<u>**Mesoscale Meteorology: Gravity Waves**</u>

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Introduction

Here, we primarily consider *internal gravity waves*, or waves that propagate in a density-stratified fluid (nominally, a stably-stratified fluid, with lower density air residing above higher density air) under the influence of *buoyancy* forces. Indeed, it is buoyancy that acts as the *restoring force* for internal gravity waves; i.e., a parcel perturbed upward or downward will be restored to, or oscillate beyond, its initial location as a function of its buoyancy relative to its surroundings.

Recall that buoyancy is defined as:

$$B = -\frac{\rho'}{\rho} g \approx \frac{\theta_v'}{\overline{\theta_v}} g \approx \frac{\theta'}{\overline{\theta}} g$$

The first approximate form holds in the absence of hydrometeors given greater variation in density due to temperature rather than pressure perturbations. The second approximate form holds for dry dynamics; e.g., when moisture is altogether neglected. If base-state potential temperature increases with height, a parcel perturbed upward (downward) by an infinitesimal distance will have negative (positive) perturbation potential temperature; buoyancy restores the parcel downward (upward).

Mathematical Framework

Internal gravity waves are one of many allowable solutions to the primitive equations. To consider the structure and propagation of internal gravity waves in the atmosphere without considering full atmospheric complexity, we first make a few simplifying assumptions:

- The base-state environment is in hydrostatic balance (i.e., no vertical parcel accelerations).
- All parcel displacements are adiabatic (i.e., conserving potential temperature).
- The Coriolis and frictional forces are neglected.
- The Boussinesq approximation is made, where density is assumed constant (i.e., $\rho = \rho_0$, or incompressible) except where it appears with buoyancy. This eliminates the restoring force for acoustic/sound waves, thus eliminating these non-meteorological waves from the primitive equations. We must retain density variations with buoyancy, however, to obtain internal gravity waves.
- We assume that the salient wave dynamics can be represented by two-dimensional planar waves in the *x*-*z* direction. In general, the *x* direction can be taken to be any direction along which the waves propagate; i.e., internal gravity waves do not always propagate to the east or west.
- The relevant meteorological variables can be decomposed into base state and perturbation components ($u = \overline{u} + u'$, $w = w'$, $\rho = \overline{\rho}(z) + \rho'$, $p = \overline{p}(z) + p'$, and $\theta = \overline{\theta}(z) + \theta'$) to aid in *linearizing* the primitive equations. Note that the density decomposition applies only with buoyancy, while we assume no base-state vertical motion such that $w = w'$ only.

Given these simplifying assumptions, we can linearize the primitive equations: namely, the *x*- and *z*-direction momentum, continuity, and thermodynamic equations. Doing so, we obtain:

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\theta'}{\bar{\theta}}g$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \theta'}{\partial t} + \bar{u}\frac{\partial \theta'}{\partial x} + w'\frac{\partial \bar{\theta}}{\partial z} = 0$$

This set of four equations contains four unknowns: *u'*, *w'*, *p'*, and *θ'*. To solve for these variables, we collapse the system into a single equation for a single unknown by progressively rewriting each unknown in terms of one or more of the other variables. Most commonly, we collapse this system into a single equation for *w'*, from which we can obtain *u'*, then *p'* and *θ'*.

We assume that solutions for each unknown variable take a two-dimensional plane wave form:

$$f'(x,z,t) = \hat{f}\exp\left[i(kx + mz - \omega t)\right]$$

where *k* is the zonal wavenumber, *m* is the vertical wavenumber, and *ω* is the oscillation frequency. Note that:

$$k = \frac{2\pi}{\lambda_x} \qquad\qquad m = \frac{2\pi}{\lambda_z}$$

In other words, wavenumber is an inverse function of wavelength. Small wavenumbers imply long wavelengths, and vice versa. Note that $kx + mz - \omega t$ is defined as the wave phase ϕ.

If we combine the above equation set into a single equation for *w'*, we can obtain the *dispersion relation*, representing an expression for the oscillation frequency *ω*:

$$\omega = k\bar{u} \pm \frac{Nk}{\left(k^2 + m^2\right)^{1/2}}$$

Here, *N* is the *Brunt-Vaisala frequency* and is equal to the square root of the *static stability* N^2:

$$N^2 = \frac{g}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z}$$

Static stability is zero when the environmental lapse rate is dry adiabatic ($\bar{\theta}$ constant with height) and is a positive value when the environmental potential temperature increases with height. Thus,

ω depends on the base-state wind, horizontal and vertical wavelengths (through k and m), and the base-state static stability (through N).

The period of oscillation for any given wave is equal to:

$$\tau = \frac{2\pi}{\omega}$$

For an internal gravity wave, if $\omega \sim N$, which is somewhat justified from the dispersion relation, and assuming $\bar{\theta} = 300\,\text{K}$ and $\frac{\partial \bar{\theta}}{\partial z} = 6$ K km^{-1}, then τ = 448.8 s ~ 7.5 min. In other words, internal gravity waves are high frequency waves.

The *phase speed* of an internal gravity wave, describing the speed and direction of motion of wave phase (e.g., crests and troughs) is given by:

$$c_{px} = \frac{\omega}{k} = \bar{u} \pm \frac{N}{\left(k^2 + m^2\right)^{1/2}}$$

$$c_{pz} = \frac{\omega}{m} = \frac{k\bar{u}}{m} \pm \frac{kN}{m\left(k^2 + m^2\right)^{1/2}}$$

The *group velocity* of an internal gravity wave, describing the speed and direction of motion of the wave and its energy, is given by:

$$c_{gx} = \frac{\partial \omega}{\partial k} = \bar{u} \pm \frac{Nm^2}{\left(k^2 + m^2\right)^{3/2}}$$

$$c_{gz} = \frac{\partial \omega}{\partial m} = \pm \frac{-Nkm}{\left(k^2 + m^2\right)^{3/2}}$$

The phase speed is a function of wavenumber and does not equal the group velocity. Thus, internal gravity waves are dispersive; e.g., different wavelength waves comprising the full wave form have different propagation characteristics. In other words, a wave’s phase propagates in a different sense than does its energy.

If we assume two-dimensional plane wave solutions for all unknown variables (u', w', p', and θ'; e.g., $u' = \hat{u}\exp\left[i\left(kx + mz - \omega t\right)\right]$), substitute into the simplified primitive equation set, simplify, and solve for the unknowns in terms of each other, we obtain the *polarization relations*:

$$w' = -\frac{k}{m}u'$$

$$\theta' = \pm \frac{i\left(k^2 + m^2\right)^{1/2}}{Nm}\frac{\partial \bar{\theta}}{\partial z}u'$$

$$p' = \pm \frac{N\rho_0}{\left(k^2 + m^2\right)^{1/2}} u'$$

We can use these solutions and information about the phase speed and group velocity to infer the structure and propagation characteristics of internal gravity waves.

This discussion is based upon the diagram on the second slide of the lecture materials. This discussion is akin to but slightly different than that in the course textbook. The concepts here and in the course text are generalizable to any orientation of phase line so long as the appropriate signs of k and m are deduced and the interpretation changed to match.

Consider *phase lines*, or lines along which phase $kx + mz$ = constant, sloping up and to the right as depicted on the aforementioned diagram. Manipulating this equation, we find that:

$$z = -\frac{k}{m} x + \text{constant}$$

The slope of the phase lines is thus equal to $-k/m$. Given phase lines with positive slope, this means that k and m have opposite sign. How do we know which sign they have? Consider c_{px} and c_{pz}. If we consider the positive root for each and assume $N > 0$, then $c_{px} > 0$ and $c_{pz} < 0$. This describes a phase speed directed down and to the east. Recalling the definitions of k and m, this implies that k is positive (λ_x in the positive x-direction) and m is negative (λ_z in the negative z-direction) for the drawn phase lines.

We can show that parcel oscillations are *along* phase lines, with frequency given by the dispersion relation. We also can show that the phase speed vector is perpendicular to the phase lines. Consider its slope (e.g., rise over run):

$$\frac{c_{pz}}{c_{px}} = \frac{m}{k}$$

This is the negative of the inverse of the slope of the phase lines themselves.

We can also show that the group velocity is along the phase lines and that the group velocity vector is directed up and to the right. Assuming $N > 0$ and taking the positive root, $c_{gx} > 0$. For k and m of opposite sign, and again assuming $N > 0$ and taking the positive root, $c_{gz} > 0$. The group velocity vector's slope is given by:

$$\frac{c_{gz}}{c_{gx}} = -\frac{k}{m}$$

This is identical to the slope of the phase lines themselves. Together, these indicate a group velocity along the phase lines directed up and to the right. The waves propagate up and to the east (as given by the group velocity), while the individual crests and troughs along a wave propagate down and to the east (as given by the phase speed). In general, internal gravity waves propagate *upward and away from their sources*, where for this diagram the source can be envisioned to be at the origin.

Given k and m of opposite sign, we know that w' has equal sign to u'; i.e., eastward-moving parcels ascend while westward-moving parcels descend. For $N > 0$ and the positive root (for eastward-moving waves), p' has equal sign to u'; eastward-moving parcels have high perturbation pressure while westward-moving parcels have low perturbation pressure. Finally, given $m < 0$, $N > 0$, and base-state potential temperature increasing with height, θ' has opposite sign to and is 90° out of phase (due to the leading i) from u' or, equivalently, w'. This can be understood physically: an ascending (descending) air parcel is denser (less dense) than its surroundings. At its maximum upward (downward) displacement, which occurs when w' is equal to zero, the air parcel is colder (warmer) than its surroundings.

Ducted Mesoscale Gravity Waves

Gravity waves are ubiquitous atmospheric features, and the majority propagate up and away from their origins while steadily decreasing in amplitude.

Refraction and *reflection* occur as any wave approaches the interface between two distinct layers with different refractive indices. For atmospheric gases, refractive index is directly proportional to density. Thus, a simple application to the atmosphere is that of a wave approaching the interface between two layers of varying density; e.g., a layer near the ground with higher density beneath a layer with lower density. Radar beam refraction is an example of these concepts. Decreasing atmospheric density with height results in radar beam refraction. *Subrefraction* (*superrefraction*) occurs when density decreases less (more) rapidly with height than normal such that the change in refractive properties between layers is smaller (larger) than normal.

Upward-propagating internal gravity waves may also become trapped, or *ducted*, when the change in refractive properties across the interface between high and low density layers is large enough to result in downward wave refraction. We now wish to determine the conditions leading to gravity wave ducting, where wave refraction at the top and bottom of a higher-density layer results in trapped (i.e., primarily horizontally-propagating) waves. While wind discontinuities may result in ducting, the primary cause of ducting, and thus our focus here, is on ducting fostered by vertical density variations. For simplicity, we assume a gravity wave in the x-z direction, although gravity waves can and do propagate in any horizontal direction.

The *Richardson number* is a measure of the ratio of buoyancy forcing to vertical wind shear, i.e.,

$$Ri = \frac{N^2}{\left\| \frac{\partial \bar{u}}{\partial z} \right\|^2} = \frac{\frac{g}{\bar{\theta}} \frac{\partial \bar{\theta}}{\partial z}}{\left\| \frac{\partial \bar{u}}{\partial z} \right\|^2}$$

Here, N^2 is the static stability. The denominator is the squared magnitude of the zonal vertical wind shear. The Richardson number is small when vertical wind shear is large and large when the static stability is large (e.g., with a large increase in potential temperature with height). This Richardson number formally applies only in the case of subsaturated conditions; for saturated conditions, the static stability should be replaced by the moist static stability N_m^2, which is a function of θ_e.

Ducting potential is assessed by computing the Richardson number over the less dense layer; i.e., for the layer atop the actual ducting layer which has higher density. For lower tropospheric gravity waves, this implies that the ground is the lower bound on the ducting layer. Given this, the following insight can be obtained:

- For ***Ri* < 0.25**, the internal gravity wave is refracted in the ducting layer with extraction of energy from the mean flow. Nominally, this describes an upper layer with static stability approaching zero (i.e., lapse rate near dry adiabatic) and large vertical wind shear.
- For **0.25 < *Ri* < 2**, the internal gravity wave is refracted in the ducting layer with loss of energy to the mean flow, more so for larger *Ri*. Nominally, this describes an upper layer with somewhat greater static stability and/or reduced vertical wind shear when compared to the previous case.
- For ***Ri* > 2**, internal gravity wave refraction and ducting is unlikely; the wave rapidly loses energy to the mean flow.

We require that the gravity wave horizontal phase speed c_p exceed the base-state horizontal wind $\bar{u}$ at all levels within the ducted layer. In other words, a *critical level*, where $c_p = \bar{u}$, cannot exist within the ducted layer. Upon approaching a critical level, the restoring force (related to buoyancy) becomes infinitesimally small, resulting in parcel oscillations becoming increasingly horizontal and the gravity wave becoming part of the mean flow.

We know that any wave, including gravity waves, can be represented as the superposition of many waves of varying wavelength and amplitude. For ducted gravity waves, it can be shown that the wave with the *longest* wavelength is that which is most likely to be ducted. Thus, ducted gravity waves have wavelengths of tens to hundreds of kilometers, as compared to typical gravity waves with wavelengths one to two orders of magnitude smaller.

We can also show that the stable ducting layer has a minimum depth, given the longest wavelength wave, for ducted gravity waves to occur:

$$c_p - \bar{u} = \frac{2ND}{\pi}, \text{ or } D = \frac{\pi\left(c_p - \bar{u}\right)}{2N}$$

Here, Brunt-Vaisala frequency N is evaluated within the stable layer and the depth of the stable layer is given by D. The gravity wave's horizontal phase speed is given by c_p (assuming no vertical propagation) and the base-state horizontal wind is given by $\bar{u}$. The necessary depth is larger for smaller static stability within the stable layer and/or larger difference between the phase speed and the base-state horizontal wind.

Gravity Wave Characteristics and Environments

Thus, ducted gravity waves are most likely when (a) there is a layer with comparatively high stability and high density beneath a layer with comparatively low stability and low density, (b) the higher density layer is sufficiently thick to allow for ducting, and (c) the base-state horizontal wind speed is smaller than the gravity wave's phase speed throughout the higher density layer. In general, ducted gravity waves occur within stable layers of sufficient depth (and low base-state horizontal wind speed) that reside beneath layers with lapse rates that are nearly dry adiabatic. A frontal inversion, whether ahead of a retreating warm front or behind an advancing cold front, can serve as an effective ducting layer. Many mesoscale gravity waves have been documented when such environments are collocated with jet streaks, where the associated horizontal flow imbalance can trigger inertia-gravity waves (for which the Coriolis force is also important).

Ducted gravity waves are characterized by collocated cold/high and warm/low temperature and pressure anomalies. For an eastward-moving wave with no mean flow, convergence and ascent are maximized east (west) of the high (low) pressure anomaly. Divergence and descent are maximized west (east) of the high (low) pressure anomaly. Vertical motion is zero at the pressure anomalies. Maximum westerly (easterly) wind speed is found with the high (low) pressure anomaly. Pressure anomalies with ducted gravity waves are of magnitude 1-10 hPa. As noted previously, their horizontal wavelengths can be tens to hundreds of kilometers. We also quantified an approximate wave period of 7.5 min, which compares favorably to the periods of observed ducted gravity waves that range between 5-30 min. Given typical pressure anomaly magnitudes and wave periods, local surface pressure tendency can be of order 10 hPa h^{-1}.

There are many potential triggers for gravity waves, ducted or otherwise. The most common is the *geostrophic adjustment* process, wherein gravity waves act to restore geostrophic balance between the mass (or pressure) and wind fields. The wind field adjusts to the pressure field for large-scale disturbances, the pressure field adjusts to the wind field for small-scale disturbances, and the wind and pressure fields mutually adjust to each other for disturbances of intermediate scale. Gravity waves redistribute mass, modulate velocity, or both to restore geostrophic balance. Flow along or over sharply-sloping terrain and the release of shearing instability can also trigger gravity waves.

Thunderstorms, or deep, moist convection, can also result in gravity waves through two means. In the first, updrafts extending into the high-stability region of the tropopause can trigger buoyancy oscillations associated with gravity waves. These are depicted in satellite imagery as pulsating waves that move radially outward from the updraft. In the second, downdrafts and their associated density currents can force otherwise stable air upward over the even more stable density current, triggering a gravity wave-like buoyancy oscillation known as a *bore*. The dynamics of bores differ slightly from those of pure gravity waves, however. In conditionally unstable environments, lifting associated with the upward crests of ducted gravity waves may be sufficient to bring air to its LCL, if not also its LFC, and thus can also serve as a trigger for thunderstorm development.