Lecture I: Coxeter Groups

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1. Coxeter Systems

- A *Coxeter system* is a pair (W, S), where W is a group, and $S \subset W$ is a set of generators subject to relations

$$(ss')^{m(s,s')} = 1, \quad s, s' \in S,$$

where $m(s,s) = 1$ and $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$. If no relation occurs for s, s', then $m(s,s') = \infty$. The rank of (W, S) is $|S|$.

- The *Coxeter graph* Γ associated with (W, S) is an undirected graph with S as the vertex set, joining vertices s and s' by an edge labeled $m(s,s')$ whenever $m(s,s') \geq 3$. If $s \neq s'$ are not joined, then $m(s,s') = 2$. To simplify notation, $m(s,s') = 3$ may be omitted.

1. Coxeter Systems

- Example 1.1. The pair (S_n, S) is a Coxeter group, where S_n is the permutation group of $\{1, 2, \ldots, n\}$, and

$$S = \{(i, i+1) | 1 \leq i \leq n-1\},$$

where $(i, i+1)$ is the permutation that interchanges i and $i+1$. The Coxeter graph for this Coxeter system is

Γ:

- Example 1.2. If $|S| = 2$, the Coxeter group W is the Dihedral group D_m.

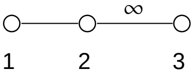
1. Coxeter Systems

- **Example 1.3.** Let $S = \{s_1, s_2, s_3\}$ and

$$m(s_1, s_2) = 3, \quad m(s_1, s_3) = 2, \quad m(s_2, s_3) = \infty.$$

The Coxeter graph is

Γ:



We have

$$W \simeq PGL(2, \mathbb{Z}) = GL(2, \mathbb{Z})/\{\pm 1\},$$

where the isomorphism is given by

$$\varphi : \quad s_1 \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 \longrightarrow \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s_3 \longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. Length Function

- For $w \in W$, let $\ell(w)$ be the number of terms in a reduced expression (shortest expression) $w = s_1 s_2 \cdots s_r$. Reduced expression is not unique.

- Proposition 2.1. We have
  1. $\ell(1) = 0$.
  2. $\ell(w) = \ell(w^{-1})$.
  3. $\ell(w) = 1 \Leftrightarrow w \in S$.
  4. $\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w')$.
  5. $\ell(ws) = \ell(w) \pm 1$ for $s \in S$. Similarly for $\ell(sw)$.

- There is an onto homomorphism $\varepsilon : W \longrightarrow \{\pm 1\}$ defined by $\varepsilon(w) = (-1)^{\ell(w)}$.

3. Geometric Representation of W

- Given a Coxeter system (W, S), let V be a vector space over $\mathbb{R}$ with basis $\{\alpha_s \mid s \in S\}$. Define a symmetric bilinear form B on V by

$$B(\alpha_s, \alpha_{s'}) = -cos\frac{\pi}{m(s, s')}, \quad s, s' \in S.$$

For each $s \in S$, define a reflection $\sigma_s : V \longrightarrow V$ by

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

Then $\sigma_s(\alpha_s) = -\alpha_s$, σ_s fixes

$$H_s = \{x \in V \mid B(\alpha_s, x) = 0\}$$

pointwise, $o(\sigma_s) = 2$, and $B(\sigma_s x, \sigma_s y) = B(x, y)$ for $x, y \in V$.

3. Geometric Representation of W

- Proposition 3.1. There is a unique faithful representation

$$\sigma : \quad W \longrightarrow GL(V)$$

sending s to σ_s such that $\sigma(W)$ preserves the form B on V.

4. Positive and Negative Roots

- We call
$$\Phi = \{w(\alpha_s) \mid w \in W, s \in S\}$$
the root system of W and call the elements of Φ roots. If $\alpha \in \Phi$, we can write
$$\alpha = \sum_{s \in S} c_s \alpha_s, \quad c_s \in \mathbb{R}.$$
Write $\alpha > 0$ (resp. < 0) if all $c_s \geq 0$ (resp. ≤ 0), and let
$$\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\}, \quad \Phi^- = \{\alpha \in \Phi \mid \alpha < 0\}.$$

- Theorem 4.1. Let $w \in W$ and $s \in S$. If $\ell(ws) > \ell(w)$, then $w(\alpha_s) > 0$. If $\ell(ws) < \ell(w)$, then $w(\alpha_s) < 0$.

4. Positive and Negative Roots

- Corollary 4.2. We have $\Phi = \Phi^+ \cup \Phi^-$ with $\Phi^+ \cap \Phi^- = \emptyset$ and $|\Phi^+| = |\Phi^-|$.
- For $I \subset S$, the subgroup W_I of W generated by I is called a *parabolic subgroup* of W.
- Theorem 4.3. (a) For each subset I of S, the pair (W_I, I) with the given values $m(s, s')$ is a Coxeter system.
  (b) If $w \in W_I$ and $w = s_1 \cdots s_r$ ($s_i \in S$) is a reduced expression, then all $s_i \in I$. In particular, $\ell|_{W_I} = \ell_I$.

5. Geometric Interpretation of the Length Function

- **Proposition 5.1.** (a) If $s \in S$, then $s(\alpha_s) = -\alpha_s$ and

$$s(\Phi^+ - \{\alpha_s\}) = \Phi^+ - \{\alpha_s\}.$$

  (b) For any $w \in W$, $\ell(w)$ is the number of positive roots sent by w to negative roots.

6. Roots and Reflections

- We can associate a reflection in $GL(V)$ with $\alpha \in \Phi$. If $\alpha = w(\alpha_s)$, then

$$\begin{aligned} wsw^{-1}(\lambda) &= w[w^{-1}(\lambda) - 2B(w^{-1}(\lambda), \alpha_s)\alpha_s] \\ &= \lambda - 2B(w^{-1}(\lambda), \alpha_s)w(\alpha_s) \\ &= \lambda - 2B(\lambda, w(\alpha_s))\alpha \\ &= \lambda - 2B(\lambda, \alpha)\alpha. \end{aligned}$$

  Let $s_\alpha = wsw^{-1}$, then $\alpha \longrightarrow s_\alpha$ is one-to-one for $\alpha \in \Phi^+$.

- Lemma 6.1. If $\alpha, \beta \in \Phi$ and $\beta = w(\alpha)$ for some $w \in W$, then $ws_\alpha w^{-1} = s_\beta$.

  Let

$$T = \bigcup_{w \in W} wSw^{-1}.$$

- Proposition 6.2. Let $w \in W$, $\alpha \in \Phi^+$. Then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) > 0$.

- Example 6.3. Consider $W = S_n$, the permutation group of $\{1, 2, \ldots, n\}$. Take $S = \{(i, i+1) \mid 1 \le i \le n-1\}$. Let $s_i = (i, i+1)$ and $\alpha_i = \alpha_{s_i}$. We have

$$m(s_i, s_j) = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } |i-j| > 1, \\ 3 & \text{if } |i-j| = 1. \end{cases}$$

Thus

$$s_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } i = j, \\ \alpha_j & \text{if } |i-j| > 1, \\ \alpha_i + \alpha_j & \text{if } |i-j| = 1. \end{cases}$$

Hence

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k} \mid 1 \le i \le n-1,\ 0 \le k \le n-1-i\}.$$

7. Strong Exchange Condition

- Theorem 7.1. (Strong Exchange Condition) Let $w = s_1 \cdots s_r$, $s_i \in S$, not necessarily reduced. If $t \in T$ satisfies $\ell(wt) < \ell(w)$, then there is an index i for which $wt = s_1 \cdots \hat{s}_i \cdots s_r$ (omitting s_i). If the expression is reduced, then i is unique.

- If we require that $t \in S$, then the resulting weaker statement is called the *Exchange Condition*.

- Corollary 7.2. (a) *Deletion Condition*. If $w = s_1 \cdots s_r$ with $\ell(w) < r$, then there are $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$. (b) If $w = s_1 \cdots s_r$, then a reduced expression for w may be obtained by omitting certain s_i (even number of terms).

8. Bruhat Ordering

- For $w, w' \in W$, write $w' \longrightarrow w$ if $w = w't$ for some $t \in T$ with $\ell(w) > \ell(w')$. Define $w' < w$ if there is a sequence

$$w' = w_0 \longrightarrow w_1 \longrightarrow \cdots \longrightarrow w_m = w.$$

  Then the resulting relation $w' \leq w$ is a partial ordering of W called the *Bruhat ordering*. If we insist that $t \in S$, the ordering is called the weak ordering.

- Example 8.1. Consider the dihedral group D_m, $m \leq \infty$. All elements of distinct lengths are comparable in the Bruhat ordering (but not in the weak ordering):

$$w' < w \Longleftrightarrow \ell(w') < \ell(w).$$

- Example 8.2. For $W = S_n$, any $\sigma \in S_n$ can be represented by $(\sigma(1), \ldots, \sigma(n))$. Then $\sigma \leq \tau \iff \tau$ is obtained from σ by a sequence of transposition (i, j), where $i < j$ and i occurs to the left of j in σ. For example

$$(12345) \longrightarrow (12354) \longrightarrow (13254).$$

- Proposition 8.3. Let $w' \leq w$ and $s \in S$. Then either $w's \leq w$ or else $w's \leq ws$ (or both).

8. Bruhat Ordering

- We consider the Möbius function of the Bruhat ordering.
- Recall that if a partially ordered set $(X, \leq)$ satisfies the following finiteness condition: for each $y \in X$, the set $\{x \in X | x \leq y\}$ is finite, then for any $I := \{(x, y) \in X \times X | x \leq y\}$ there is a unique function $\nu : I \to \mathbb{Z}$ (Möbius function) such that
$$\sum_{x \leq z \leq y} \nu(x, z) = \delta_{x,y}, \quad \forall (x, y) \in I.$$
- Proposition 8.4. If $x < y$ in W, then $\nu(x, y) = \varepsilon(xy)$.

9. Subexpressions

- Let $w = s_1 \cdots s_r \in W$ be a reduced expression. A subexpression of $s_1 \cdots s_r$ is of the form $s_{i_1} \cdots s_{i_q}$ such that $1 \leq i_1 < \cdots < i_q \leq r$.
- Theorem 9.1. Let $w = s_1 \cdots s_r$ be a fixed reduced expression for $w \in W$. Then $w' \leq w$ if and only if w' can be obtained as a subexpression of this fixed reduced expression.
- Corollary 9.2. If $I \subset S$, the Bruhat ordering of W agrees on W_I with the Bruhat ordering of the Coxeter group W_I.

10. Intervals in the Bruhat Ordering

- Elements of W directly adjacent in the Bruhat ordering must differ in length 1.
- Lemma 10.1. Let $w' < w$, with $\ell(w) = \ell(w') + 1$. Suppose there is $s \in S$ such that $w' < w's$ and $w's \neq w$. Then both $w < ws$ and $w's < ws$.
- Proposition 10.2. Let $w' < w$. Then there are $w_0, \ldots, w_m \in W$ such that

$$w' = w_0 < w_1 < \cdots < w_m = w,$$

and

$$\ell(w_i) = \ell(w_{i-1}) + 1, \quad 1 \leq i \leq m.$$