

Principal submatrices and minors

Let $A \in \mathbb{R}^{n\times n}$ and $I \subset \{1, \ldots, n\}$ a subset of indices.

(a) The *principal minor* is denoted by $\det A[I]$, where the rows and columns with indices in I have been **kept**.

(b) The *complementary principal minor* is denoted by $\det A(I)$, where the rows and columns with indices in I have been **deleted**.

(c) The quantity $(-1)^{|I|} \det A[I]$ is called the *signed* principal minor.

Reaction-diffusion systems and Turing Instability

Let $u = (u_1, \ldots, u_n)$, $\Omega \subset \mathbb{R}^k$ and consider both

$$\frac{du}{dt} = f(u), \quad \text{and} \quad \frac{\partial u}{\partial t} = f(u) + D\Delta u, \quad x \in \Omega,\ t \geq 0$$

supplied with initial and homogeneous Neumann boundary conditions. Let u^* be a hyperbolic equilibrium of f and let

$$A = \frac{\partial f_i}{\partial u_j}(u^*)$$

be the Jacobian of f at that point. Using the Fourier ansatz, the characteristic equation becomes

$$\det(A - \mu D - \lambda I) = 0$$

where $\mu > 0$ is an eigenvalue of the negative Laplacian with homogeneous Neumann boundary conditions.

Sufficient condition for excitability

Proposition

(Wang & Li, 2001) Let $A \in \mathbb{R}^{n\times n}$ be stable and suppose that A has a negative signed principal minor. Then A is excitable with respect to diffusion.

The result follows from the linearity of the determinant

$$\det(A - D) = \sum_{k=0}^{n} (-1)^k \sum_{|I|=k} \left(\det A(I) \prod_{i\in I} d_i \right).$$

Set the diffusion constants d_i to 0 in the set I for which $(-1)^{|I|} \det A[I] \geq 0$ and to choose them positive outside I.

Introduction of delays

We consider the delay differential equation

$$\frac{du_i}{dt} = f_i(u_1(t - \tau_{i1}), \dots, u_n(t - \tau_{in})) \quad i = 1, 2, \dots, n$$

where $\tau_{ij} \geq 0$ are the delays $1 \leq i \neq j \leq n$ and $\tau_{ii} = 0$ for all i. The linearization at the steady state u^* is

$$\frac{du_i}{dt} = \sum_{j=1}^{n} a_{ij} u_j(t - \tau_{ij}) \quad i = 1, \dots, n$$

where $a_{ij} = \dfrac{\partial f_i}{\partial u_j}(u^*)$.

Delay stability

The corresponding characteristic polynomial is

$$\det \begin{bmatrix} a_{11} - \lambda & a_{12} e^{-\lambda \tau_{12}} & \cdots \\ a_{21} e^{-\lambda \tau_{21}} & a_{22} - \lambda & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = 0. \tag{1}$$

A matrix $A \in \mathbb{R}^{n \times n}$ is called *strongly stable with respect to delay* if the characteristic polynomial has only roots with negative real parts for any choice $\tau_{ij} \geq 0$, $1 \leq i \neq j \leq n$.

Such delays are called *harmless*. Note that this definition implies that A itself is stable.

Characterization of delay stability

Let $\tilde{A}$ be the matrix obtained from A by replacing a_{ij} by $|a_{ij}|$ for all $i \neq j$. Then A is called *weakly diagonally dominant* if all principal minors of $-\tilde{A}$ are non-negative.

Theorem

(Hofbauer & So, 2000) The zero solution of the linear delay system $u' = Au_\tau$ is asymptotically stable for all choices of harmless delays if and only if $a_{ii} < 0$ for all i, $\det A \neq 0$ and A is weakly diagonally dominant.

Reaction-diffusion systems with delays

The reaction-diffusion system with delays is

$$\frac{\partial u_i}{\partial t}(x,t) = f_i(u_1(t-\tau_{i1}),\dots,u_n(t-\tau_{in})) + d_i \Delta u_i, \quad i = 1,2,\dots n,$$

and its linearization

$$\frac{\partial u_i}{\partial t}(x,t) = \sum_{j=1}^{n} a_{ij} u_j(t-\tau_{ij}) + d_i \Delta u_i$$

with characteristic polynomial (in λ)

$$\det \begin{bmatrix} a_{11} - d_1 - \lambda & a_{12} e^{-\lambda \tau_{12}} & \dots \\ a_{21} e^{-\lambda \tau_{21}} & a_{22} - d_2 - \lambda & \dots \\ \dots & \dots & \dots \end{bmatrix} = 0.$$

Persistence of delay stability

Theorem

(H. & Mincheva, 2016) If A is strongly stable with respect to delay, then $A - D$ is strongly stable with respect to delay for any diagonal matrix $D \geq 0$.

Ideas from the proof

Since A is weakly diagonally dominant, there exists a vector $c > 0$ such that

$$a_{ii}c_i + \sum_{j \neq i} |a_{ij}| c_j \leq 0 \quad i = 1, 2, \ldots, n,$$

(Fiedler & Ptak, 1962). It follows that

$$(a_{ii} - d_i)c_i + \sum_{j \neq i} |a_{ij}| c_j \leq 0 \quad i = 1, 2, \ldots, n$$

is also satisfied for any $d_i \geq 0$ and the same $c > 0$. $A - D$ is weakly diagonally dominant.

Invertibility of $A - D$

Let $r(s) = \det(A - sD)$ for $s \geq 0$. By the linearity of the determinant we have

$$\frac{d}{ds} r(s) = -\sum_{i=1}^{n} d_i \det(A(i)) + 2s \sum_{i,j=1, i\neq j}^{n} d_i d_j \det(A(i,j)) - 3s^2 \sum_{i,j,k=1, i\neq j\neq k\neq i}^{n} d_i d_j d_k \det(A(i,j,k)) + \dots + ns^{n-1}(-1)^n \prod_{i=1}^{n} d_i.$$

We show that r is monotone increasing if n is even and monotone decreasing if n is odd. Since $r(0) = \det A$ has the sign $(-1)^n$, the function r has no zero.

Delay stability implies diffusion stability

Corollary

If A is strongly stable with respect to delay, then A is strongly stable with respect to diffusion.

For any diagonal matrix $D \geq 0$, $A - D$ is strongly stable with respect to delay. Hence it is stable (all eigenvalues have real part < 0). Thus A is strongly stable with respect to diffusion.

Fast diffusion cancels delay effects

Theorem

(H. & Mincheva, 2016) If A is strongly stable stable with respect to diffusion, then $A - D$ is strongly stable with respect to delay if $d_i > 0$, $i = 1, 2, \ldots, n$ are sufficiently large.

For sufficiently large $d_i > 0$,

$$-a_{ii} + d_i > \sum_{j \neq i} |a_{ij}|, \quad i = 1, 2, \ldots, n.$$

It follows that all principal minors of $-\tilde{A} + D$ are positive (Fiedler, 1986).

Excitation of a diffusion-stable matrix by delay

Proposition

Let A be strongly stable with respect to diffusion and $a_{ii} < 0$ for all i. Suppose that $\det(-\tilde{A} + D) < 0$ for some D. Then there exist harmless delays such that the characteristic polynomial

$$\det \begin{bmatrix} a_{11} - d_1 - \lambda & a_{12} e^{-\lambda \tau_{12}} & \dots \\ a_{21} e^{-\lambda \tau_{21}} & a_{22} - d_2 - \lambda & \dots \\ \dots & \dots & \dots \end{bmatrix} = 0.$$

has a root with a positive real part.

Let

$$F_\epsilon(z) = \det \begin{bmatrix} a_{11} - d_1 - z\epsilon & a_{12} e^{-z\eta_{12}} & \dots \\ a_{21} e^{-z\eta_{21}} & a_{22} - d_2 - z\epsilon & \dots \\ \dots & \dots & \dots \end{bmatrix}.$$

where $\eta_{ij} = 1/2$ for $a_{ij} < 0$ and $\eta_{ij} = 1$ for $a_{ij} \geq 0$ and $\epsilon > 0$ is small.

Excitation of a diffusion-stable matrix by delay

We let $z = x + 2\pi i$ and

$$F_0(x) = \det \begin{bmatrix} a_{11} - d_1 & |a_{12}|e^{-x\eta_{12}} & \dots \\ |a_{21}|e^{-x\eta_{21}} & a_{22} - d_2 & \dots \\ \dots & \dots & \dots \end{bmatrix}.$$

By the assumptions and the Intermediate Value Theorem, there exists $\bar{x} > 0$ such that $F_0(\bar{x}) = 0$. By Rouché's Theorem, $F_\epsilon(z)$ has a root $\bar{z}(\epsilon)$. This gives the delays $\tau_{ij} = \dfrac{\eta_{ij}}{\epsilon}$ that result in destabilization.

Example

Simplified version of a model by Satnoianu *et al.* (2000)

$$\frac{\partial u_1}{\partial t} = \frac{1}{u_2^a(t-\tau_{12})} - u_1 + d_1 \Delta u_1,$$
$$\frac{\partial u_2}{\partial t} = u_3(t-\tau_{23}) - u_2 + d_2 \Delta u_2,$$
$$\frac{\partial u_3}{\partial t} = u_1(t-\tau_{31}) - u_3 + d_3 \Delta u_3,$$

where $a > 0$. We can think of this model as of a chain of "activations" $u_1 \nearrow u_3 \nearrow u_2$ and an "inhibition" $u_2 \searrow u_1$, provided that $a > 0$. The system has a constant steady state $u^* = (1, 1, 1)$.

Example

The Jacobian A_a of the kinetic system at the steady state $(1, 1, 1)$ is

$$A_a = \begin{bmatrix} -1 & -a & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

This matrix is nonsingular and weakly diagonally dominant for $a \in (0, 1]$. Thus A_a is strongly stable with respect to delay in that range.

Example

$$A_a = \begin{bmatrix} -1 & -a & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

- A_a is strongly stable with respect to diffusion if $0 < a < 8$
- A_a is not strongly stable with respect to delay if $a > 1$.

$A_a - D$ where $1 < a < 8$ will be strongly stable with respect to delay if the diffusion rates $d_i > 0$ are sufficiently large.

Acknowledgments

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P. Hinow, M. Mincheva, Linear Stability of Delayed Reaction-Diffusion Systems, *submitted*, (2016)

Thank you for your attention