Lecture II: Computing Kazhdan-Lusztig Polynomials

Yi Ming Zou

Department of Mathematical Sciences
University of Wisconsin-Milwaukee

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1. Hecke Algebras

- The Hecke Algebra $\mathscr{H}$ associated with a Coxeter system (W, S) is a free $R = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ module with basis $\{T_w \mid w \in W\}$ and the multiplication defined by

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w, & \text{if } \ell(ws) < \ell(w). \end{cases}$$

- As an R-algebra, $\mathscr{H}$ is generated by $\{T_s \mid s \in S\}$ subject to relations:
  1. $T_s = T_s T_1 = T_1 T_s$ and $T_s^2 = qT_1 + (q-1)T_s$, $\forall s \in S$; and
  2. $T_{s_1} T_{s_2} \cdots$ (m terms) $= T_{s_2} T_{s_1} \cdots$ (m terms), where $s_1, s_2 \in S$ and $m = o(s_1 s_2)$.

- Note that $T_s, s \in S$, is invertible and

$$T_s^{-1} = q^{-1} T_s - (1 - q^{-1}) T_1.$$

Thus T_w is invertible for all $w \in W$.

- Define an involution $\overline{}$ on $\mathscr{H}$ by requiring

$$\overline{q^{1/2}} = q^{-1/2}, \quad \overline{T_w} = (T_{w^{-1}})^{-1},$$

and $\overline{hh'} = \overline{h} \cdot \overline{h'}, \forall h, h' \in \mathscr{H}$. Let $\mathscr{H}^0$ be the set of fixed points of this involution, and let $R^0 = \mathscr{H}^0 \cap R$. Then $R^0 = \mathbb{Z}[f]$ where $f = q^{1/2} + q^{-1/2}$. Note that $\mathscr{H}^0$ is an R^0-algebra, and as an R^0-module it is free.

- Theorem 1.1. (Kazhdan-Lusztig) For $y \in W$, there is a unique element $C_y \in \mathscr{H}^0$ such that

$$C_y = \sum_{x \leq y} \varepsilon(x)\varepsilon(y) q^{(\ell(y)-2\ell(x))/2} \overline{P}_{x,y} T_x,$$

where $P_{x,y} \in R$ is a polynomial in q of degree $\leq (\ell(y) - \ell(x) - 1)/2$ for $x < y$, and $P_{y,y} = 1$.

2. Kazhdan-Lusztig Polynomials

- The polynomials $P_{x,y}$ are the so-called *Kazhdan-Lusztig polynomials*.

- Theorem 1.1 can be restated as

  Theorem 2.1. For $y \in W$, there is a unique element $C'_y \in \mathscr{H}^0$ such that

  $$C'_y = q^{-\ell(y)/2} \sum_{x \leq y} P_{x,y} T_x,$$

  where $P_{x,y} \in R$ is a polynomial in q of degree $\leq (\ell(y) - \ell(x) - 1)/2$ for $x < y$, and $P_{y,y} = 1$.

- The elements C'_y and C_y are related by $C'_y = (-1)^{\ell(y)} \sigma(C_y)$, where σ is the involution of $\mathscr{H}$ defined by

  $$\sigma(\sum a_w T_w) = \sum (-1)^{\ell(w)} \overline{a}_w q^{-\ell(w)} T_w.$$

2. Kazhdan-Lusztig Polynomials

- Conjecture 2.2. (Kazhdan-Lusztig, not "the Kazhdan-Lusztig conjecture") The coefficients of the polynomials $P_{x,y}$ are non-negative integers.
- The conjecture is true for the Weyl groups of Kac-Moody Lie algebras.
- The values of these polynomials at 1 provide character formulas for the infinite dimensional simple highest weight modules of semisimple Lie algebras over $\mathbb{C}$ (the Kazhdan-Lusztig conjecture, true for symmetrizable Kac-Moody algebras):

$$chL_w = \sum_{y \leq w} \varepsilon(yw) P_{y,w}(1) chM_y.$$

- For real Lie groups, one has the Lusztig-Vogan polynomials.

2. Kazhdan-Lusztig Polynomials

- These polynomials have a rich combinatorial theory, and there have been a lot of research activities devoted to study them.
- Question: How to compute these polynomials?
- Kazhdan and Lusztig provided a recursive method in their original paper for the computation of these polynomials via another set of polynomials $R_{x,y}$ defined by

$$\overline{T_y} = (T_{y^{-1}})^{-1} = \sum_x \overline{R}_{x,y} q^{-\ell(x)} T_x.$$

- Proposition 2.3. We have
  - $\overline{R}_{x,y} = \varepsilon(xy) q^{\ell(x)-\ell(y)} R_{x,y}$.
  - $\overline{T_y} = \sum_{x \leq y} \varepsilon(xy) q^{-\ell(y)} R_{x,y} T_x$.
  - $\sum_{x \leq z \leq y} \varepsilon(xz) R_{x,z} R_{z,y} = \delta_{x,y}$.

2. Kazhdan-Lusztig Polynomials

- The polynomials $R_{x,y}$ can be computed recursively using the formulas:
  - $R_{x,y} = 1$ if $x = y$ and 0 unless $x \leq y$,
  - $R_{x,y} = R_{sx,sy}$ if $sx < x$ and $sy < y$,
  - $R_{x,y} = R_{xs,ys}$ if $x < xs$ and $y < ys$,
  - $R_{x,y} = (q-1)R_{sx,y} + qR_{sx,sy}$ if $sx > x$ and $sy < y$.
- The polynomials $P_{x,w}$ can then be constructed using

$$q^{(\ell(w)-\ell(x))/2}\overline{P}_{x,w} - q^{(\ell(x)-\ell(w))/2}P_{x,w} = q^{(\ell(x)-\ell(w))/2}\sum_{x<y\leq w} R_{x,y}P_{y,w}.$$

2. Kazhdan-Lusztig Polynomials

- Exercise. Compute $P_{x,y}$ for $y = s_1 s_2 s_1 \in S_3$ using these recursive formulas.
- Next, we describe a closed formula due to Deodhar. The theorems, propositions, etc, in the rest of this lecture are due to Deodhar. Let us first consider an example.
- Example. Compute $P_{x,y}$ for $y = s_1 s_2 s_3 s_1 \in S_4$. Recall that $P_{x,y}$ is a polynomial in q of degree $\leq (\ell(y) - \ell(x) - 1)/2$ for $x < y$.

3. Combinatorics of Subexpressions

- Let $y \in W$. Fix a reduced expression $y = s_1 s_2 \cdots s_k$, and let $s = (s_1, s_2, \ldots, s_k)$. We give an equivalent definition of the subexpressions of y.
- Definition 3.1. We call $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ a subexpression of s if

  (1) $\sigma_0 = 1$, and

  (2) $\sigma_{j-1}^{-1}\sigma_j \in \{1, s_j\}$, $\forall 1 \le j \le k$.
- The set of all subexpressions is denoted by $\mathscr{S}$.
- Define maps ℓ, π, and $\tilde{}$ on $\mathscr{S}$ by
  1. $\ell(\sigma) = |\{j \mid \sigma_{j-1} \neq \sigma_j\}|$ (= t if σ is identified with $1 \le i_1 < i_2 < \cdots < i_t \le k$).
  2. $\pi(\sigma) = \sigma_k$.
  3. $\tilde{\sigma} = \{(\sigma_0, \ldots, \sigma_{k-1}, \sigma_k s_k\}$.

3. Combinatorics of Subexpressions

- We have
  1. $\text{Im}(\pi) = \{x \in W \mid x \le y\} := W(y)$.
  2. $\ell(\sigma) - \ell(\pi(\sigma)) \in 2\mathbb{Z}_+$.
  3. $\ell(\tilde{\sigma}) = \ell(\sigma) \pm 1$.

- Definition 3.2. The *defect* of σ is defined to be

$$d(\sigma) = |\{j \mid \sigma_{j-1}s_j < \sigma_{j-1}\}|.$$

- Example 3.3. Consider S_3. Let $y = s_1s_2s_1$. Then $s = (s_1, s_2, s_1)$ and (since $\sigma_0 = 1$, we omit it in our listing)

$$\begin{aligned}\mathscr{S} &= \{(1,1,1),(1,1,s_1),(1,s_2,1),(s_1,1,1),\\ &\quad (1,s_2,s_1),(s_1,1,s_1),(s_1,s_2,1),(s_1,s_2,s_1)\}.\end{aligned}$$

For $\sigma = (s_1, 1, s_1)$, we have $\ell(\sigma) = 2$, $\pi(\sigma) = 1$, $\tilde{\sigma} = (s_1, 1, 1)$, and $d(\sigma) = 1$.

3. Combinatorics of Subexpressions

- Proposition 3.3. We have
  1. $d(\sigma) = d(\tilde{\sigma})$.
  2. $\pi : \mathscr{D}_0 = \{\sigma \mid d(\sigma) = 0\} \longrightarrow W(y)$ is bijective.
  3. The subset $\mathscr{D}_0$ is stable under the map $\tilde{}$.
  4. $d(\sigma) \geq (\ell(\sigma) - \ell(\pi(\sigma)))/2$.
  5. $\ell(y) - \ell(\pi(\sigma)) \geq d(\sigma)$ and equality holds if and only if $\pi(\sigma) = y$.
- For $\mathscr{L} \subset \mathscr{S}$, define $h(\mathscr{L}) \in \mathscr{H}$ by

$$h(\mathscr{L}) = q^{-\ell(y)/2} \sum_{\sigma \in \mathscr{L}} q^{d(\sigma)} T_{\pi(\sigma)}.$$

Set

$$P_x(\mathscr{L}) = \sum_{\substack{\sigma \in \mathscr{L} \\ \pi(\sigma) = x}} q^{d(\sigma)}.$$

Then

$$h(\mathscr{L}) = q^{-\ell(y)/2} \sum_{x \leq y} P_x(\mathscr{L}) T_x.$$

4. Admissible Subsets

- Definition 4.1. A subset $\mathscr{E} \subset \mathscr{S}$ is called *admissible* if it satisfies the following three properties:
  (1) $\mathscr{E}$ contains the special subexpression
  $$(1, s_1, s_1 s_2, \ldots, s_1 s_2 \cdots s_k) = \pi^{-1}(y),$$
  (2) $\mathscr{E} = \tilde{\mathscr{E}}$, and
  (3) $h(\mathscr{E}) \in \mathscr{H}^0$.
- Note that $\mathscr{S}$ is admissible. Conditions (1) and (2) clearly hold. (3) can be seen from Proposition 5.1 below.
- Lemma 4.2. Let $\mathscr{L} \subset \mathscr{S}$. Then $h(\mathscr{L}) \in \mathscr{H}^0$ if and only if
  $$\sum_{x \leq z \leq y} R_{x,z} P_z(\mathscr{L}) = q^{\ell(y)-\ell(x)} \overline{P_x(\mathscr{L})} \quad \forall x \in W(y).$$
- Lemma 4.3. If $\mathscr{E}$ is admissible then $\mathscr{D}_0 \subset \mathscr{E}$.

4. Admissible Subsets

- Let R_+^0 be the set of polynomial in $f = q^{1/2} + q^{-1/2}$ with non-negative coefficients. For a pair $\mathscr{E}_1 \supset \mathscr{E}_2$ of admissible sets, we say that $\mathscr{E}_1$ dominates $\mathscr{E}_2$ if there is an $x \in W(y)$ and $g \in R_+^0$ such that

  (1) $h(\mathscr{E}_1 \setminus \mathscr{E}_2) = g \cdot C'_x$, and
  (2) $\deg(P_z(\mathscr{E}_2)) \leq (\ell(y) - \ell(z) - 1)/2$ for all $x \leq z < y$.

- Let $\preceq$ be the partial order generated by this dominance relation. Note that $\mathscr{S}$ is a maximal element for this order.

5. Another Basis of $\mathscr{H}^0$

- Theorem 2.1 implies that $C'_x, x \in W$ form a basis of the R-module $\mathscr{H}$ and hence a basis of the R^0-module $\mathscr{H}^0$.
- Also from Theorem 2.1, one has $C'_s = q^{-1/2}(T_s + T_1)$ for $s \in S$.
- For $\forall y \in W$, fix a reduced expression $y = s_1 \cdots s_k$. Let $\mathbf{s} = (s_1, \ldots, s_k)$, and let

$$D'_{\mathbf{s}} = C'_{s_1} C'_{s_2} \cdots C'_{s_k}.$$

- Proposition 5.1. (1) We have

$$D'_{\mathbf{s}} = q^{-\ell(y)/2} \sum_{\sigma \in \mathscr{S}} q^{d(\sigma)} T_{\pi(\sigma)}.$$

(2) The set $\{D'_{\mathbf{s}} \mid y \in W\}$ with $\mathbf{s}$ as described above, is a R^0 basis of $\mathscr{H}^0$.

- Let $\mu(x,z)$ be the coefficient of $q^{(\ell(z)-\ell(x)-1)/2}$ in $P_{x,z}$.
- Proposition 5.2. Let $s^\# = (s_1, \ldots, s_{k-1})$, denote the corresponding terms by using the notation #, and let

$$D'_s = C'_y + \sum_{x<y} L_x C'_x \quad \text{and} \quad D'_{s^\#} = C'_{ys_k} + \sum_{w<ys_k} L^\#_w C'_w,$$

where L_x and $L^\#_w \in R^0$. Then

$$L_x = \begin{cases} 0, & xs_k > x, \\ fL^\#_x + L^\#_{xs_k} + \sum_{\substack{zs_k>z \\ z\leq ys_k}} \mu(x,z) L^\#_z, & xs_k < x. \end{cases}$$

Furthermore, if $\mu(x,z) \geq 0$ for all $x \leq z$ ($\in W(y)$), then L_x is a polynomial in f with non-negative coefficients for all $x \in W(y)$.

- Proposition 6.1. Let $\mathscr{E}$ be an admissible set. Then there exists a unique subset $A(\mathscr{E})$ of elements strictly less than y such that

$$h(\mathscr{E}) = C'_y + \sum_{x \in A(\mathscr{E})} L_x C'_x.$$

Or equivalently,

$$P_u(\mathscr{E}) = P_{u,y} + \sum_{x \in A(\mathscr{E})} q^{(\ell(y)-\ell(x))/2} L_x P_{u,x}, \quad \forall u \leq y.$$

6. Minimal Admissible Sets

- Corollary 6.2. Let $\mathscr{E}$ be an admissible set.
  (a) The following are equivalent.

  (1) $A(\mathscr{E}) = \emptyset$.
  (2) $h(\mathscr{E}) = C'_y$.
  (3) $\deg P_u(\mathscr{E}) \leq (\ell(y) - \ell(u) - 1)/2$ for all $u < y$.

  (b) If the conditions of part (a) hold, then $\mathscr{E}$ is minimal for the order $\preceq$.

- Definition 6.3. An element $y \in W$ is called *good* if for any minimal admissible set $\mathscr{E}_{min}$, $h(\mathscr{E}_{min}) = C'_y$.

- Proposition 6.4. Let y be good, let $\mathscr{E}$ be an admissible set, and let x_0 be an element such that

$$\deg P_z(\mathscr{E}) \leq \frac{\ell(y) - \ell(z) - 1}{2}, \quad \forall x_0 < z < y,$$

and

$$\deg P_{x_0}(\mathscr{E}) > \frac{\ell(y) - \ell(x_0) - 1}{2}.$$

Then there is an admissible set $\mathscr{E}'$ and $g \in R_+^0$ such that $\mathscr{E}'$ is dominated by $\mathscr{E}$ and $h(\mathscr{E} \backslash \mathscr{E}') = gC'_{x_0}$.

6. Minimal Admissible Sets

- **Theorem 6.5.** If the coefficients of the Kazhdan-Lusztig polynomials $P_{x,z}$ are non-negative for all $x \leq z \leq y$, then y is good and hence

$$P_{x,y} = \sum_{\substack{\sigma \in \mathscr{E}_{min} \\ \pi(\sigma)=x}} q^{d(\sigma)} \quad (= P(\mathscr{E}_{min})),$$

  where $\mathscr{E}_{min}$ is any minimal admissible set.

- **Corollary 6.6.** If (W, S) is the Weyl group of a Kac-Moody Lie algebra, then every element of W is good and hence Theorem 6.5 provides a closed formula as well as an algorithm for all the Kazhdan-Lusztig polynomials.

- Algorithm 7.1. Let y be good and fix a reduced expression $y = s_1 \cdots s_k$.

  (1) Start with $\mathscr{S}$. If

$$\deg P_z(\mathscr{S}) \leq \frac{\ell(y) - \ell(z) - 1}{2}, \quad \forall z < y,$$

  then $h(\mathscr{S}) = C'_y$, and hence $P_z(\mathscr{S}) = P_{z,y}$ for all $z \leq y$.

  (2) If this is not the case, then $\mathscr{S}$ is not minimal, and one can choose an x_0 such that

$$\deg P_z(\mathscr{S}) \leq \frac{\ell(y) - \ell(z) - 1}{2}, \quad \forall x_0 < z < y,$$

  and

$$\deg P_{x_0}(\mathscr{S}) > \frac{\ell(y) - \ell(x_0) - 1}{2}.$$

  (Cont'd next page)

7. Algorithm

(Algorithm cont'd)

By Proposition 6.4, there is an admissible set $\mathscr{E}_1$ which is dominated by $\mathscr{S}$ and

$$h(\mathscr{E}_1) = h(\mathscr{S}) - gC'_{x_0}.$$

(3) One can identify the element g as follows. Let g_1 (resp. g_2) be the sum of terms in P_{x_0} of degree $> (\ell(y) - \ell(x_0) - 1)/2$ (resp. $> (\ell(y) - \ell(x_0))/2$), then

$$g = q^{-(\ell(y)-\ell(x_0))/2} g_1 + q^{(\ell(y)-\ell(x_0))/2} \overline{g_2}.$$

(4) Argue with $\mathscr{E}_1$ in place of $\mathscr{S}$ and repeat steps (1) through (3).

It is clear that the algorithm terminates, and one can get a minimal admissible set and hence $P_{x,y}$ for all $x \leq y$.

8. Examples

- Example 8.1. Consider S_3 and $y = s_1 s_2 s_1$. Write $\ell(y, \sigma) = (\ell(y) - \ell(\pi(\sigma)) - 1)/2$. We have

$\mathscr{S}$	$\pi(\sigma)$	$d(\sigma)$	$\ell(y,\sigma)$
$1, 1, 1$	1	0	1
$1, 1, s_1$	s_1	0	$1/2$
$1, s_2, 1$	s_2	0	$1/2$
$s_1, 1, 1$	s_1	1	$1/2$
$1, s_2, s_1$	$s_2 s_1$	0	0
$s_1, 1, s_1$	1	1	1
$s_1, s_2, 1$	$s_1 s_2$	0	0
s_1, s_2, s_1	$s_1 s_2 s_1$		

So $\mathscr{S}$ is not minimal. Take $x_0 = s_1$. Then $g = 1$ and $\mathscr{E}_1 = \mathscr{S} - \{(s_1, 1, 1), (s_1, 1, s_1)\}$ $(= \mathscr{D}_0)$. Thus all $P_{x,y} = 1$ for $x \leq y$.

8. Examples

- Example 8.2. Consider S_4 and $y = s_2 s_1 s_3 s_2$. We have

$\mathscr{S}$	$\pi(\sigma)$	$d(\sigma)$	$\ell(y,\sigma)$
$1, 1, 1, 1$	1	0	$3/2$
$1, 1, 1, s_2$	s_2	0	1
$1, 1, s_3, 1$	s_3	0	1
$1, s_1, 1, 1$	s_1	0	1
$s_2, 1, 1, 1$	s_2	1	1
$1, 1, s_3, s_2$	$s_3 s_2$	0	$1/2$
$1, s_1, 1, s_2$	$s_1 s_2$	0	$1/2$
$s_2, 1, 1, s_2$	1	1	$3/2$
$1, s_1, s_3, 1$	$s_1 s_3$	0	$1/2$
$s_2, 1, s_3, 1$	$s_2 s_3$	0	$1/2$
$s_2, s_1, 1, 1$	$s_2 s_1$	0	$1/2$
$1, s_1, s_3, s_2$	$s_1 s_3 s_2$	0	0
$s_2, 1, s_3, s_2$	$s_2 s_3 s_2$	0	0
$s_2, s_1, 1, s_2$	$s_2 s_1 s_2$	0	0
$s_2, s_1, s_3, 1$	$s_2 s_1 s_3$	0	0
s_2, s_1, s_3, s_2			

8. Examples

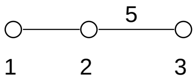
- Example 8.2-cont'd

  Hence $\mathscr{S}$ is minimal and $P_{x,y} = P_x(\mathscr{S})$. We see that all $P_{x,y} = 1$ except

$$P_{1,y} = P_{s_2,y} = 1 + q.$$

- Example 8.3. Consider H_3. Its Coxeter graph is

Γ:



Take $y = s_3 s_2 s_1 s_3 s_2 s_3$, then

$$P_{s_3 s_2 s_3,y} = P_{s_3 s_2,y} = P_{s_2 s_3,y} = P_{s_2,y} = 1 + q,$$
$$P_{s_3,y} = P_{1,y} = 1 + q + q^2,$$

and all others are 1.