SKEW DIFFERENTIAL OPERATORS ON COMMUTATIVE RINGS
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Dedicated to the memory of Professor Robert B. Warfield, Jr. (1940-1989)

Section 0: Introduction

In this paper we define a ring of skew differential operators \( D(A; \phi) \) on a commutative algebra \( A \) associated to an automorphism \( \phi \) of \( A \) (the commutators used in the definition are twisted by powers of \( \phi \)), derive some of its basic properties, and compute some examples. These operators provide a higher order generalization of skew derivations. The definition and calculations contained herein can be considered as an experiment directed toward determining how to define a “twisted” or a “quantum” analog of ordinary differential operators.

Two potential applications of our definition immediately suggest themselves. First, just as with ordinary differential operators, skew differential operators may be helpful in the study of algebraic varieties: the automorphism may be useful in exploiting symmetries or in focusing on singular points (which must be permuted by it). Second, rings of skew differential operators are connected to the theory of quantum groups. For instance, in [4] skew-derivations are shown to play an important role in that theory, while in example 4.3 we see that a \( q \)-analog of the Weyl algebra arises as a ring of skew differential operators. However in some ways our results are disappointing. Section 3 shows that the rings we obtain are often not too far removed from Ore extensions in a single variable: for example it appears unlikely that one can obtain a \( q \)-analog of the Weyl algebra \( A_n \) for \( n > 1 \) as a ring of skew differential operators. Thus another version of skew differential operators may be desirable. (Perhaps we should not restrict ourselves to commutative base algebras \( A \), but even in the case of ordinary differential operators, the commutator definition does not seem appropriate for non-commutative rings — the first order operators no longer are derivations.)

Roughly speaking, when \( A \) has \( n \) independent variables, the ring of ordinary differential operators in characteristic 0 generally has \( 2n \); however, when \( \phi \) has infinite order, the ring of skew differential operators frequently has \( n + 1 \) independent variables, regardless of the characteristic. We show in section 3 that when \( \phi \) has infinite order, skew differential operators are often determined by their action on the powers of a single element and that if \( A \) is a domain with quotient field \( Q \), the ring \( D(A; \phi) \) is isomorphic to a subring of the Ore extension \( Q[x; \phi] \) (this is the ring generated by \( Q \) and the indeterminate \( x \) with multiplication determined by the rule \( xq = \phi(q)x \)). In this case \( D(A; \phi) \) is a domain but not a simple ring.

When \( \phi \) has finite order, on the other hand, the ring of skew differential operators contains the ring of ordinary differential operators. Moreover, if \( \phi \neq \text{id} \), then \( D(A; \phi) \) is not a domain, but it is simple whenever the ring of ordinary differential operators is simple. In the final section we work out some examples, among them a calculation of the possible rings of skew differential operators for a polynomial ring in one variable. Our examples show that even in nice cases, the ring of skew differential operators need not be either affine or Noetherian.
Section 1: Definitions and basic properties

Throughout, $k$ is a field, $A$ is a commutative $k$-algebra with identity, and $\phi$ is a $k$-linear automorphism of $A$. The ring of ordinary differential operators on $A$ is defined to be the set of those $k$-linear maps from $A$ to $A$ which vanish after repeated commutation with elements of $A$. (For more on the theory of rings of differential operators, see for example [5,7,8].)

We wish to introduce a “skew” version of this construction, based on the automorphism $\phi$; the main difference is that we must use a different commutator at each level. To test if a linear map $f$ is a skew differential operator of order $n$, we form a commutator involving $\phi^n$, then another involving $\phi^{n-1}$, and we continue until we form one involving $\phi^0$ (an ordinary commutator): $f$ is a skew differential operator of order at most $n$ if the end result is 0. (Thus when $\phi = \text{id}$, our definition reduces to the usual one.) I would like to thank Peter Malcolmson for inspiring this paper by suggesting that one could use a twisted commutator in defining differential operators (although the twisting we use is not the one he suggested).

We note that in [6] a definition of differential operators involving an endomorphism $\phi$ is given; however, the differential operators obtained there are the elements of $\phi \circ D(A)$, and so do not form a subalgebra of $\text{End}_k(A)$.

For $a \in A$, define $\mu_a \in \text{End}_k(A)$ by $\mu_a(x) = ax$. We use $\sum_{i=0}^{n} a_i \phi^i$ to denote $\sum_{i=0}^{n} \mu_{a_i} \circ \phi^i$. For $n \in \mathbb{Z}$, define $D_n(A; \phi) \subseteq \text{End}_k(A)$ by

$$D_n(A; \phi) = \begin{cases} 0 & \text{if } n < 0; \\ \{ f \in \text{End}_k(A) \mid \mu_{\phi^n(a)} \circ f - f \circ \mu_a \in D_{n-1}(A; \phi) \text{ for all } a \in A \} & \text{if } n \geq 0. \end{cases}$$

When no confusion can arise, we will write $D_n$ for $D_n(A; \phi)$.

The next proposition gives some of the basic properties of skew differential operators: except for (e) the properties here are the same as those of ordinary differential operators. Part (e) and its later amplification in section 3 marks a crucial difference in the theory of skew differential operators. The proof of this proposition indicates why we define the twisting of the commutator as we do.

**Proposition 1.1.**

(a) Each $D_n(A; \phi)$ is a $k$-subspace of $\text{End}_k(A)$.

(b) $D_n(A; \phi) \circ D_m(A; \phi) \subseteq D_{n+m}(A; \phi)$ for all $n, m \in \mathbb{Z}$.

(c) $D_0(A; \phi) = \{ \mu_a | a \in A \}$.

(d) $D_n(A; \phi) \subseteq D_m(A; \phi)$ whenever $n \leq m$.

(e) For any $n \geq 0$ and $a_0, \ldots, a_n \in A$, we have $\sum_{i=0}^{n} a_i \phi^i \in D_n(A; \phi)$.

**Proof.** (a) This follows easily by induction on $n$.

(b) This is clear if either $n$ or $m$ is negative. Thus we can assume $n, m \geq 0$. We proceed by induction on $n + m$. Suppose $f \in D_n$, $g \in D_m$, $a \in A$. Then

$$\mu_{\phi^{n+m}(a)} \circ f \circ g - f \circ g \circ \mu_a = (\mu_{\phi^n(\phi^m(a))} \circ f - f \circ \mu_{\phi^m(a)}) \circ g + f \circ (\mu_{\phi^m(a)} \circ g - g \circ \mu_a).$$

The transformations in parentheses on the right hand side are in $D_{n-1}$ and $D_{m-1}$ respectively, and so by induction the two summands on the right hand side are in $D_{n+m-1}$. By (a), their sum is in $D_{n+m-1}$; this shows $f \circ g \in D_{n+m}(A; \phi)$.

(c) This is easy.
(d) This is obviously true for $n < 0$. We show by induction that $D_n \subseteq D_{n+1}$ for $n \geq 0$. Let $f \in D_n$ and $a \in A$. Then $\mu_{\phi_{n+1}(a)} \mu_a \in D_0$ by (c), so $\mu_{\phi_{n+1}(a)} \circ f - f \circ \mu_a \in D_n$ by (a) and (b), whence $f \in D_{n+1}$.

(e) This follows from the previous parts, since it is clear that $\phi \in D_1$.

Thus $D(A; \phi) = \cup_{n \in \mathbb{Z}} D_n(A; \phi)$ is a (nonnegatively) filtered subalgebra of $\text{End}_k(A)$ whose zero part is isomorphic to $A$. We call $D(A; \phi)$ the ring of $\phi$-differential-operators on $A$. We will identify $A$ with $\{ \mu_a | a \in A \}$, in which case (e) says that $D(A; \phi)$ contains $A[\phi]$ and (b) implies each $D_n(A; \phi)$ is an $A - A$ submodule of $D(A; \phi)$. We will write $D_n(A) = D_n(A; \text{id})$ and $D(A) = D(A; \text{id})$: this is the ordinary ring of differential operators on $A$. The order of an element $f \in D(A; \phi)$ is the smallest $n$ for which $f \in D_n(A; \phi)$.

(Note that the word “order” has two meanings in this paper. One is the definition just given; the other is the order of the automorphism $\phi$. This should cause no confusion, since as a skew differential operator, $\phi$ always has order 1 if $\phi \neq \text{id}$.)

**Remark.** We have chosen to give an inductive definition of skew differential operators, and we will use an inductive approach when studying them. However, one can give a direct characterization as follows; we leave its proof to the reader. If $f \in \text{End}_k(A)$, then $f \in D_n(A; \phi)$ if and only if $f$ satisfies the following product rule: for all $a_0, \ldots, a_n \in A$,

$$f(a_0 \cdots a_n) = \sum_{i=0}^{n} (-1)^{n-i} \left( \sum_{0 \leq j_1 < \cdots < j_i \leq n} a_0 \phi(a_1) \cdots \phi^j(a_{j_1}) \cdots \phi^n(a_n) \right) f(a_{j_1} \cdots a_{j_i})$$

(where $\cdot$ indicates a factor to be omitted, and an empty product is defined to be 1). This rule is of course a generalization of the product rule for ordinary differential operators — see for example [5, Proposition 1.1].

As an example, let $f \in D_1$. Then $\mu_{f(1)} \in D_0$, so $f' = f - \mu_{f(1)}$ is still in $D_1$ and satisfies $f'(1) = 0$. Now for any $a \in A$, $\mu_{\phi(a)} \circ f' - f' \circ \mu_a \in D_0$, so by proposition 1.1(c),

$$\phi(a) f'(b) - f'(ab) = (\phi(a) f'(1) - f'(a1)) b = -f'(a) b$$

for any $b \in B$.

Simplifying this equation leads to $f'(ab) = \phi(a) f'(b) + f'(a) b$, so $f'$ is a $\phi$-derivation on $A$. Thus every $f \in D_1$ can be written uniquely in the form $f = f' + \mu_a$, where $f'$ is a $\phi$-derivation on $A$ and $a = f(1) \in A$; conversely, every such sum is an element of $D_1$.

If $f \in \text{End}_k(A)$ and $a \in A$, and the order $n$ is understood, we will use $f_a$ to denote $\mu_{\phi^n(a)} \circ f - f \circ \mu_a$. In this notation the definition of $D_n$ becomes: $f \in D_n$ if and only if $f_a \in D_{n-1}$ for all $a \in A$. It is not hard to see that $f_{\alpha a + \beta b} = \alpha f_a + \beta f_b$ and $f_{ab} = \mu_{\phi^n(a)} \circ f_b + f_a \circ \mu_b$ for $\alpha, \beta \in k$, $a, b \in A$. Thus if $f_a, f_b \in D_{n-1}$, we see $f_{\alpha a + \beta b}, f_{ab} \in D_{n-1}$. As $f_1 = 0$, this proves the following.

**Lemma 1.2.** Let $A$ be generated by the subset $X$ as a $k$-algebra and let $f \in \text{End}_k(A)$. Then $f \in D_n(A; \phi)$ if and only if $\mu_{\phi^n(a)} \circ f - f \circ \mu_a \in D_{n-1}(A; \phi)$ for all $a \in X$.

The following result shows that a skew differential operator of order $n$ is completely determined by its values on products of at most $n$ elements from any generating set for
A (this will be expanded to cover localizations in lemma 2.1). This parallels a standard result for ordinary differential operators. In section 3, we will see that a much stronger result frequently holds in $D(A; \phi)$.

**Lemma 1.3.** Let $n \geq 0$ and let $B$ be the subalgebra of $A$ generated by the subset $X$. If $f, g \in D_n(A; \phi)$ and $f(x_1 \cdots x_m) = g(x_1 \cdots x_m)$ whenever $0 \leq m \leq n$ and $x_1, \ldots, x_m \in X$, then $f(b) = g(b)$ for all $b \in B$.

**Proof.** It is enough to show that if each $f(x_1 \cdots x_m) = 0$, then $f(b) = 0$ for all $b \in B$: we proceed by induction on $n$. The result follows from proposition 1.1(c) when $n = 0$, so we may suppose $n \geq 1$.

For $x \in X$, define $f_x = \mu f^{n(x)} \circ f - f \circ \mu x \in D_{n-1}$. Clearly $f_x(y_1 \cdots y_{m-1}) = 0$ for $1 \leq m \leq n$, $y_1, \ldots, y_{m-1} \in X$, and so by induction on $n$, we have that $f_x(b') = 0$ for all $b' \in B$. Thus $\phi^n(x)f(b') = f(xb')$ for all $x \in X, b' \in B$. Since $X$ generates $B$ as an algebra, this implies $\phi^n(b)f(b') = f(bb')$ for all $b, b' \in B$. Hence $f(b) = \phi^n(b)f(1) = 0$ for all $b \in B$.

**Corollary 1.4.** If $A$ is finitely generated as a $k$-algebra, then each $D_n(A; \phi)$ is finitely generated as both a left and right $A$-module.

**Lemma 1.5.** Let $a \in A$ be regular (not a zero divisor) and let $n \in \mathbb{Z}$. Then

(a) $\mu_a$ is regular in $D(A; \phi)$.

(b) If $f \in \text{End}_k(A)$ and $\mu_a \circ f \in D_n(A; \phi)$, then $f \in D_n(A; \phi)$.

(c) If $f \in D(A; \phi)$ and $f \circ \mu_a \in D_n(A; \phi)$, then $f \in D_n(A; \phi)$.

**Proof.** We prove (b) and leave the others to the reader. Regularity of $a$ implies that if $\mu_a \circ f = 0$, then $f = 0$. Thus (b) holds if $n < 0$. We now assume $n \geq 0$ and proceed by induction. Clearly for any $b \in A$ we have $\mu_a \circ f_b = (\mu_a \circ f)_b \in D_{n-1}$, and so by induction we may conclude $f_b \in D_{n-1}$. This implies $f \in D_n$.

The next lemma fails if $B$ is not a domain, even if $b_n = 1$: see example 4.1.

**Lemma 1.6.** Suppose $A$ is a subalgebra of the domain $B$, $b_0, \ldots, b_n \in B$ with $b_n \neq 0$, and $f = \sum_{i=0}^n b_i \phi^i$ has the property that $f(A) \subseteq A$. Then $f \in D(A; \phi)$ and if the order of $\phi$ as an automorphism is greater than $n$, then $f$ has order exactly $n$ in $D(A; \phi)$.

**Proof.** For any $a \in A$, the function $f_a = \mu \phi^n(a) \circ f - f \circ \mu_a = \sum_{i=0}^{n-1} b_i (\phi^n(a) - \phi^i(a)) \phi^i$ also maps $A$ into itself, and so by induction $f_a \in D_{n-1}(A; \phi)$. Hence $f \in D_n(A; \phi)$. We now show that if $\phi$ has order greater than $n$, then $f \notin D_m(A; \phi)$ for $m < n$.

Among all choices of $b_0, \ldots, b_n \in B$ with $b_n \neq 0$ and $f = \sum_{i=0}^n b_i \phi^i \in D_m(A; \phi)$, choose a set so that $m$ is as small as possible. This makes sense: such an $m$ cannot be negative, since Dedekind’s lemma that distinct homomorphisms into a field are linearly independent implies that $\sum_{i=0}^n b_i \phi^i$ is not identically zero on $A$ for any choice of $b_0, \ldots, b_n \in B$ with $b_n \neq 0$. Thus $n \geq m \geq 0$. Now for any $a \in A$, we know that $\mu \phi^m(a) \circ f - f \circ \mu_a = \sum_{i=0}^m b_i (\phi^m(a) - \phi^i(a)) \phi^i$ is in $D_{m-1}(A; \phi)$. By choice of $m$, this can happen only if $b_n (\phi^m(a) - \phi^i(a)) = 0$, i.e., $\phi^m(a) = \phi^i(a)$ for all $a \in A$. Since $0 \leq m \leq n < \text{order}(\phi)$, this implies $m = n$. 

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The next lemma shows that $D(A; \phi^k)$ is a subalgebra of $D(A; \phi)$ for all $k \geq 1$. In particular, if $\phi$ has finite order then $D(A)$ is a subalgebra of $D(A; \phi)$.

**Lemma 1.7.** Let $n, m \in \mathbb{Z}$, $m \geq 1$. Then $D_n(A; \phi^m) \subseteq D_{nm}(A; \phi)$.

**Proof.** This is clearly true for $n \leq 0$. We now proceed by induction on $n$. Let $f \in D_n(A; \phi^m)$ and $a \in A$. Then by induction

$$\mu_{(\phi^m)^n}(a) \circ f - f \circ \mu_a \in D_{n-1}(A; \phi^m) \subseteq D_{(n-1)m}(A; \phi) \subseteq D_{nm-1}(A; \phi).$$

Thus $f \in D_{mn}(A; \phi)$.

The following is an analog of lemma 1.6 when $\phi$ has finite order.

**Lemma 1.8.** Suppose that $A$ is a domain and $\phi$ has order $n$ as an automorphism. If $f$ is a nonzero element of $D(A)$ of order $m$ and $0 \leq k < n$, then $f \circ \phi^k$ and $\phi^k \circ f$ have order $nm + k$ in $D(A; \phi)$.

**Proof.** We proceed by induction on $m$, the case $m = 0$ following from lemma 1.6. Let $f \circ \phi^k$ have order $qn + r$ in $D(A; \phi)$, where $0 \leq r < n$; by lemmas 1.7 and 1.1, we have $qn + r \leq mn + k$. For any $a \in A$, we have $(f \circ \phi^k)_a \in D_{qn+r-1}(A; \phi)$, where

$$(f \circ \phi^k)_a = \mu_{\phi^n+r(a)} \circ f \circ \phi^k - f \circ \phi^k \circ \mu_a = \mu_{\phi^n(a) - \phi^k(a)} \circ f \circ \phi^k + (\mu_{\phi^k(a)} \circ f \circ \phi^k) \circ \phi^k.$$

Now $\mu_{\phi^k(a)} \circ f \circ \phi^k$ has order at most $m - 1$ in $D(A)$, and there is an $a = a_0$ for which it has order exactly $m - 1$. Hence in $D(A; \phi)$, $(\mu_{\phi^k(a)} \circ f \circ \phi^k) \circ \phi^k$ has order at most $ln + k$ where $l < m$. Now suppose that $k \neq r$; then there is an $a \in A$ such that $\phi^r(a) - \phi^k(a)$ is regular and lemma 1.5 implies that $\mu_{\phi^r(a) - \phi^k(a)} \circ f \circ \phi^k$ has order $qn + r$. Since $(f \circ \phi^k)_a$ has order less than $qn + r$, this implies $(\mu_{\phi^k(a)} \circ f \circ \phi^k) \circ \phi^k$ has order $qn + r$, which by the above implies that $k = r$. Thus we must have $k = r$. By now choosing $a = a_0$ and applying induction, we see that $(f \circ \phi^k)_a = (\mu_{\phi^k(a)} \circ f \circ \phi^k) \circ \phi^k$ has order $(m - 1)n + k$. Thus $(m - 1)n + k < qn + k \leq mn + k$, proving $m = q$. A similar proof works for $\phi^k \circ f$.

Recall that $D(A)$ is simple when $A$ is affine and regular (and in some other cases as well) and $k$ has characteristic 0. This is not always true of $D(A; \phi)$; the next proposition gives one case when it is true. The converse to (b) is not valid: see examples 4.1 and 4.2.

**Proposition 1.9.** (a) Every nonzero ideal of $D(A; \phi)$ contains an element of the form $a\phi^n$ where $a \in A$ is nonzero and $n$ is the smallest order of nonzero elements of the ideal. If $\phi$ has finite order, then $n = 0$.

(b) If $\phi$ has finite order and $D(A)$ is simple, then $D(A; \phi)$ is simple.

**Proof.** Let $I$ be a nonzero ideal of $D(A; \phi)$, let $n$ be the smallest order of any nonzero element of $I$, and let $f$ be a nonzero element of $I$ of order $n$. Then for any $b \in A$, $f_b = \mu_{\phi^n}(b) \circ f - f \circ \mu_b \in f \cap D_{n-1}$, whence $f_b = 0$. This implies $f(b) = \phi^n(b)f(1)$ for all $b \in A$, and so $f = \mu_{f(1)} \circ \phi^n$. Since $f \neq 0$, $f(1) \neq 0$. If $\phi$ has finite order, then $\phi^{-n}$ is a positive power of $\phi$ and so lies in $D(A; \phi)$. Thus $I$ contains $f \circ \phi^{-n} = \mu_{f(1)}$. 5
To prove (b), note that \( I \cap D(A) \) is an ideal of \( D(A) \), and it is nonzero by (a). Thus it is all of \( D(A) \), whence \( 1 \in I \cap D(A) \), and so \( I = D(A; \phi) \).

Suppose \( A' \) is a \( k \)-algebra and \( \phi' \) is an algebra automorphism of \( A' \), and suppose \( \theta : A \to A' \) is an algebra isomorphism with \( \theta \circ \phi = \phi' \circ \theta \). Then the map \( f \mapsto \theta \circ f \circ \theta^{-1} \) is easily seen to be a filtration-preserving isomorphism from \( D(A; \phi) \) onto \( D(A'; \phi') \). In particular, if \( \phi \) and \( \phi' \) are automorphisms of \( A \) which are conjugate in \( \text{Aut}_k(A) \), we can naturally identify \( D(A; \phi) \) with \( D(A; \phi') \).

Similarly, if \( \psi \) is an algebra automorphism of \( A \) that commutes with \( \phi \), then we can define a filtration-preserving algebra automorphism \( \tilde{\psi} \) of \( D(A; \phi) \) by \( \tilde{\psi}(f) = \psi \circ f \circ \psi^{-1} \). For example, \( \phi \) extends to an automorphism \( \tilde{\phi} \) of \( D(A; \phi) \) having the same order as \( \phi \).

The next result gives an application of these observations. Recall that an element \( x \) of a ring \( R \) is said to normal if \( xR = Rx \). Example 4.1 shows that the assumption that \( A \) is a domain cannot be omitted.

**Proposition 1.10.** (a) \( \phi \) is a normal element of \( D(A; \phi) \), and it is not a unit in \( D(A; \phi) \) if \( \phi \) has infinite order and \( A \) is a domain.

(b) If \( \phi \) has infinite order and \( A \) is a domain, then \( D(A; \phi) \) is not simple.

**Proof.** (a) The remarks just before the proposition show \( \phi \circ f \circ \phi^{-1} \) and \( \phi^{-1} \circ f \circ \phi \) are in \( D(A; \phi) \) whenever \( f \in D(A; \phi) \). This proves \( \phi \) is a normal element.

Suppose now that \( \phi \) has infinite order, \( A \) is a domain, and \( \phi^{-1} \in D(A; \phi) \). Let \( \phi^{-1} \in D_n \) where \( n \) is minimal and let \( a \in A \) be such that \( \phi^{n+1}(a) \neq a \). Then \( f = \mu_{\phi^n(a)} \circ \phi^{-1} - \phi^{-1} \circ \mu_a \) lies in \( D_{n-1} \), and

\[
f(x) = \phi^n(a)\phi^{-1}(x) - \phi^{-1}(ax) = \phi^{-1}(\phi^{n+1}(a) - a)\phi^{-1}(x).
\]

Thus \( \phi^{-1}(\phi^{n+1}(a) - a)\phi^{-1} \in D_{n-1} \). By lemma 1.5(b), this implies \( \phi^{-1} \in D_{n-1} \), violating minimality of \( n \). This contradiction shows \( \phi^{-1} \notin D(A; \phi) \).

(b) This follows immediately from (a).

**Section 2: Localization**

Next we consider the connection between \( D(B; \phi) \) and \( D(A; \phi) \) when \( B = AC^{-1} \) is a localization of \( A \) at a multiplicatively closed set \( C \) of regular elements such that \( \phi \) extends to an automorphism of \( B \) (which we still denote by \( \phi \)). We show that \( D(A; \phi) \) can be naturally identified with the elements of \( D(B; \phi) \) that map \( A \) into itself. When \( A \) is finitely generated as a \( k \)-algebra, \( D(B; \phi) \) is a localization of \( D(A; \phi) \).

We begin with a result which is valid whenever \( A \) is itself a localization.

**Lemma 2.1.** Suppose \( A \) is a localization of the subalgebra \( A' \) and \( f \in D_n(A; \phi) \).

(a) If \( f(a') = 0 \) for all \( a' \in A' \), then \( f = 0 \).

(b) If \( A' \) is generated by a subset \( X \), then \( f \) is completely determined by its values on \( \{x_1 \cdots x_m \mid x_1, \ldots, x_m \in X, 0 \leq m \leq n\} \). If \( \phi(A') \subseteq A' \) and each \( f(x_1 \cdots x_m) \in A' \), then \( f(A') \subseteq A' \).
Proof. (a) We proceed by induction on \( n \), the case \( n < 0 \) being trivial. If \( a \in A \), then by hypothesis \( a = a'c^{-1} \) for some \( a', c \in A' \). Define \( f_c = \mu_{\phi^n(c)} \circ f - f \circ \mu_c \in D_{n-1}(A; \phi) \), and note that \( f_c(a') = 0 \) for all \( a' \in A' \). Thus by induction we have
\[
0 = f_c(a) = \phi^n(c)f(a) - f(a') = \phi^n(c)f(a).
\]
Since \( \phi^n(c) \) is a unit in \( A \), this implies \( f(a) = 0 \).

(b) The first claim in (b) follows from (a) and lemma 1.3. To prove the second claim, suppose \( f(x_1 \cdots x_m) \in A' \) whenever \( 0 \leq m \leq n \) and \( x_1, \ldots, x_m \in X \). We proceed as in the proof of lemma 1.3: for each \( x \in X \), \( f_x(y_1 \cdots y_{m-1}) \in A' \) for all \( y_1, \ldots, y_{m-1} \in X \), where \( f_x \in D_{n-1}(A; \phi) \) is defined as usual. Thus by induction on \( n \), \( f_x(A') \subseteq A' \). The formula for \( f_{ab} \) given before lemma 1.2 together with the fact that \( X \) generates \( A' \) as an algebra now implies \( f_{a'}(A') \subseteq A' \) for all \( a' \in A' \). In particular, \( \phi^n(a')f(1) - f(a') = f_a(1) \in A' \). Since \( \phi^n(a'), f(1) \in A' \), this implies \( f(a') \in A' \).

Let \( B = AC^{-1} \): certainly \( \phi \) extends to \( B \) if \( \phi(C) = C \). Conversely, if \( \phi \) extends to \( B \) and we let \( C' \) denote the multiplicatively closed subset of \( A \) generated by all the subsets \( \phi^n(C), n \in \mathbb{Z}, \) then \( \phi(C') = C' \) and \( B = AC'^{-1} \). For this reason we will always assume \( \phi(C) = C \). When a denominator set \( C \) contains zero divisors, one may achieve the localization \( AC^{-1} \) by first factoring out the \( C \)-torsion from \( A \) and then localizing at a corresponding set of regular elements. Thus it is not too great a simplification to assume \( C \) consists of regular elements, and we shall do so.

Now define \( D_n(B; A; \phi) \) to be \( \{ f \in D_n(B; \phi) \mid f(A) \subseteq A \} \) and \( D(B; A; \phi) \) to be \( \{ f \in D(B; \phi) \mid f(A) \subseteq A \} \). Clearly \( D(B; A; \phi) \) is a subalgebra of \( D(B; \phi) \). Our main aim in this section is to show that this subalgebra can be naturally identified with \( D(A; \phi) \).

**Lemma 2.2.** If \( B = AC^{-1} \) where \( C \) is a multiplicatively closed set of regular elements of \( A \) with \( \phi(C) = C \), then the map \( f \mapsto f|_A \) is an injective filtration-preserving homomorphism from \( D(B; A; \phi) \) to \( D(A; \phi) \). This homomorphism maps \( D_0(B; A; \phi) \) onto \( D_0(A; \phi) \).

**Proof.** Suppose \( f \in D_n(B; A; \phi) \). Then \( f|_A \in \text{End}_k(A) \) and we show by induction on \( n \) that \( f|_A \in D_n(A; \phi) \). This is certainly true for \( n \leq 0 \).

If \( a \in A \subseteq B \), then \( \mu_{\phi^n(a)} \circ f - f \circ \mu_a \in D_{n-1}(B; \phi) \); since \( f(A) \subseteq A \), we have \( (\mu_{\phi^n(a)} \circ f - f \circ \mu_a)(A) \subseteq A \). Thus by induction \( \mu_{\phi^n(a)} \circ f|_A - f|_A \circ \mu_a \in D_{n-1}(A; \phi) \). This shows \( f|_A \in D_n(A; \phi) \).

The fact that \( f = f|_A \) for \( f \in D_0(B; \phi) \) shows the map is onto at the zero level. Clearly the mapping is an algebra homomorphism. If \( f \in D(B; A; \phi) \) and \( f|_A = 0 \), then lemma 2.1 shows that \( f = 0 \), whence the mapping is injective. ■

The map in lemma 2.2 is actually an isomorphism. To prove this we need to show how to extend an element of \( D(A; \phi) \) to an element of \( D(B; \phi) \).

Suppose \( f \in D_n(A; \phi) \): we will define an extension \( \tilde{f} \in \text{End}_k(B) \). If \( n < 0 \), define \( \tilde{f} = 0 \); if \( n = 0 \), define \( \tilde{f} = \mu_{f(1)} \). Clearly this \( \tilde{f} \) is in \( D_n(B; \phi) \) and has the property that \( \tilde{f}|_A = f \), so \( \tilde{f} \in D(B; A; \phi) \). By lemma 2.1, \( \tilde{f} \) is the unique element of \( D(B; \phi) \) with \( \tilde{f}|_A = f \). Suppose now that for each \( g \in D_m(A; \phi) \) with \( m < n \) we have constructed a \( \tilde{g} \in D_m(B; \phi) \) with \( \tilde{g}|_A = g \). Then we inductively define \( \tilde{f} \) for \( f \in D_n(A; \phi) \) by
\[
(*) \quad \tilde{f}(ac^{-1}) = [\tilde{f}_c(ac^{-1}) + f(a)]\phi^n(c)^{-1} \quad \text{for} \ a \in A, c \in C,
\]
where as usual $f_c = \mu_{\phi^n(c)} \circ f - f \circ \mu_c \in D_{n-1}(A; \phi)$.

We first show that $\tilde{f}$ is well-defined. Since it is always possible to find a common denominator, it is enough to show the definition of $\tilde{f}$ gives the same result when applied to either $ac^{-1}$ or $(ad)(cd)^{-1}$, where $d \in C$. Now

$$\tilde{f}((ad)(cd)^{-1}) = [\tilde{f}_{cd}((ad)(cd)^{-1}) + f(ad)]\phi^n(cd)^{-1} \sim \tilde{f}_{cd}(ac^{-1})\phi^n(c)^{-1}\phi^n(d)^{-1} + f(ad)\phi^n(c)^{-1}\phi^n(d)^{-1}.$$  

We want this to equal $(\tilde{f}_c(ac^{-1}) + f(a))\phi^n(c)^{-1}$, so we need to show

$$\tilde{f}_{cd}(ac^{-1}) + f(ad) = [\tilde{f}_c(ac^{-1}) + f(a)]\phi^n(d), \quad \text{i.e.,}$$

$$\tilde{f}_{cd}(ac^{-1}) = \phi^n(d)\tilde{f}_c(ac^{-1}) + f_d(a).$$

We showed in the remarks before lemma 1.2 that $f_{dc} = \mu_{\phi^n(d)} \circ f_c + f_d \circ \mu_c$, so $\tilde{f}_{cd} - \mu_{\phi^n(d)} \circ \tilde{f}_c - \tilde{f}_d \circ \mu_c$ vanishes on $A$. Thus by lemma 2.1 it vanishes on $B$. This yields

$$\tilde{f}_{cd}(ac^{-1}) = \phi^n(d)\tilde{f}_c(ac^{-1}) + \tilde{f}_d(ac^{-1}) = \phi^n(d)\tilde{f}_c(ac^{-1}) + f_d(a),$$

as required. This proves $\tilde{f}$ is a well-defined element of $\operatorname{End}_k(A)$. Moreover,

$$\tilde{f}(a^{-1}) = [\tilde{f}_1(a^{-1}) + f(a)]\phi^n(1)^{-1} = f_1(a) + f(a) = f(a),$$

so $\tilde{f}|_A = f$.

We now wish to show that $\tilde{f} \in D_n(B; \phi)$. The following formula will be useful. For any $x, y \in A$, we claim

$$(**): \mu_{\phi^n(x)} \circ \tilde{f}_y - \tilde{f}_y \circ \mu_x = \mu_{\phi^n(y)} \circ \tilde{f}_x - \tilde{f}_x \circ \mu_y.$$  

To prove this, note that $(\phi^n(x)f_y(a) - f_y(xa)) = (\phi^n(y)f_x(a) - f_x(ya))$ for all $a \in A$, so that $(**)$ is valid when the domain of the functions is restricted to $A$. But both sides of $(**)$ lie in $D_{n-1}(B; \phi)$, so by lemma 2.1, the equality $(**)$ is valid on all of $B$.

We now proceed in two steps. First fix $a \in A$ and define $(\tilde{f})_a \in \operatorname{End}_k(B)$ by $(\tilde{f})_a(y) = \phi^n(a)\tilde{f}(y) - \tilde{f}(ay)$. Suppose $y = bc^{-1}$ where $b \in A$, $c \in C$. Then using $(**)$ we see that

$$(\tilde{f})_a(y) = \phi^n(a)\tilde{f}_c(bc^{-1}) + f(b)]\phi^n(c)^{-1} = \phi^n(a)\tilde{f}_c(y) - \tilde{f}_c(ay)]\phi^n(c)^{-1} + \phi^n(a)f(b) - f(ab)]\phi^n(c)^{-1}$$

$$= \phi^n(a)\tilde{f}_c(y) - \tilde{f}_c(ay)]\phi^n(c)^{-1} + f_a(b)]\phi^n(c)^{-1} = \tilde{f}_a(y).$$

Thus $(\tilde{f})_a = \tilde{f}_a \in D_{n-1}(B; \phi)$.

Next let $x = ac^{-1} \in B$ for $a \in A$, $c \in C$, and define $g = (\tilde{f})_x \in \operatorname{End}_k(B)$ by $g(y) = \phi^n(x)\tilde{f}(y) - \tilde{f}(xy)$. Then using the first step we see that

$$g(y) = \phi^n(ac^{-1})\tilde{f}(y) - \tilde{f}(ayc^{-1}) = \phi^n(a)\tilde{f}(y)\phi^n(c)^{-1} = \tilde{f}_c(ayc^{-1}) + \tilde{f}(ay)]\phi^n(c)^{-1}$$

$$= \phi^n(a)\tilde{f}(y) - \tilde{f}(ay)]\phi^n(c)^{-1} - \tilde{f}_c(xy)]\phi^n(c)^{-1} = \tilde{f}_a(y)\phi^n(c)^{-1} - \tilde{f}_c(xy)]\phi^n(c)^{-1}.$$  

Hence $g = \mu_{\phi^n(c)^{-1}} \circ \tilde{f}_a - \mu_{\phi^n(c)^{-1}} \circ \tilde{f}_c \circ \mu_x$. By induction $\tilde{f}_a, \tilde{f}_c \in D_{n-1}(B; \phi)$, and the other factors are in $D_0(B; \phi)$, so $g \in D_{n-1}(B; \phi)$. This proves $\tilde{f} \in D_n(B; \phi)$.

This finishes the induction showing the existence of $\tilde{f}$. In addition, if $g \in D_n(B; \phi)$ and $g(A) \subseteq A$, then $g|_A \in D_n(A; \phi)$, and so $g|_A \in D_n(B; \phi)$. As $g$ and $g|_A$ agree on $A$, they are equal. This completes the proof of the following proposition.
In this section we show that if elements of \( A \) are inverse filtration-preserving isomorphisms between the \( k \)-algebras \( D(A; \phi) \) and \( D(B; A; \phi) \).

Remark. One can verify that the following is an explicit formula for \( \tilde{f} \) when \( f \in D_n(A; \phi) \).

\[
\tilde{f}(ac^{-1}) = c^{-1} \phi(c)^{-1} \cdots \phi^n(c)^{-1} \left\{ a \phi(c) \phi^2(c) \cdots \phi^n(c) f(1) + \sum_{i=1}^{n} (-1)^i \left( \sum_{1 \leq j_1 < \ldots < j_{n-i} \leq n} \phi^{j_1}(c) \cdots \phi^{j_{n-i}}(c) \right) (af(c^i) - cf(ac^{i-1})) \right\}. 
\]

The last item of business in this section is to decide when all elements of \( D_n(B; \phi) \) have the form \( \mu_{c^{-1}} \circ \tilde{f} \) or \( \tilde{f} \circ \mu_{c^{-1}} \) for some \( f \in D_n(A; \phi) \). This is the case if \( A \) is finitely generated as an algebra, as the next proposition shows.

Proposition 2.4. Suppose \( B = AC^{-1} \) where \( C \) is a multiplicatively closed set of regular elements of \( A \) with \( \phi(C) = C \). Then the maps \( f \mapsto \tilde{f} \) (where \( \tilde{f} \) is defined as above) and \( g \mapsto g|_A \) are inverse filtration-preserving isomorphisms between the \( k \)-algebras \( D(A; \phi) \) and \( D(B; A; \phi) \).

Proof. (a) Given \( f \in D_n(A; \phi) \) and \( c \in C \), we wish to find \( c', c'' \in C \) and \( f', f'' \in D_n(A; \phi) \) with \( \mu_c \circ f' = f \circ \mu_{c'} \) and \( f'' \circ \mu_c = \mu_{c''} \circ f \). We proceed by induction on \( n \), the case \( n \leq 0 \) being clear. First set \( d = \phi^{-n}(c) \in C \) and define \( f_d \in D_{n-1} \) as usual. Then by induction we can find \( g \in D_{n-1}, d' \in C \) with \( \mu_c \circ g = f_d \circ \mu_{d'} = \mu_{\phi^n(d')} \circ f \circ \mu_{d'} - f \circ \mu_d \circ \mu_{d'} \), whence \( \mu_c \circ (f \circ \mu_{d'} - g) = f \circ \mu_{d'd'} \), as required. This proves that the right Ore condition is satisfied; the left Ore condition is verified in a similar manner.

(b) Since \( \{ \mu_c | c \in C \} \) is an Ore set in \( D(A; \phi) \) whose elements are units in \( D(B; \phi) \), it is enough to show that for each \( f \in D_n(B; \phi) \), \( (\mu_c \circ f)(A) \subseteq A \) for some \( c \in C \). Let \( X \) be a finite generating set for \( A \). If we consider the finitely many elements of \( B \) of the form \( f(x_1 \cdots x_m) \) with \( 0 \leq m \leq n, x_1, \ldots, x_m \in X \), then we can find a single common denominator \( c \) for them, so \( cf(x_1 \cdots x_m) \) always lies in \( A \). Now \( \mu_c \circ f \in D_n(B; \phi) \), and the containment we have just stated implies, by lemma 2.1(b), that \( (\mu_c \circ f)(A) \subseteq A \). This completes the proof.

Section 3: Operators determined by their effect on the powers of a single element

In this section we show that if \( t \) is an element of \( A \) such that \( \phi^i(t) - t \) is regular for all \( i > 0 \), then an element of \( D_n(A; \phi) \) is completely determined by its effect on \( 1, t, \ldots, t^n \). If in fact \( \phi^i(t) - t \) is a unit for all \( i > 0 \), then we can find an element of \( D_n \) which takes \( 1, t, \ldots, t^n \) to any elements of \( A \) we desire, and in this case, \( D(A; \phi) = A[\phi] \) can be naturally identified with the Ore extension \( A[x; \phi] \). In case \( A \) has (Krull) dimension bigger than 1, this is quite different from what happens for \( D(A) \).
Lemma 3.1. Suppose $m, n \in \mathbb{Z}, m \geq 0, f \in D_n(A; \phi), t \in A$. Suppose also that $\phi^i(t) - t$ is regular for $1 \leq i \leq \max(m, n)$ and that $f(ta) = \phi^m(t) f(a)$ for all $a \in A$. Then $f = f(1) \phi^m$, and if $m > n$, we have $f = 0$.

Proof. This is trivial if $n < 0$. We now proceed by induction on $n \geq 0$, assuming $f \neq 0$. Since reducing $n$ does no harm, we may assume $f \in D_n, f \notin D_{n-1}$.

Our assumption on $f(ta)$ implies that $\mu_{\phi^m(t)} f - f \circ \mu_t \in D_{n-1}$ and we know that $\mu_{\phi^m(t)} - f \circ \mu_t \in D_{n-1}$, whence we obtain $\mu_{\phi^m(t)} - \phi^n(t) f \in D_{n-1}$. The regularity hypothesis implies that $\phi^m(t) - \phi^n(t)$ is regular if $m \neq n$, and so $f \in D_{n-1}$ by lemma 1.5(b), contrary to our assumption on $n$. Thus $m = n$.

Now for each $b \in A$, we have $f_b = \mu_{\phi^n(b)} f - f \circ \mu_b \in D_{n-1}$. Clearly $f_b(ta) = t^m f_b(a)$ for all $a \in A$. Since $m > n - 1$, it follows by induction that $f_b = 0$, so $\phi^n(b) f(a) = f(ba)$ for all $a, b \in A$. Hence $f(b) = \phi^n(b) f(1)$ for all $b \in A$, i.e., $f = f(1) \phi^m$.

This leads to the following surprising result.

Proposition 3.2. Suppose $n \geq 0, f, g \in D_n(A; \phi)$, and $t \in A$ is such that $\phi^i(t) - t$ is regular for $1 \leq i \leq n$. If $f(t^k) = g(t^k)$ for $0 \leq k \leq n$, then $f = g$.

Proof. We may assume $g = 0$, so $f(t^k) = 0$ for $0 \leq k \leq n$; we will show $f = 0$ by induction on $n$. This follows from proposition 1.1(c) for $n = 0$. Define $f_t \in D_{n-1}$ as usual and note that $f_t(t^k) = 0$ if $0 \leq k \leq n - 1$. By the induction assumption $f_t = 0$, and so $f(ta) = \phi^n(t) f(a)$ for all $a \in A$. By lemma 3.1, this implies $f = 0$, since $f(1) = 0$.

Proposition 3.2 is of course very useful in calculating $D(A; \phi)$. We give one example of its use. Given $A$ and $\phi$, let $B$ be another $k$-algebra, and note that $\phi \otimes \text{id}$ is an automorphism of $A \otimes_k B$. If there exists $t \in A$ such that $\phi^i(t) - t$ regular for every $i > 0$, then $D(A \otimes_k B; \phi \otimes \text{id}) = D(A; \phi) \otimes_k B$. The reason for this is that $(\phi \otimes \text{id})^i(t \otimes 1) = (\phi^i(t) - t) \otimes 1$ is regular in $A \otimes B$ for every $i > 0$, so elements of $D(A \otimes_k B; \phi \otimes \text{id})$ are determined by their action on $A$. We leave the details to the interested reader. This example shows how proposition 3.2 causes the theory of skew differential operators to differ from the theory of ordinary differential operators, since $D(A \otimes_k B)$ has operators which act nontrivially on $B$ and annihilate $A$.

Corollary 3.3. Suppose there is a $t \in A$ such that either $\phi^i(t) - t$ regular for all $i > 0$ or $A$ is a localization of the subalgebra generated by $t$. If for each $n \geq 0$ there is a $\delta_n \in D_n(A; \phi)$ with $\delta_n(t^i) = 0$ for $i < n$ while $\delta_n(t^n) = 1$, then

$$D_n(A; \phi) = \oplus_{i=0}^n A \delta_i = \oplus_{i=0}^n \delta_i A \quad \text{and} \quad D(A; \phi) = \oplus_{i=0}^\infty A \delta_i = \oplus_{i=0}^\infty \delta_i A.$$  

Proof. Proposition 3.2 or lemma 2.1 implies that $D_n(A; \phi) = \oplus_{i=0}^n A \delta_i$, and $D(A; \phi) = \oplus_{i=0}^\infty A \delta_i$. Since $A \delta_i \equiv \delta_i A \pmod{D_{i-1}}$, we also have $D_n(A; \phi) = \oplus_{i=0}^n \delta_i A$ and $D(A; \phi) = \oplus_{i=0}^\infty \delta_i A$.

Theorem 3.4. If $n \geq 0, p_0, \ldots, p_n, t \in A$, and $\phi^i(t) - t$ is a unit for $1 \leq i \leq n$, then there is a unique $f \in D_n(A; \phi)$ such that $f(t^k) = p_k$ for all $k$ with $0 \leq k \leq n$, and in fact, $f = \sum_{i=0}^n a_i \phi^i$ for some $a_0, \ldots, a_n \in A$.  

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Proof. Uniqueness follows from proposition 3.2. To prove the existence of \( f \) we need to solve the system
\[
\sum_{j=0}^{n} \phi^j(t^i)a_j = p_i, \quad 0 \leq i \leq n
\]
of \( n+1 \) linear equations in the \( n+1 \) unknowns \( a_0, \ldots, a_n \). The determinant of this system is the determinant of the matrix
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
t & \phi(t) & \cdots & \phi^n(t) \\
\vdots & \vdots & \ddots & \vdots \\
t^n & \phi(t)^n & \cdots & \phi^n(t)^n
\end{pmatrix},
\]
which is \( \prod_{0 \leq i < j \leq n} (\phi^j(t) - \phi^i(t)) \). This is a unit by our hypothesis, and so the system has a solution.

\[ \square \]

**Corollary 3.5.** If for each \( n \geq 0 \) there is a \( t_n \in A \) such that \( \phi^i(t_n) - t_n \) is a unit for \( 1 \leq i \leq n \), then for any \( n \geq 0 \), \( D_n(A; \phi) = \bigoplus_{i=0}^{n} A\phi^i \), and \( D(A; \phi) = A[\phi] \).

\[ \square \]

**Corollary 3.6.** (a) If \( A \) is a field extension of \( k \) and \( \phi \) has infinite order, then for any \( n \geq 0 \), \( D_n(A; \phi) = \bigoplus_{i=0}^{n} A\phi^i \), and \( D(A; \phi) = A[\phi] \).
(b) If \( A \) is a domain with quotient field \( Q \) and \( \phi \) has infinite order, then \( D_n(A; \phi) = \{ f = \sum_{i=0}^{n} q_i\phi^i | q_0, \ldots, q_n \in Q \text{ and } f(A) \subseteq A \} \) and \( D(A; \phi) = \{ f \in Q[\phi] | f(A) \subseteq A \} \).

**Proof.** (a) Since \( \phi^n \neq \text{id} \), there is a \( t_n \) with \( \phi^n(t_n) \neq t_n \), whence \( \phi^i(t_n) \neq t_n \) for \( 1 \leq i \leq n \). Now apply Corollary 3.5.
(b) This follows from (a) and proposition 2.3.

\[ \square \]

Hence if \( \phi \) has infinite order and \( A \) is a field, \( D(A; \phi) \) is naturally isomorphic to the Ore extension \( A[x; \phi] \) (and so is a Noetherian domain). In general, if there is a \( t \in A \) with \( \phi^i(t) - t \) regular for all \( i > 0 \), then if we let \( C \) be the multiplicatively closed set generated by \( \{ \phi^i(t) - \phi^j(t) | i > j \geq 0 \} \), the results above show that \( A[\phi] \subseteq D(A; \phi) \subseteq AC^{-1}[\phi] \).

**Corollary 3.7.** Suppose \( \phi \neq \text{id} \). Then \( D(A; \phi) \) is a domain if and only if \( A \) is a domain and \( \phi \) has infinite order. If \( A \) is finitely generated as a \( k \)-algebra and \( D(A; \phi) \) is a domain, then it is an Ore domain.

**Proof.** If \( A \) is a domain with quotient field \( Q \) and \( \phi \) has infinite order, then the last corollary tells us that \( D(A; \phi) \) is a subalgebra of the Noetherian domain \( D(Q; \phi) \). If \( A \) is finitely generated, proposition 2.4 shows \( D(Q; \phi) \) is a localization of \( D(A; \phi) \); this implies \( D(A; \phi) \) is an Ore domain.

If \( A \) is not a domain, then \( D(A; \phi) \) cannot be either since it contains \( A \) as a subalgebra. If \( A \) is a domain and \( \phi \) has finite order \( k > 1 \), then
\[
(\phi - \text{id})(\phi^{k-1} + \cdots + \phi + \text{id}) = 0,
\]
and neither factor is 0 by lemma 1.6.

\[ \square \]
Remark. If \( A \) is a domain and \( \phi \) has finite order \( n \), there is a \( t \in A \) with \( t, \phi(t), \ldots, \phi^{n-1}(t) \) distinct. To see this, note that any element of the quotient field of \( A \) can be written with a denominator which is fixed by \( \phi \). Thus we may pass to the quotient field and so assume \( A \) is a field. Now \( A \) is a Galois extension of dimension \( n \) of the fixed field \( A^\phi \), and so by the normal basis theorem, \( A \) has a basis over \( A^\phi \) consisting of \( t, \phi(t), \ldots, \phi^{n-1}(t) \) for some \( t \in A \). Thus if \( A \) is a field, this shows \( D_{n-1}(A; \phi) = \oplus_{i=0}^{n-1} A\phi^i \); however \( D_n(A; \phi) \) will contain the elements of \( D_1(A) \) since \( \phi^n = \text{id} \).

Section 4: Some examples

In this section we work out some examples, several of which show the difference between skew differential operator rings and ordinary differential operator rings. Example 4.5, for example, shows that even if \( A \) is the coordinate ring of a smooth variety over a field of characteristic 0, \( D(A; \phi) \) need not be Noetherian or finitely generated as a \( k \)-algebra. In example 4.3, we work out the possibilities for \( D(A; \phi) \) when \( A \) is the polynomial ring in one variable over \( k \).

We first consider two finite-dimensional examples which show that some previous results cannot be extended.

Example 4.1. Let \( A = k[t]/(t^2) \) and denote the image of \( t \) by \( \bar{t} \). Suppose \( \lambda \in k \) is not a root of unity and define an automorphism \( \phi \) of \( A \) by \( \phi(\bar{t}) = \lambda \bar{t} \), so that \( \phi \) has infinite order. It is easily checked that for any integer \( m \), \( \phi^m = (\lambda - 1)^{-1}[(\lambda^m - 1)\phi - (\lambda^m - \lambda)\text{id}] \), whence \( \phi^m \in D_1(A; \phi) \). In particular, \( \phi \) is a unit in \( D(A; \phi) \) in spite of having infinite order as an automorphism, and \( \phi^m \) has order less than \( m \) as a skew differential operator for \( m \geq 2 \). One can check that the map \( f \in \text{End}_k(A) \) defined by \( f(\alpha \bar{t} + \beta) = \alpha \) is in \( D_2(A; \phi) \) but not \( D_1(A; \phi) \), so \( f \) is not a linear combination of powers of \( \phi \). Also \( D(A; \phi) = D_2(A; \phi) = \text{End}_k(A) \cong M_2(k) \), and so \( D(A; \phi) \) is simple in spite of \( \phi \)'s having infinite order.

Example 4.2. Let \( A = k \times k \) and let \( \phi \) be the automorphism defined by \( \phi((x, y)) = (y, x) \), so \( \phi \) has order 2. Now one easily checks that \( D(A) = A \), whence \( D(A) \) is not simple. However, \( D(A; \phi) = A \oplus A\phi = \text{End}_k(A) \) is simple.

Recall that if \( A \) is the coordinate ring of a nonsingular variety or of a curve over a field of characteristic 0, the ordinary ring of differential operators \( D(A) \) is Noetherian and is finitely generated as a \( k \)-algebra. Before presenting the remaining examples, we discuss a general situation in order to develop a method to show certain rings of skew differential operators are not Noetherian. Let \( A \) be an integral domain and suppose that there exist \( \delta_m \in D_m(A; \phi) \) for each \( m \geq 0 \) such that \( D_n(A; \phi) = \oplus_{m=0}^n A\delta_m \) for all \( n \), and such that for all \( i, j \), \( \delta_i \delta_j = c_{ij} \delta_{i+j} \) for some \( c_{ij} \in A \). Now let \( I_m \) be the left ideal of \( D(A; \phi) \) generated by \( \delta_1, \ldots, \delta_m \). If it is the case that \( \delta_m \notin I_{m-1} \) for infinitely many \( m \), then of course \( D(A; \phi) \) will not be left Noetherian.

Suppose that \( \delta_m = \sum_{j=1}^{m-1} d_j \delta_j \) for some \( d_1, \ldots, d_{m-1} \in D(A; \phi) \). Using the direct
sum decomposition of $D(A; \phi)$, we may write each $d_j = \sum_{i=0}^{\infty} a_{ij} \delta_i$. Thus

$$
\delta_m = \sum_{j=1}^{m-1} \sum_{i=0}^{\infty} a_{ij} \delta_i \delta_j = \sum_{j=1}^{m-1} \sum_{i=0}^{\infty} a_{ij} c_{ij} \delta_{i+j}.
$$

As $D(A; \phi)$ is a direct sum and $A$ is an integral domain, this implies $\sum_{j=1}^{m-1} a_{m-j,j} c_{m-j,j} = 1$. Thus if for infinitely many values of $m$, the elements $c_{m-1,1}, c_{m-2,2}, \ldots, c_{1,m-1}$ generate a proper ideal of $A$, the ring $D(A; \phi)$ will not be left Noetherian. Reversing the sides gives a condition for $D(A; \phi)$ to fail to be right Noetherian.

In the next example we work out in detail all the possibilities for $D(A; \phi)$ when $A$ is a polynomial ring in one variable. One of the conclusions that follows from our calculations is that in this case the following statements are equivalent: (1) $D(A; \phi)$ is (right or left) Noetherian; (2) $D(A; \phi)$ is finitely generated as a $k$-algebra; (3) either $k$ has characteristic 0 or $\phi$ has infinite order.

**Example 4.3.** Let $A = k[t]$ be the polynomial ring in one variable and let $\phi$ be an automorphism of $A$. It is easy to show that $\phi$ is conjugate either to a dilation or to translation by 1; we will describe $D(A; \phi)$ in detail for these possibilities.

Before we consider separate cases, we show that for any $\phi$, for each $n \geq 0$ there exists $\delta_n \in D_n$ such that $\delta_n(t^i) = 0$ for $i < n$ and $\delta_n(t^n) = 1$. It then follows from corollary 3.3 that $D_n = \bigoplus_{i=0}^{n} A \delta_i = \bigoplus_{i=0}^{n} \delta_i A$.

Clearly we may take $\delta_0 = 1$. Suppose $n \geq 1$ and we have defined $\delta_0, \ldots, \delta_{n-1}$. Then we can define a unique linear map $\delta_n \in \text{End}_k(A)$ recursively from the relations

$$
\delta_n(1) = 0 \quad \text{and} \quad \delta_n(tp) = \phi^n(t) \delta_n(p) + \delta_{n-1}(p) \quad \text{for } p \in A.
$$

Moreover, $(\delta_n)_t = \mu_{\phi^n(t)} \circ \delta_n - \delta_n \circ \mu_t = -\delta_n-1$. Thus by lemma 1.2, $\delta_n \in D_n$. It is easy to see that $\delta_n(t^i) = 0$ if $i < n$, while $\delta_n(t^n) = 1$; hence the required $\delta_n$ exists. The inductive definition also shows that if $p \in A$ has degree $d \geq n$, then $\delta_n(p)$ has degree at most $d - n$. Thus $\delta_i \delta_j(t^m)$ is 0 if $m < i + j$ and is a scalar if $m = i + j$. Since elements of $D_{i+j}$ are determined by their effect on $1, t, \ldots, t^{i+j}$, it follows that $\delta_i \delta_j = c_{ij} \delta_{i+j}$ for some scalar $c_{ij}$.

We note that by employing the above recursive definition one can derive an explicit formula for $\delta_n$. Thus $\delta_n(t^i) = 0$ for $i < n$, while if $k \geq 0$,

$$
\delta_n(t^{n+k}) = \sum_{i_0 + \cdots + i_n = k} t^{i_0} \phi(t)^{i_1} \cdots \phi^n(t)^{i_n},
$$

where the exponents $i_0, \ldots, i_n$ are nonnegative integers.

We now consider the separate cases.

(a) Suppose $\phi(t) = qt$ where $q \in k$ is not a root of unity. Define $\delta \in \text{End}_k(A)$ by $\delta(a) = (\phi(a) - a)/(\phi(t) - t)$, i.e., $\delta = (q-1)^{-1} t^{-1} (\phi - \text{id}) \in D_1$. (This $\delta$ is called the Eulerian derivative corresponding to $q$ or the $q$-difference operator.) Since $\delta(1) = 0, \delta(t) = 1$, we
may take $\delta_1 = \delta$. Now $\delta(t^i) = [(q^i - 1)/(q - 1)]t^{i-1}$, so $\delta^m(t^i) = 0$ for $i < m$ and $\delta^m(t^n) = \prod_{i=1}^m (q^i - 1)/(q - 1) \in k \setminus \{0\}$. Thus we may take $\delta_m = (\prod_{i=1}^m (q^i - 1)/(q - 1))^{-1} \delta^m$. It follows that $D(A; \phi) = A[\delta]$, and this is clearly isomorphic to the Ore extension $A[x; \phi, \delta]$ (that is, the ring generated by $A$ and the indeterminate $x$ with multiplication determined by the rule $xa = \phi(a)x + \delta(a)$), which in turn is isomorphic to the algebra $k[t, x | xt - qtx = 1]$. This is a “quantized” version of the Weyl algebra; this algebra is discussed in [1; 2, section 8]. It is a Noetherian domain which is finitely generated as a $k$-algebra; the element $w = tx + (q - 1)^{-1}$, which corresponds to $\phi$, is normal. The results in [2] show that every nonzero ideal of $D(A; \phi)$ contains a power of $\phi$ and that $D(A; \phi)/(\phi) \cong k[t^{\pm 1}]$.

Note that if $B = k[t^{\pm 1}]$ and we extend $\phi$ to $B$, then $\phi^n(t) - t$ is a unit in $B$ for any $n > 0$, so $D[B; \phi] = B[\phi] \cong k[t^{\pm 1}, x | t^{-1}xt = qx]$.

(b) Suppose $\phi(t) = qt$ where $q \in k$ is a primitive $n$th root of unity. Then by lemma 1.7, $D(A; \phi)$ contains $D(A)$. Now $D(A)$ is always a simple ring (see [7] for facts about $D(A)$, especially when $k$ has positive characteristic), so by proposition 1.9, $D(A; \phi)$ is simple. If $q = 1$, then $D(A)$ is the first Weyl algebra $A_1$ in case $k$ has characteristic 0; this is a simple Noetherian domain. If $k$ has positive characteristic, then $D(A)$ is neither Noetherian nor a domain.

Assume now that $q \neq 1$: then we can define $\delta_1 = \delta$ to be the Eulerian derivative as in (a), and use this to define $\delta_1, \ldots, \delta_{n-1}$. However, $\delta^n = 0$, and so in fact $\delta_j = 0$ whenever $i, j < n \leq i + j$. This implies that $D_{n-1}$ is actually a subalgebra of $D(A; \phi)$, which is finitely generated as a left or right module over $A$, and so $D_{n-1}$ is a Noetherian ring.

Note also that $\delta_{n-1}(t^i) = t^{i-(n-1)}$ if $i \equiv -1 \pmod n$ and 0 otherwise. We can now define $\delta_n \in \text{End}_k(A)$ by $\delta_n(t^i) = [\frac{i}{n}]t^{i-n}$ (where $[x]$ denotes the greatest integer function of $x$). Since

$$(\delta_n)_t(t^i) = \phi^n(t)\delta_n(t^i) - \delta_n(t^{i+1}) = [\frac{i}{n}]t^{i+1-n} - [\frac{i}{n} + 1]t^{i+1-n} = -\delta_{n-1}(t^i),$$

we see $\delta_n \in D_n$. Clearly $\delta_n(t^i) = 0$ if $i < n$ and $\delta_n(t^n) = 1$. A direct calculation shows that $\delta\delta_n = \delta_n\delta$. The rest of the analysis splits into two cases depending on the characteristic of $k$.

(i) Suppose $k$ has characteristic 0. If $m \geq 0$, write $m = na + r$ where $a, r \in \mathbb{Z}$, $0 \leq r < n$, and define $f = (1/a!)[\delta, \delta^a] \in D_m$. It is easily checked that $f(t^i) = 0$ if $i < m$ and $f(t^m) = 1$. Thus we may take $\delta_m = f$; this agrees with our previous definitions if $m \leq n$. It follows that $D(A; \phi) = D_{n-1}[\delta_n]$, whence $D(A; \phi)$ is generated as an algebra by $t, \delta, \delta_n$.

We now show that $D(A; \phi) \cong M_n(A_1)$, whence $D(A; \phi)$ is Noetherian. If we let $X = \delta_n, Y = t^n$, then $D(A; \phi)$ is generated as a $k$-algebra by $t, \delta, X, Y$ subject to the relations

$$\delta^n = 0, t^n = Y, \delta t - q t\delta = 1, X\delta = \delta X, Y\delta = \delta Y,$$

$$Xt - tX = (\prod_{i=1}^{n-1} \frac{q^i - 1}{q - 1})^{-1}\delta^n - 1, XY - YX = 1.$$
Then the map defined by

\[ X \mapsto xI, \; Y \mapsto yI, \; t \mapsto e_{21} + \cdots + e_{n,n-1} + ye_{1n}, \; \delta \mapsto \sum_{i=1}^{n-1} \frac{q^i - 1}{q - 1} e_{i,i+1} \]

yields an isomorphism from \( D(A; \phi) \) onto \( M_n(A_1) \). (The author would like to thank Ian Musson for pointing this out to him in the case \( n = 2 \).

Now let \( B = k[t^{\pm 1}] \) and extend \( \phi \) to \( B \), and also to \( D(A) \) and \( D(B) \) as discussed after proposition 1.9 (so \( \phi(t) = qt \) and \( \phi\left( \frac{d}{dt} \right) = q^{-1} \frac{d}{dt} \)). Then by proposition 2.4, \( D(B; \phi) = D(A; \phi)[t^{-1}] \). It is easy to see that there is an \( f = \sum a_i \delta_i \in D_n(A; \phi) \) with \( f(t^i) = it^{i-1} \) for \( 0 \leq i \leq n-1 \). Set \( g = \frac{1}{n} t^{-(n-1)} \left( \frac{d}{dt} - f \right) \in D_n(B; \phi) \); then \( g(t^i) = 0 \) if \( 0 \leq i \leq n-1 \) and \( g(t^n) = 1 \). It follows that \( g = \delta_n \). This in turn implies that \( D(B; \phi) = D(B)[\phi] \), which is isomorphic to the skew group ring \( D(B) * \langle \phi \rangle = (D(A) * \langle \phi \rangle)[t^{-1}] \) by lemma 1.8. Since no non-identity power of \( \phi \) is inner on \( D(A) \), if follows from [3, Corollary 3.18] that \( D(A) * \langle \phi \rangle \) is a simple ring. As \( D(A; \phi) \) is strictly contained between this simple ring and its localization \( D(B; \phi) \) (since for example \( \delta \notin D(A) * \langle \phi \rangle \)), it follows that \( D(A; \phi) \) cannot be finitely generated as a module over \( D(A) * \langle \phi \rangle \), and hence that \( D(A; \phi) \) is not finitely generated as a module over \( D(A) \). (The author would like to thank the referee for this comment.)

(ii) Suppose \( k \) has characteristic \( p > 0 \). Then \( \delta_n^p = 0 \), so \( D_{np-1} \) is a subalgebra. In fact each \( D_{np^s-1} \) is a subalgebra, which implies \( D(A; \phi) \) is a strictly increasing union of subalgebras, and so cannot be finitely generated as a \( k \)-algebra. In addition \( \delta_i \delta_j = 0 \) whenever \( i, j < np^s \leq i+j \); the discussion following example 4.2 thus implies that \( D(A; \phi) \) is neither left nor right Noetherian.

The claims in the last paragraph are proved by induction on \( s \), employing the following four induction hypotheses:

1. \( \delta_{np^s}(t^i) = \left\lfloor \frac{i}{np^s} \right\rfloor t_i^{i-np^s} \);

2. \( \delta_{np^s-1}(t^i) = \begin{cases} t_i^{i-(np^s-1)} & \text{if } i \equiv -1 \pmod{np^s}; \\ 0 & \text{otherwise}; \end{cases} \)

3. if \( m = np^s a + r < np^{s+1} \), where \( a, r \in \mathbb{Z} \), \( 0 \leq a < p \), \( 0 \leq r < np^s \), then \( \delta_m = (1/\alpha)!\delta_r \delta_{np^s}^a \);

4. \( \delta_i \delta_j = 0 \) whenever \( i, j < np^s \leq i+j \).

All of these assumptions are valid when \( s = 0 \) by our discussion at the start of (b) above. We omit the details of the induction, as they are mostly computations.

(c) Finally, suppose \( \phi(t) = t+1 \). If \( k \) has characteristic \( 0 \), \( \phi^n(t) - t = n \) is a unit for any \( n > 0 \), and so we know \( D(A; \phi) = A[\phi] \), which is isomorphic to the Ore extension \( A[x; \phi] \). This in turn is isomorphic to the algebra \( k[x, t | xt - tx = x] \), which is the universal enveloping algebra of the unique non-Abelian two-dimensional Lie algebra. This is a Noetherian domain which is not simple (obviously \( x \) is a normal element).

If \( k \) has positive characteristic \( p \), then \( \phi \) has order \( p \), and so \( D(A; \phi) \) contains \( D(A) \). As in (b), this implies \( D(A; \phi) \) is a simple ring which is not a domain. By [7, Corollary 2.2], \( D(A) \) contains an infinite orthogonal set of nonzero idempotents, and so \( D(A; \phi) \) also
contains such a set. This implies \( D(A; \phi) \) is neither right nor left Noetherian. We now show that each subspace \( D_{p^s-1} \) is a subalgebra, which implies that \( D(A; \phi) \) is not finitely generated as a \( k \)-algebra.

Let \( Q \) be the quotient field of \( A \); then by proposition 2.3, it is enough to show each \( D_{p^s-1}(Q; \phi) \) is a subalgebra of \( D(Q; \phi) \). Let \( n = pa + r \) be any nonnegative integer, where \( 0 \leq r < p \). By proposition 2.4 and the results above we know that \( D_n(Q; \phi) = \bigoplus_{m=0}^{r} Q\delta_m \) for some \( \delta_m \). It follows from [7] that there is an element \( f \in D(Q) \) with order exactly \( a \), and lemma 1.8 implies that \( f \circ \phi^n \) has order \( n \). Since \( Q \) is a field, we see that \( D_n(Q; \phi) = Q(f \circ \phi^n) + D_{n-1}(Q; \phi) \). It now follows by induction on \( n \) that \( D_n(Q; \phi) = \sum_{i=0}^{r} D_{a}(Q)\phi^i + \sum_{i=r+1}^{p-1} D_{a-1}(Q)\phi^i \). In particular, \( D_{p^s-1}(Q; \phi) = \sum_{i=0}^{p-1} D_{p^s-1-1}(Q)\phi^i \).

Recall that if \( k \) is algebraically closed of characteristic 0 and \( A \) is the coordinate ring of a smooth surface and that \( \phi \) has infinite order. By Corollary 3.7, \( D_{p^s-1}(Q; \phi) \) is a subalgebra of \( D(Q; \phi) \), it follows that \( D_{p^s-1}(Q; \phi) \) is a subalgebra of \( D(Q; \phi) \).

Example 4.4. Suppose \( k \) has characteristic 0; let \( A = k[s, t] \) and let \( \phi \) be defined by \( \phi(s) = s, \phi(t) = t + 1 \). Since \( \phi^n(t) - t \) is always a unit, \( D(A; \phi) = A[\phi] = A'[\phi'][s] \), where \( A' = k[t], \phi'(t) = t + 1 \) (see example 4.3(c)). Since \( A \) is the coordinate ring of affine 2-space, all the results mentioned in the last paragraph apply to \( D(A) \). However, they do not apply to \( D(A; \phi) \).

(1) The Gelfand-Kirillov, Krull, and global dimensions of \( D(A; \phi) \) are all 3 (since for example \( D(A; \phi) \) is isomorphic to the enveloping algebra of a 3-dimensional solvable Lie algebra); for \( A \) these dimensions are all 2.

(2) As a left \( D(A; \phi) \)-module with the natural action, \( A \) is not simple; in fact, it is not Artinian. The \( D(A; \phi) \)-submodules of \( A \) are precisely the ideals generated by polynomials in \( s \).

Our final example is of a nice ring \( A \) with an automorphism \( \phi \) of infinite order such that \( D(A; \phi) \) is not Noetherian and is not finitely generated as an algebra. This example can be regarded as the "generic" version of example 4.3(a).

Example 4.5. Let \( k \) be any field, let \( A = k[q^\pm t] \) (where \( q \) and \( t \) are indeterminates), and let \( \phi \) be the automorphism of \( A \) defined by \( \phi(q) = q, \phi(t) = qt \). Note that \( A \) is the coordinate ring of a smooth surface and that \( \phi \) has infinite order. By Corollary 3.7, \( D(A; \phi) \) is an Ore domain.

As in example 4.3, we take \( \delta_1 = \delta \) to be the Eulerian derivative defined by \( \delta(a) = (\phi(a) - a)/(\phi(t) - t) \). It is easy to see that this makes sense: \( (qt)^i - t^i = (q^i - 1)t^i \), so \( \phi(p) - p \) is always divisible by \( (q - 1)t \) for \( p \in A \). Clearly \( \delta^m(t^r) = 0 \) if \( r < m \), while

\[
\delta^m(t^r) = (\prod_{s=0}^{m-1} \frac{q^{r-s} - 1}{q - 1})t^{r-m} \quad \text{if } r \geq m.
\]
It follows from corollary 3.3 that if we set

$$
\delta_m = (\prod_{s=1}^{m} \frac{q^s - 1}{q - 1})^{-1} \delta^m \in D_m,
$$

we have $D_n = \oplus_{m=0}^{n} A \delta_m = \oplus_{m=0}^{n} \delta_m A$ for all $n$.

Now $\delta_i(\delta_j(t')) = 0$ if $r < i + j$, while

$$
\delta_i(\delta_j(t^{i+j})) = \delta_i(\lbrace \prod_{s=1}^{j} \frac{q^s - 1}{q - 1} \rbrace^{-1} \prod_{s=0}^{j-1} \frac{q^{i+j-s} - 1}{q - 1} t^i) = \prod_{s=0}^{j-1} \frac{q^{i+j-s} - 1}{q^{s+1} - 1}.
$$

Since both $\delta_i \delta_j$ and $\delta_{i+j}$ are in $D_{i+j}$, this implies $\delta_i \delta_j = c_{ij} \delta_{i+j}$, where $c_{ij} = \prod_{s=0}^{j-1} \frac{q^{i+j-s} - 1}{q^{s+1} - 1}$.

It follows that $c_{m-1,1}, c_{m-2,2}, \ldots, c_{1,m-1}$ all are divisible by the $m^{th}$ cyclotomic polynomial $\Phi_m(q)$. (To see this, note that the primitive complex $m^{th}$ roots of unity are roots of the factor $q^m - 1$ in the numerator of each $c_{ij}$, but are not roots of any of the factors $q^{s+1} - 1$ in the denominator, since $0 \leq s \leq m - 2$. Thus $\Phi_m(q)$ divides each $c_{ij}$ in $C[q]$. As $\Phi_m(q)$ has integer coefficients and each of the polynomials $c_{ij}$ has integer coefficients, $\Phi_m(q)$ divides $c_{ij}$ in $Z[q]$, and hence in $k[q]$ for any field $k$.) Since $\Phi_m(q)$ is not a unit for $m \geq 2$, these elements lie in a proper ideal of $A$. It follows from the discussion before example 4.3 that $D(A; \phi)$ is not left Noetherian.

Since $\phi(q) = q$, we have $\delta_m(qa) = q \delta_m(a)$ for all $a$ and all $\delta_m$, whence $\delta_i \delta_j = \delta_{i+j} c_{i+j}$ as well. Since $D_n = \oplus_{m=0}^{n} \delta_m A$, we can conclude that $D(A; \phi)$ is not right Noetherian. A similar argument shows that $D(A; \phi)$ is not finitely generated as a $k$-algebra.

References