Mesoscale Meteorology: Supercell Dynamics

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Overview

Supercell thunderstorms are long-lived single-cell thunderstorms, with longevities ranging from 1 to over 6 h. In contrast to single-cell thunderstorms, which have no appreciable mid-tropospheric rotation, supercell thunderstorms are characterized by a persistent mid-tropospheric *mesocyclone*, or meso-γ-scale (i.e., cell-scale; O(1-10 km)) area of large vertical vorticity ($O(10^{-2}\ s^{-1})$) collocated with the cell's updraft over a significant vertical depth within the middle troposphere. Supercell thunderstorms occur within environments of large lower-to-middle tropospheric vertical wind shear; generally, $\geq$ 20 m s^{-1} over the 0-6 km layer. As we will soon demonstrate, the horizontal vorticity associated with the vertically-sheared flow is the mesocyclone vertical vorticity source.

Large vertical wind shear magnitudes imply that supercells are favored in environments with long hodographs. Hodographs in supercell-supporting environments often have clockwise curvature (or veering winds with height) that favors cyclonic supercell mesocyclones. Counterclockwise-curved hodographs (or backing winds with height) favors anticyclonic supercell mesocyclones. Recalling that backing winds are associated with cold air advection through thermal wind, since supercell thunderstorms require large surface-based moisture and instability, anticyclonic mesocyclones are less common and often found at higher altitudes (where cold air advection is less deleterious to surface-based instability) than their cyclonic counterparts.

Whereas the motion of single-cell and, to lesser extent, multicell thunderstorms has a significant component associated with the mean wind over the lower-to-middle troposphere, supercell motion often deviates substantially from this mean wind vector. For clockwise-curved hodographs, storm motion typically lies to the right of the hodograph; the opposite is true for counterclockwise-curved hodographs. The relatively large depth ($\geq$ 6 km) of the vertical wind shear layer, plus the magnitude of vertical wind shear within this layer ($\geq$ 20 m s^{-1}), gives rise to significant storm-relative winds over much of the troposphere.

These storm-relative winds play two important roles in supercell maintenance and structure. First, the storm-relative winds modulate where precipitation falls out relative to the lower-tropospheric updraft. Recall that buoyancy is reduced by hydrometeor mass; if this hydrometeor mass is located away from the updraft, then precipitation does not significantly reduce buoyancy. In general, the middle-to-upper tropospheric storm-relative flow is downwind of the lower-tropospheric updraft. This is responsible for supercells appearing elongated downwind in radar reflectivity data.

Second, the storm-relative winds modulate how the gust front propagates outward relative to the lower-tropospheric updraft. For single-cell and multicell thunderstorms, gust front propagation is away from the updraft. For supercell thunderstorms, however, the updraft and gust front remain collocated, fostering quasi-persistent upward-directed parcel acceleration. The storm-relative flow is generally directed upward of the lower-tropospheric updraft, opposing the outward propagation of the gust front, thereby fostering the synergistic collocation of the gust front and updraft.

Supercell thunderstorms form in environments with surface-based CAPE values from 500-5000 J kg^{-1}. As supercell updrafts originate at the surface, supercell thunderstorms inherently are surface-based features. Supercell thunderstorms typically form in the local late afternoon and early evening hours in environments of minimal surface-based CIN. Synoptic-scale forcing for ascent in supercell environments is typically only large enough to lead to convection initiation if superposed with local forcing for ascent. If the synoptic-scale ascent is too weak, convection may not initiate. If the synoptic-scale ascent is too strong, too much convection may initiate, and the interaction between these thunderstorms and their cold pool leads to their upscale growth into an organized mesoscale complex. This can occur in the central United States as day becomes night and the nocturnal low-level jet develops, or in the presence of strong lifting along an advancing cold front.

Supercell Structure

Supercell thunderstorms are characterized by a single updraft that extends through the depth of the troposphere. Collocated with this updraft over a substantial fraction of its depth is a mesocyclone. Maximum updraft velocity typically exceeds 20 m s^{-1}, with velocities of up to 50 m s^{-1} possible in the most intense supercells in the most unstable environments (given that the maximum-theoretical updraft velocity is directly proportional to CAPE). A towering cumulonimbus cloud characterizes a supercell's visual appearance on the storm-scale whereas on the updraft-scale, the mesocyclone results in significant cloud curvature in proximity to the updraft. A wall cloud may form downwind of the updraft as rain-cooled and -humidified air is brought inward toward the updraft, ascending and reaching its LCL at a lower altitude than the comparatively less humid environmental air.

In radar reflectivity data, a supercell updraft is characterized by a bounded weak echo region, or a region where the updraft is sufficiently strong as to keep hydrometeors (i.e., radar targets) elevated. In the lower troposphere, the weak echo region may not be bounded on all sides by precipitation. The upwind precipitation echo bounding the updraft is responsible for the archetypal hook echo signature of supercell thunderstorms. In radar velocity data, mesocyclones are characterized by a couplet of adjacent inbound and outbound velocities. Dual-polarization data can be used to identify hail (collocated high Z, low CC, and negative to large Z_{DR}) or tornadoes (collocated strong rotation, low CC, and negative or small Z_{DR}) if present in conjunction with a given supercell thunderstorm.

Supercell thunderstorms are characterized by two primary updraft regions, one each in the forward and rearward directions. The forward-flank downdraft (FFD) results from downwind hydrometeor deposition by the storm-relative wind. Here, hydrometeor evaporation, melting, and sublimation result in latent cooling, the development of negative buoyancy, and thus downward-directed parcel accelerations. The intensity of the negative buoyancy, and thus cold pool strength (here defined as $T_{sfc}^{FFD} - T_{sfc}^{env}$), is a function of the environmental moisture profile as it influences both entrainment and evaporation potential. The outward spread of the gust front associated with the FFD is to some extent restrained by the upwind-directed lower tropospheric storm-relative flow.

The causes of the rear-flank downdraft (RFD) are less well-understood. One potential cause is the entrainment of relatively dry environmental air in the middle-to-upper troposphere. As in the FFD, this promotes latent cooling, negative buoyancy, and downward-directed parcel accelerations. If the hydrometeor mass becomes too large within a column to support it remaining suspended, its fallout may also result in a downdraft. Finally, a downward-directed vertical perturbation pressure gradient force on the rear flank of the supercell may also result in downdraft formation. Which is dominant likely varies between supercells. The resulting cold pool strength varies; latent cooling typically results in stronger cold pools than hydrometeor loading or pressure gradient forcing.

A supercell's lower-tropospheric updraft is sufficiently strong to lift both warm environmental air as well as rain-cooled (and thus stabilized) FFD air. In the latter case, mechanical lift is sufficiently large as to overcome the negative buoyancy generated by evaporation, melting, and sublimation. The near-surface inflow region of a supercell thunderstorm is characterized by a dynamic pressure minimum of 1-3 hPa known as an inflow low that straddles the FFD. This pressure minimum is associated with the strong inflow (and thus rearward-directed horizontal parcel accelerations) into the lower-tropospheric updraft. This connection can be demonstrated using a Bernoulli equation, which assumes steady-state (or irrotational) flow that is frictionless and not affected by the Coriolis force. Under these assumptions, the following expression holds along a surface ($z = 0$) streamline:

$$\frac{\rho v^2}{2} + p = \text{constant}$$

Here, v is the wind speed and not the meridional wind component. Assuming constant density, if v is large, p must be small (and vice versa). Thus, relatively fast inflow is associated with relatively low surface pressure. Differentiating this equation along a streamline (the s direction), we obtain:

$$\Delta p = p_{env} - p_{near-storm} = \frac{\rho}{2}\left(v_{env}^2 - v_{near-storm}^2\right)$$

For a wind speed increase from 5-20 m s^{-1} between the ambient and near-storm environments, and $\rho = 1$ kg m^{-3}, a surface pressure minimum of 1.88 hPa results, consistent with observations.

To this point, we have considered what can be called 'classic' supercells. Variants on this structure exist in the form of high-precipitation (HP) and low-precipitation (LP) supercells, distinguished by the favored location for precipitation fallout relative to the updraft. For LP supercells, most precipitation fallout occurs well downwind of the updraft, where it typically evaporates within the subsaturated sub-cloud layer before reaching the ground. The FFD tends to be weak, whereas the RFD may be absent. For HP supercells, most precipitation fallout occurs near the updraft within the hook echo and on the backside of the updraft, such that the updraft is often obscured by rain. LP supercells are favored when the upper-tropospheric storm-relative wind is relatively fast (> 30 m s^{-1}) and, to some extent, in drier ambient environments, whereas HP supercells are favored when the upper-tropospheric storm-relative wind is relatively slow (< 20 m s^{-1}) and when many storms exist close to one another.

Supercell Mesocyclone Development

To understand supercell mesocyclone formation, we use the vertical vorticity tendency equation. Just as vorticity is the curl of the velocity, vorticity tendency is the curl of the velocity tendency manifest in the form of the momentum equation. As we are interested in rotation in the *x-y* plane, we begin by considering only the component of the vorticity tendency in this plane:

$$\frac{\partial \zeta}{\partial t} \equiv \mathbf{k} \cdot \frac{\partial \vec{\omega}}{\partial t} = -\mathbf{v} \cdot \nabla(\zeta + f) + \vec{\omega} \cdot \nabla w + f \frac{\partial w}{\partial z} + \mathbf{k} \cdot (\nabla \times \mathbf{F})$$

The Boussinesq approximation was made to obtain this vertical vorticity tendency equation. The first right-hand side term represents advection, the second and third right-hand side terms represent the combined effects of tilting horizontal vorticity into the vertical and stretching vertical and planetary vorticity, and the last right-hand side term represents friction. On supercell scales, the Coriolis force is typically unimportant and frictional generation is typically small. Neglecting these forcings, we obtain:

$$\frac{\partial \zeta}{\partial t} = -\mathbf{v} \cdot \nabla \zeta + \vec{\omega} \cdot \nabla w = -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} - w \frac{\partial \zeta}{\partial z} + \xi \frac{\partial w}{\partial x} + \eta \frac{\partial w}{\partial y} + \zeta \frac{\partial w}{\partial z}$$

where $\vec{\omega} \equiv (\xi \mathbf{i} + \eta \mathbf{j} + \zeta \mathbf{k}) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$.

Substituting, we obtain:

$$\frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} - w \frac{\partial \zeta}{\partial z} + \frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The fourth and fifth right-hand side terms represent tilting horizontal vorticity into the vertical; the sixth right-hand side term represents stretching of vertical vorticity. The first three terms represent vertical vorticity advection in both horizontal and vertical directions. Advection serves to transport vertical vorticity; stretching amplifies pre-existing vertical vorticity; and tilting can generate new vertical vorticity from pre-existing horizontal vorticity (such as with the vertically-sheared flow).

To simplify our evaluation, we can develop a linearized form of this equation. Let $u = \bar{u}(z) + u'$, $v = \bar{v}(z) + v'$, and $w = w'$. Overbar terms represent the ambient environment; e.g., a horizontally-homogeneous vertically-sheared horizontal wind with zero vertical velocity. Prime terms represent storm-scale departures from the ambient environment. Note that given the definitions for *u* and *v*, we can show that:

$$\zeta = \frac{\partial}{\partial x}(\bar{v} + v') - \frac{\partial}{\partial y}(\bar{u} + u') = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \equiv \zeta'$$

since the overbar terms vary in the vertical direction only.

If we substitute these definitions into the vertical vorticity tendency equation, we obtain:

$$\frac{\partial \zeta}{\partial t} = -(\bar{u} + u')\frac{\partial \zeta'}{\partial x} - (\bar{v} + v')\frac{\partial \zeta'}{\partial y} - w'\frac{\partial \zeta'}{\partial z} + \frac{\partial w'}{\partial x}\left(\frac{\partial w'}{\partial y} - \frac{\partial(\bar{v} + v')}{\partial z}\right) + \frac{\partial w'}{\partial y}\left(\frac{\partial(\bar{u} + u')}{\partial z} - \frac{\partial w'}{\partial x}\right) + \zeta'\frac{\partial w'}{\partial z}$$

Neglecting the products of perturbation terms, this simplifies to:

$$\frac{\partial \zeta}{\partial t} = -\bar{u}\frac{\partial \zeta'}{\partial x} - \bar{v}\frac{\partial \zeta'}{\partial y} - \frac{\partial w'}{\partial x}\frac{\partial \bar{v}}{\partial z} + \frac{\partial w'}{\partial y}\frac{\partial \bar{u}}{\partial z}$$

The first two right-hand side terms represent horizontal advection. The second two right-hand side terms represent tilting of horizontal vorticity associated with the vertically-sheared flow into the vertical. Several terms, notably the stretching term, are not present as they are non-linear forcing terms. We will revisit the stretching term shortly. Combining the two advection terms, the above equation can be written as:

$$\frac{\partial \zeta}{\partial t} = -\bar{\mathbf{v}} \cdot \nabla_h \zeta' - \frac{\partial w'}{\partial x}\frac{\partial \bar{v}}{\partial z} + \frac{\partial w'}{\partial y}\frac{\partial \bar{u}}{\partial z}$$

If the updraft moves with constant velocity **c**, the storm-relative (updraft-relative) vertical vorticity tendency can be written as:

$$\left(\frac{\partial \zeta}{\partial t}\right)_{sr} = -\left(\bar{\mathbf{v}} - \mathbf{c}\right) \cdot \nabla_h \zeta' - \frac{\partial w'}{\partial x}\frac{\partial \bar{v}}{\partial z} + \frac{\partial w'}{\partial y}\frac{\partial \bar{u}}{\partial z}$$

Note that **c** only enters the advection term, and then only for the horizontal velocity. Subtracting a constant value across the entire domain (x, y, z) does not change the horizontal or vertical gradients of a given quantity. If one point initially has a value of 1 and the adjacent point initially has a value of 5, such that their difference is 4, subtracting 3 from each point does not change their difference: the points now have values of -2 and 2, the difference of which remains 4.

As noted before, advection cannot generate vertical vorticity. It acts only on ζ', such that absent storm-scale vertical vorticity (as is true in the ambient environment), the advection is zero. Tilting can generate vertical vorticity, however. Consider an environment of only westerly vertical wind shear, such that the tilting term simplifies to the last right-hand side term above. Next, consider an isolated updraft ($w' > 0$) embedded within this environment.

- North of the updraft, $\frac{\partial w'}{\partial y} < 0$, such that $\frac{\partial w'}{\partial y}\frac{\partial \bar{u}}{\partial z} < 0$, generating anticyclonic vorticity.
- South of the updraft, $\frac{\partial w'}{\partial y} > 0$, such that $\frac{\partial w'}{\partial y}\frac{\partial \bar{u}}{\partial z} > 0$, generating cyclonic vorticity.

The above example can be generalized to any vertical wind shear, unidirectional or otherwise, and to downdrafts. Since $w = 0$ at the ground (no vertical velocity through a rigid surface), the vorticity tendency holds for the lower to middle troposphere. Earlier in the semester, we demonstrated that the horizontal vorticity vector points 90° left of the vertical wind shear vector. Tilting generates localized vertical vorticity anomalies that lie *along* the horizontal vorticity vector, with both the horizontal vorticity vector and perturbation vertical vorticity gradient pointing from south to north in this example.

Once vertical vorticity has been generated by tilting, advection by the storm-relative wind becomes important. Given the relationship between the horizontal vorticity vector and perturbation vertical vorticity gradient that we just developed and the definition of the dot product, the following insight can be obtained:

- For storm-relative flow parallel to and along the horizontal vorticity vector, the storm-relative flow will advect the cyclonic vertical vorticity anomaly underneath the updraft.
- For storm-relative flow parallel to and opposing the horizontal vorticity vector, the storm-relative flow will advect the anticyclonic vertical vorticity anomaly underneath the updraft.
- For storm-relative flow perpendicular to the horizontal vorticity vector, the storm-relative flow will advect neither vertical vorticity anomaly underneath the updraft.

These are **streamwise vorticity**, **antistreamwise vorticity**, and **crosswise vorticity**, respectively. For storm motion along a straight hodograph, the horizontal vorticity is entirely crosswise. For storm motion off the hodograph and/or when the hodograph is curved, streamwise vorticity is non-zero. The greater the deviation of storm motion off the hodograph, the greater the streamwise vorticity; supercellular motion in a direction other than that given by the mean wind over the lower-to-middle troposphere increases the streamwise vorticity.

Here is where the non-linear stretching term becomes important. In its basic form, it represents the product of the vertical vorticity and the vertical variation in vertical velocity. Consider an updraft with zero vertical velocity at the ground and tropopause and maximum velocity at mid-levels. This vertical distribution for w is fairly common for deep, moist convection due to continuity and given a buoyancy maximum in the middle troposphere (since parcel vertical accelerations are directly proportional to buoyancy). For storm-relative flow advecting a cyclonic vertical vorticity anomaly underneath the updraft, stretching amplifies the cyclonic vertical vorticity throughout the lower to middle troposphere!

From this, it stands to follow that cyclonically-rotating supercell mesocyclones acquire cyclonic vertical vorticity first by tilting horizontal vorticity into the vertical plane, advecting it beneath the updraft, and then by amplifying it through updraft stretching. Note that this is *not* the same process that leads to tornadogenesis, or rotation at the ground, which is beyond the scope of this class. The requirement of tilting requires there to be ambient vertical wind shear, particularly in the lower to middle troposphere, and the requirement of tilting requires that the storm-relative flow be able to advect the tilted vorticity beneath the updraft. Single-cell convection, given weak ambient vertical wind shear, cannot generate vertical vorticity by tilting. Multicell convection, given the continual disruption of the lower tropospheric updraft by the advancing gust front, cannot continually advect the tilted vorticity beneath the updraft to be amplified via stretching. Only supercells, with large ambient vertical wind shear and restrained gust front advancement, can complete the process.

Given the importance of streamwise vorticity to supercell mesocyclone development, we desire to obtain a quantitative estimate of streamwise vorticity within the ambient environment. Generally, *helicity* is a measure of the degree to which fluid motion is aligned with the vorticity of the fluid, must as we demonstrated with the storm-relative advection term to the linearized vertical vorticity tendency equation. At a single level, helicity is given by:

$$H = \mathbf{v} \cdot \vec{\omega}$$

Or, vertically-integrated from the surface (recalling that supercells are surface-based) to a height *d* taken to represent the top of the storm inflow layer (typically 1-3 km above ground level),

$$H = \int_0^d \mathbf{v} \cdot \vec{\omega} dz$$

In the ambient environment, storm-scale perturbations are zero, such that *u* and *v* are functions of *z* only and *w* = 0. Thus,

$$\vec{\omega} = -\frac{\partial \bar{v}}{\partial z}\mathbf{i} + \frac{\partial \bar{u}}{\partial z}\mathbf{j} \equiv \overrightarrow{\omega_h}$$

and helicity becomes:

$$H = \int_0^d \bar{\mathbf{v}} \cdot \overrightarrow{\omega_h} dz$$

or, in the storm-relative sense,

$$H = \int_0^d \left(\bar{\mathbf{v}} - \mathbf{c}\right) \cdot \overrightarrow{\omega_h} dz$$

where the environmental horizontal vorticity is unchanged by subtracting the storm motion **c**.

Recall that the dot product between two vectors is zero when they are perpendicular, is a maximum positive value when they are parallel and in the same direction, and is a maximum negative value when they are parallel and in the opposite direction. Since streamwise vorticity is defined by the projection of the horizontal vorticity onto the storm-relative motion vector, the helicity above is a vertically-integrated measure of streamwise vorticity known as **storm-relative helicity**.

For an observed or estimated **c**, and since the ambient horizontal vorticity is only a function of the vertically-sheared flow, we can calculate the storm-relative helicity if we know the vertical profile of wind within the storm environment. On a hodograph, we can estimate the storm-relative helicity from the area between the curves cut out by the surface storm-relative motion, the storm-relative motion at the top of the inflow layer (e.g., 1 or 3 km), and the hodograph itself. This is perhaps the way by which the statement above regarding how deviant motion from the mean wind increases the streamwise vorticity can be best visualized. For a clockwise-curved hodograph, storm motion progressively further to the right of the mean wind increases the area cut out by the aforementioned curves, thus increasing the streamwise vorticity and storm-relative helicity.

Above, we emphasized the tilting of ambient horizontal vorticity associated with the vertically-shear flow into the vertical. However, baroclinically-generated horizontal vorticity associated with the FFD may also be important for supercell mesocyclone development. Air within the FFD is less buoyant than that to its south within the ambient environment, establishing a negative meridional gradient in buoyancy across the FFD gust front. Just as we developed a vertical vorticity tendency equation, so too can we develop a tendency equation for horizontal vorticity across the gust front:

$$\frac{\partial \xi}{\partial t} \equiv \mathbf{i} \cdot \frac{\partial \vec{\omega}}{\partial t} = -\mathbf{v} \cdot \nabla \xi + \vec{\omega} \cdot \nabla u + f\frac{\partial u}{\partial z} + \frac{\partial B}{\partial y} + \mathbf{i} \cdot \left(\nabla \times \mathbf{F}\right)$$

The right-hand side forcing terms represent advection, stretching and tilting of horizontal vorticity, tilting of planetary vorticity into the horizontal, baroclinic generation, and friction, respectively.

Neglecting all but the baroclinic generation term for simplicity, we obtain:

$$\frac{\partial \xi}{\partial t} \propto \frac{\partial B}{\partial y}$$

For the given meridional buoyancy distribution, ξ decreases (or becomes more negative) with time, implying a westward-directed horizontal vorticity vector from baroclinic generation along the gust front's leading edge. This horizontal vector represents northerly flow at the surface, ascent along the gust front's leading edge, and southerly flow aloft, or southerly vertical wind shear consistent with the horizontal vorticity vector.

Since storm-relative inflow is typically directed toward the updraft, which lies to the west of the FFD's leading edge, the storm-relative motion vector and horizontal vorticity vector associated with baroclinic generation are largely parallel and pointed in the same direction – e.g., streamwise! Given that a portion of supercell inflow originates within the rain-cooled and -humified air along the FFD (since the supercell updraft is typically strong enough to lift this air to its LFC despite its non-zero CIN), baroclinic generation of horizontal vorticity can be an important contributor to the tilting, advection, and stretching processes important for supercell mesocyclone development.

Supercell Propagation

As we will demonstrate, perturbation vertical pressure gradient forces found within supercells are responsible for their propagation characteristics, including splitting and deviant motion. To do so, we first need to develop the appropriate mathematical relationships. We start with the momentum equation:

$$\frac{d\mathbf{v}}{dt} = -2\vec{\Omega} \times \mathbf{v} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}$$

Neglecting the vertical component of the Coriolis force and a $2\Omega w \cos \phi$ term in the expansion of the -2**Ω** x **v** term, this term is equal to -f**k** x **v**, such that:

$$\frac{d\mathbf{v}}{dt} = -f\mathbf{k} \times \mathbf{v} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}$$

If we expand the total derivative, neglect friction, and note that **g** = g**k**, we obtain:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -f\mathbf{k} \times \mathbf{v} - \frac{1}{\rho}\nabla p + g\mathbf{k}$$

For a base-state that is in hydrostatic balance with $p = \overline{p}(z) + p'$, it can be shown that:

$$-\frac{1}{\rho}\nabla p + g\mathbf{k} = -\frac{1}{\rho}\nabla p' + B\mathbf{k}$$

The transformation of the horizontal pressure gradient terms is straightforward; $\overline{p}$ is not a function of *x* or *y*, such that partial derivatives in these directions are zero. The transformation of the vertical pressure gradient term follows from the derivation presented in the previous lecture notes. If we also make the Boussinesq approximation, then we obtain:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -f\mathbf{k} \times \mathbf{v} - \alpha_0 \nabla p' + B\mathbf{k}$$

Next, we wish to take the divergence, or $\nabla \cdot$, of this equation. Doing so, we obtain:

$$\frac{\partial (\nabla \cdot \mathbf{v})}{\partial t} + \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla \cdot (f\mathbf{k} \times \mathbf{v}) - \nabla \cdot (\alpha_0 \nabla p') + \nabla \cdot B\mathbf{k}$$

The Boussinesq approximation requires that $\nabla \cdot \mathbf{v} = 0$, such that the time tendency term is zero. The buoyancy term evaluates to $\frac{\partial B}{\partial z}$. Since α_0 is constant, the perturbation pressure gradient term evaluates to $-\alpha_0 \nabla^2 p'$, where the gradient operator dotted into a gradient results in the Laplacian. Thus, we obtain:

$$\alpha_0 \nabla^2 p' = -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) - \nabla \cdot (f\mathbf{k} \times \mathbf{v}) + \frac{\partial B}{\partial z}$$

Expansion of the first right-hand side term results in:

$$-\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = -\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial w}{\partial z}\right)^2 - 2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right)$$

Noting that some terms in the initial expansion of this term are zero because $\nabla \cdot \mathbf{v} = 0$.

Expansion of the second right-hand side term results in:

$$-\nabla \cdot (f\mathbf{k} \times \mathbf{v}) = -\nabla \cdot (-fv\mathbf{i} + fu\mathbf{j}) = -\frac{\partial}{\partial x}(-fv) - \frac{\partial}{\partial y}(fu) = f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - u\frac{\partial f}{\partial y} = f\zeta - u\beta$$

Thus, we obtain:

$$\alpha_0 \nabla^2 p' = -\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial w}{\partial z}\right)^2 - 2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right) + f\zeta - u\beta + \frac{\partial B}{\partial z}$$

We will neglect the two Coriolis terms $f\zeta$ and $u\beta$ for being small on the mesoscale. If we define a horizontally-homogeneous base-state with vertical wind shear that is in hydrostatic balance, such that $u = \overline{u}(z) + u'$, $v = \overline{v}(z) + v'$, and $w = w'$, and substitute into the above, we obtain:

$$\alpha_0 \nabla^2 p' = -\left(\frac{\partial u'}{\partial x}\right)^2 - \left(\frac{\partial v'}{\partial y}\right)^2 - \left(\frac{\partial w'}{\partial z}\right)^2 - 2\left(\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y} + \frac{\partial w'}{\partial x}\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial y}\frac{\partial v'}{\partial z}\right) - 2\left(\frac{\partial w'}{\partial x}\frac{\partial \overline{u}}{\partial z} + \frac{\partial w'}{\partial y}\frac{\partial \overline{v}}{\partial z}\right) + \frac{\partial B}{\partial z}$$

In the above are fluid extension, non-linear dynamical, linear dynamical, and buoyancy forcings.

We wish to simplify the non-linear dynamical forcing terms. First, note that the deformation tensor applied to the perturbation wind can be written as:

$$\begin{aligned}
\varepsilon_{ij}^2 &= \frac{1}{4}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)^2 \\
&= \frac{1}{2}\left(\left[\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right]+\left[\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right]+\left[\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right]+\left[\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right]\right) \\
&= \frac{1}{2}\left(\left[\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right]+\left[\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right]+\left[\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right]\right)
\end{aligned}$$

where the divergence is again zero for the Boussinesq approximation. The three sets of terms on the third line are deformation terms. If we assume that vertical vorticity (i.e., rotation) dominates over deformation within the supercell updraft, each of these terms are assumed to be zero. Thus,

$$\frac{\partial u'}{\partial y} = -\frac{\partial v'}{\partial x} \qquad \frac{\partial u'}{\partial z} = -\frac{\partial w'}{\partial x} \qquad \frac{\partial v'}{\partial z} = -\frac{\partial w'}{\partial y}$$

Further, if we assume that horizontal vorticity is zero within the supercell updraft, then:

$$\xi = \frac{\partial w'}{\partial y}-\frac{\partial v'}{\partial z} = 0 \Rightarrow \frac{\partial w'}{\partial y} = \frac{\partial v'}{\partial z} \qquad \text{and} \qquad \eta = \frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x} = 0 \Rightarrow \frac{\partial w'}{\partial x} = \frac{\partial u'}{\partial z}$$

Comparing these sets of relations, we find that the third in the first and the first in the second set imply relationships of opposing sign. The same is true for the second in the first and the second in the second set. The only way for these sets of equalities to be true is if each term in them is zero. Substituting this into our pressure perturbation equation, we obtain:

$$\alpha_0\nabla^2 p' = -\left(\frac{\partial u'}{\partial x}\right)^2 - \left(\frac{\partial v'}{\partial y}\right)^2 - \left(\frac{\partial w'}{\partial z}\right)^2 - 2\left(\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}\right) - 2\left(\frac{\partial w'}{\partial x}\frac{\partial \bar{u}}{\partial z}+\frac{\partial w'}{\partial y}\frac{\partial \bar{v}}{\partial z}\right)+\frac{\partial B}{\partial z}$$

The remaining non-linear term can be rewritten using the first relation in the first set above and a binomial expansion (via a clever algebraic rewrite of this forcing term):

$$\begin{aligned}
-2\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y} &= \frac{1}{2}\left(\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}-2\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}-\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial x}-2\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}-\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)^2 = \frac{1}{2}\zeta'^2
\end{aligned}$$

Substituting this and neglecting the fluid extension terms as comparatively small within supercell updrafts, we obtain:

$$\alpha_0 \nabla^2 p' = \frac{1}{2}\zeta'^2 - 2\left(\frac{\partial w'}{\partial x}\frac{\partial \bar{u}}{\partial z} + \frac{\partial w'}{\partial y}\frac{\partial \bar{v}}{\partial z}\right) + \frac{\partial B}{\partial z}$$

Noting that $\nabla^2 p' \propto -p'$, we obtain:

$$p' \propto -\frac{1}{2}\zeta'^2 + 2\left(\frac{\partial w'}{\partial x}\frac{\partial \bar{u}}{\partial z} + \frac{\partial w'}{\partial y}\frac{\partial \bar{v}}{\partial z}\right) - \frac{\partial B}{\partial z}$$

The right-hand side terms here represent non-linear dynamical forcing associated with spin, linear dynamical forcing associated with the environmental vertical wind shear, and buoyancy forcing. In non-supercell environments, the buoyancy term forcing dominates. In supercell environments, however, the dynamical forcing terms are of comparable magnitude to the buoyancy forcing term. Here, we are chiefly interested in the dynamical forcing for vertical parcel accelerations, such that we will neglect the buoyancy term.

The non-linear forcing term is responsible for splitting supercells that move to the left and right of the mean wind. The impact of the linear forcing term varies depending upon hodograph curvature:

- **Clockwise**: favors the right-splitting supercell; the left-splitting supercell decays.
- **Counterclockwise**: favors the left-splitting supercell; the right-splitting supercell decays.
- **Straight**: neither the right- or left-splitting supercell is favored over the other.

Let us consider the non-linear forcing term first. For simplicity, consider a westerly environmental vertical wind shear with purely zonal flow, such that the hodograph is straight, the vertical wind shear vector is pointed to the east, and the horizontal vorticity vector is pointed to the north. (The insight below holds for any straight hodograph, not just one with only zonal flow.) Via tilting, the supercell updraft results in the development of cyclonic perturbation vertical vorticity to the south and anticyclonic perturbation vertical vorticity to the north of the updraft maximized at mid-levels.

Since $p' \propto -0.5\zeta'^2$, both vertical vorticity anomalies are associated with $p' < 0$, with largest values found in the middle troposphere. This establishes upward-directed perturbation pressure gradient forces north and south of the main supercell updraft that promote supercell splitting. Splitting is accelerated when a hydrometeor- or evaporation-induced downdraft is found in proximity to the initial supercell updraft. Splitting typically occurs within 0.5-1 h of supercell initiation and can occur multiple times during a supercell's lifecycle.

Splitting results in distinct updrafts that move to the left and right of the mean wind toward the vertical vorticity anomalies. Note that this is distinct from the advection of the vertical vorticity anomalies beneath the updraft that we said was important for supercell mesocyclone development; indeed, for the straight hodograph assumed in this example with storm motion along the mean wind vector, horizontal vorticity was initially crosswise. However, deviant storm motion fostered by splitting results in horizontal vorticity no longer being only crosswise: there is a streamwise component for the right-mover and an antistreamwise component for the left-mover.

The two new, distinct updrafts themselves then tilt environmental horizontal vorticity (which is still predominantly crosswise in this example) into the vertical. The resulting vertical perturbation pressure gradient force promotes the splitting of each of these cells. The left-split from the original left-split and right-split from the original right-split promote continued deviant propagation, while the right-split from the original left-split and the left-split from the original right-split may interact with each other in either constructive (e.g., merger and upscale growth) or destructive ways.

What about the case where the environmental hodograph is curved and only streamwise vorticity is present? Here, as the original supercell updraft tilts the horizontal vorticity into the vertical, the cyclonic vertical vorticity anomaly is assumed to near-instantaneously be advected by the storm-relative motion underneath the updraft, wherein stretching amplifies it. This results in an upward-directed perturbation pressure gradient force that is collocated with the original updraft, reinforcing the updraft rather than resulting in storm splitting. Consequently, the ability for a supercell to split is directly proportional to the fraction of environmental horizontal vorticity that is crosswise.

Now, let us consider the linear forcing term. For simplicity, consider a westerly environmental vertical wind shear with purely zonal flow, such that the hodograph is straight and the vertical wind shear vector is pointed to the east. Thus, the linear forcing term in this example simplifies to:

$$p' \propto \frac{\partial w'}{\partial x}\frac{\partial \bar{u}}{\partial z}$$

At the surface, $w' = 0$, such that horizontal gradients of w' are also zero. In the middle troposphere, $w' >> 0$, such that $p' \propto \frac{\partial w'}{\partial x}\frac{\partial \bar{u}}{\partial z} > 0$ west and $p' \propto \frac{\partial w'}{\partial x}\frac{\partial \bar{u}}{\partial z} < 0$ east of the updraft.

Therefore, p_d' decreases with height to the east of the updraft and increases with height to the west of the updraft. This establishes a downward-directed perturbation pressure gradient force west and an upward-directed perturbation pressure gradient force east of the updraft. This also establishes a horizontal perturbation pressure gradient force directed toward the east in the middle troposphere. For supercell propagation along the mean westerly wind, the perturbation pressure gradient forces do not impact supercell motion, nor promote the left- or right-splitting cells that may form due to non-linear forcing.

What about if the environmental hodograph is curved? Consider the case of easterly surface winds veering to southerly and then westerly through the lower to middle troposphere. For simplicity, we will also assume that wind speed remains constant with height. This defines a clockwise-curved hodograph that takes the shape of a semicircle centered on the origin. In this case, the local vertical wind shear vector is southerly near the surface, westerly in the lower troposphere, and northerly in the middle troposphere. For storm motion equal to the mean wind over this layer (e.g., slightly to the north), the horizontal vorticity is predominantly but not entirely streamwise.

The full linear forcing term must be considered in this evaluation; recall that it takes the form:

$$p' \propto 2\left(\frac{\partial w'}{\partial x}\frac{\partial \bar{u}}{\partial z} + \frac{\partial w'}{\partial y}\frac{\partial \bar{v}}{\partial z}\right)$$

Near the surface, $p' > 0$ south and $p' < 0$ north of the updraft. In the lower troposphere, $p' > 0$ west and $p' < 0$ east of the updraft. Finally, in the middle troposphere, $p' < 0$ south and $p' > 0$ north of the updraft. The combination of the near-surface and middle-tropospheric pressure anomalies sets up perturbation vertical pressure gradients: upward to the south and downward to the north of the updraft.

Recalling that this hodograph contains a small amount of crosswise vorticity, splitting is likely. As a result, the linear dynamic term promotes right-split maintenance, with additive upward-directed perturbation pressure gradient forces. The canceling perturbation vertical pressure gradient forces promote the decay of the left-split. Deviant motion to the right of the mean-wind by the right-split decreases the amount of crosswise vorticity, such that splitting becomes less likely; however, the linear dynamical forcing still acts, resulting in a single, non-splitting supercell that propagates to the right of the mean wind. Thus, for a clockwise-curved hodograph, the linear dynamical forcing term results in deviant motion to the right of the mean wind.

What about the case of a counterclockwise-curved hodograph? For supercell environments, such a hodograph is typically found in the lower-to-middle troposphere, with southwesterly winds atop the boundary layer that veer to westerly in the middle troposphere before backing to southwesterly at higher altitudes. Thus, the local environmental vertical wind shear vector is northerly atop the boundary layer, westerly in the middle troposphere, and southerly at higher altitudes. From the full linear forcing term, $p' < 0$ south and $p' > 0$ north of the updraft atop the boundary layer, whereas $p' > 0$ south and $p' < 0$ north of the updraft at higher altitudes. This establishes perturbation vertical pressure gradients as well: downward to the south and upward to the north of the updraft. This can promote the maintenance of a left-split and the decay of a right-split. However, as such hodographs also typically have strong clockwise curvature in the lower troposphere, there is forcing for both left- and right-splits to be maintained in this scenario. This often leads to messy storm evolutions.