Synoptic Meteorology II: The Quasi-Geostrophic Height Tendency Equation

Readings: Sections 2.4 and 2.5 of *Midlatitude Synoptic Meteorology*.

The Quasi-Geostrophic Vorticity/Thermodynamic Equation System

Including friction, the quasi-geostrophic vorticity equation is given by:

$$\frac{\partial \zeta_g}{\partial t} = -\vec{\mathbf{v}}_g \cdot \nabla \zeta_g - \beta v_g + f_0 \frac{\partial \omega}{\partial p} - K\zeta_g \tag{1}$$

Note that *K*, the frictional coefficient, is a function of pressure; it is highest near the surface and is positive-definite.

Recall that the geostrophic relative vorticity ζ_g can be written in terms of the geopotential, i.e.,

$$\zeta_g = \frac{1}{f_0} \nabla_p^2 \Phi \tag{2}$$

Substituting (2) into the left-hand side of (1), commuting the partial derivatives, and multiplying both sides by f_0, we obtain:

$$\nabla^2 \left(\frac{\partial \Phi}{\partial t} \right) = f_0 \left(-\vec{\mathbf{v}}_g \cdot \nabla \zeta_g \right) - f_0 \beta v_g + f_0^2 \frac{\partial \omega}{\partial p} - f_0 K \zeta_g \tag{3}$$

Because $\beta = df/dy$, the first two terms on the right-hand side of (3) can be combined into a single term, i.e.,

$$f_0 \left(-\vec{\mathbf{v}}_g \cdot \nabla \zeta_g \right) - f_0 \beta v_g = f_0 \left(-\vec{\mathbf{v}}_g \cdot \nabla \left(\zeta_g + f \right) \right) \tag{4}$$

This works because *f* does not vary in the *x*-direction – i.e., $-f_0 u_g \, \partial f / \partial x = 0$.

Substituting (4) into (3), we obtain:

$$\nabla^2 \left(\frac{\partial \Phi}{\partial t} \right) = f_0 \left(-\vec{\mathbf{v}}_g \cdot \nabla \left(\zeta_g + f \right) \right) + f_0^2 \frac{\partial \omega}{\partial p} - f_0 K \zeta_g \tag{5}$$

Likewise, recall that the quasi-geostrophic thermodynamic equation is given by the following:

$$\frac{\partial T}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla T - S_p \omega = \frac{1}{c_p} \frac{dQ}{dt} \tag{6}$$

S_p is the static stability and equals $\sigma p/R$. We desire to rewrite (6) in terms of potential temperature rather than temperature. To do so, we make use of Poisson's relationship,

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \tag{7}$$

It can be shown that a function *h* can be defined as:

$$h(p) = \frac{R}{p_0}\left(\frac{p_0}{p}\right)^{\frac{c_v}{c_p}} \tag{8}$$

Plugging (7) and (8) into (6), the quasi-geostrophic thermodynamic equation becomes:

$$\frac{\partial(h\theta)}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla(h\theta) - \sigma\omega = \frac{R}{pc_p}\frac{dQ}{dt} \tag{9}$$

where

$$\sigma = -h\frac{d\theta_0}{dp} \tag{10}$$

Likewise, plugging (7) and (8) into the hydrostatic relationship, we obtain:

$$\frac{\partial\Phi}{\partial p} = -h\theta \tag{11}$$

If we plug (11) into the left-hand side of (9) and commute the partial derivatives, we obtain:

$$-\frac{\partial}{\partial p}\left(\frac{\partial\Phi}{\partial t}\right) + \vec{\mathbf{v}}_g \cdot \nabla(h\theta) - \sigma\omega = \frac{R}{pc_p}\frac{dQ}{dt} \tag{12}$$

Rearranging (12) to leave only the term involving Φ on the left-hand side, we obtain:

$$-\frac{\partial}{\partial p}\left(\frac{\partial\Phi}{\partial t}\right) = -\vec{\mathbf{v}}_g \cdot \nabla(h\theta) + \sigma\omega + \frac{R}{pc_p}\frac{dQ}{dt} \tag{13}$$

The quasi-geostrophic vorticity (5) and thermodynamic (13) equations represent two equations containing two unknowns – vertical motion ω and geopotential height Φ. While variables such as the geostrophic relative vorticity ζ_g and potential temperature θ are also present in (5) and (13), we can diagnose these variables using (2) for ζ_g and (11) for θ once we know ω and Φ.

To obtain the quasi-geostrophic height tendency equation, we wish to combine (5) and (13) in a way that eliminates ω, leaving a single equation for Φ that describes how the geopotential field on an isobaric surface evolves in time. We will follow a similar procedure when obtaining the quasi-geostrophic omega equation next lecture, except eliminating Φ between the two equations to leave a single equation for ω.

The Quasi-Geostrophic Height Tendency Equation

First, take $-f_0^2 \frac{\partial}{\partial p}\frac{1}{\sigma}$ of the quasi-geostrophic thermodynamic equation (13):

$$f_0^2 \frac{\partial}{\partial p}\left[\frac{1}{\sigma}\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right)\right] = -f_0^2 \frac{\partial}{\partial p}\left[\frac{h}{\sigma}\left(-\vec{\mathbf{v}}_g \cdot \nabla \theta\right)\right] - f_0^2 \frac{\partial \omega}{\partial p} - \frac{f_0^2 R}{c_p}\frac{\partial}{\partial p}\left[\frac{1}{\sigma p}\frac{dQ}{dt}\right] \qquad (14)$$

If we then add (5) and (14), the terms involving $\partial\omega/\partial p$ cancel out, leaving only one equation for the time tendency of the geopotential height. Doing so, we obtain the *quasi-geostrophic height tendency equation*, as given by:

$$\nabla^2 \frac{\partial \Phi}{\partial t} + f_0^2 \frac{\partial}{\partial p}\left[\frac{1}{\sigma}\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right)\right] = f_0\left(-\vec{\mathbf{v}}_g \cdot \nabla\left(\zeta_g + f\right)\right) - f_0^2 \frac{\partial}{\partial p}\left[\frac{h}{\sigma}\left(-\vec{\mathbf{v}}_g \cdot \nabla \theta\right)\right] - f_0 K \zeta_g - \frac{f_0^2 R}{c_p}\frac{\partial}{\partial p}\left(\frac{1}{\sigma p}\frac{dQ}{dt}\right) \qquad (15)$$

Equation (15) is a partial differential equation describing the local change of the geopotential height Φ on an isobaric surface with respect to time. There are four forcing terms on the right-hand side of (15): geostrophic vorticity advection, differential potential temperature advection, friction, and differential diabatic heating.

Note that this equation is applied to the study of troughs and ridges in the middle troposphere – often at 500 hPa – and not at the surface. In later lectures, we will discuss appropriate frameworks for studying the evolution of surface cyclones and anticyclones.

If we make the substitution that $\chi = \partial\Phi/\partial t$, then (15) becomes:

$$\nabla^2\chi + f_0^2\frac{\partial}{\partial p}\left[\frac{1}{\sigma}\frac{\partial\chi}{\partial p}\right] = f_0\left(-\vec{\mathbf{v}}_g\cdot\nabla\left(\zeta_g + f\right)\right) - f_0^2\frac{\partial}{\partial p}\left[\frac{h}{\sigma}\left(-\vec{\mathbf{v}}_g\cdot\nabla\theta\right)\right] - f_0K\zeta_g - \frac{f_0^2R}{c_p}\frac{\partial}{\partial p}\left(\frac{1}{\sigma p}\frac{dQ}{dt}\right) \tag{16}$$

The left-hand side of (16) expresses χ in terms of the Laplacian (∇^2) and a second partial derivative with respect to pressure. Because the second partial derivative of a local maximum is negative and that of a local minimum is positive, the left-hand side of (16) can be approximated as:

$$\nabla^2\chi + f_0^2\frac{\partial}{\partial p}\left[\frac{1}{\sigma}\frac{\partial\chi}{\partial p}\right] \propto -\chi \tag{17}$$

The $\propto$ symbol means “is proportional to,” such that the left-hand side of (16) is proportional to $-\chi$. Therefore, where the right-hand side of (16) is positive, χ is negative, implying a local decrease in the geopotential height with time on a given isobaric surface. Likewise, where the right-hand side of (16) is negative, χ is positive, implying a local increase in the geopotential height with time on a given isobaric surface.

Interpretation of the Quasi-Geostrophic Height Tendency Equation

We now wish to interpret the contributions to the local geopotential height tendency from each of the four forcing terms on the right-hand side of (16).

Geostrophic Vorticity Advection

The contribution to the local geopotential height tendency exclusively due to geostrophic vorticity advection can be expressed by:

$$\chi \propto -f_0\left(-\vec{\mathbf{v}}_g\cdot\nabla\left(\zeta_g + f\right)\right) \tag{18}$$

The term inside of the outermost set of parentheses is an advection term. It depicts the advection *by the geostrophic wind* of the geostrophic relative (ζ_g) and planetary (f) vorticity.

Cyclonic geostrophic vorticity advection is defined by $-\vec{\mathbf{v}}_g\cdot\nabla\left(\zeta_g + f\right) > 0$. Because f_0 is positive in the Northern Hemisphere, cyclonic geostrophic vorticity advection on an isobaric surface results in $\chi < 0$, implying a *local* decrease in geopotential height with time.

Likewise, anticyclonic geostrophic vorticity advection is defined by $-\vec{\mathbf{v}}_g \cdot \nabla(\zeta_g + f) < 0$. Thus, anticyclonic geostrophic vorticity advection on an isobaric surface results in $\chi > 0$, implying a *local* increase in geopotential height with time.

In the idealized scenario in Fig. 1, geostrophic relative vorticity is maximized (minimized) in the base (apex) of each trough (ridge). The westerly geostrophic wind in the base of the trough results in height falls to the east (cyclonic advection) and height rises to the west (anticyclonic advection). Conversely, the westerly geostrophic wind in the apex of the ridge results in height rises to the east (anticyclonic advection) and height falls to the west (cyclonic advection). In this sense, geostrophic vorticity advection does not change trough/ridge amplitude in this example; rather, it only results in the movement of the trough/ridge pattern. This interpretation is identical to that offered with the advection terms in the quasi-geostrophic vorticity equation.

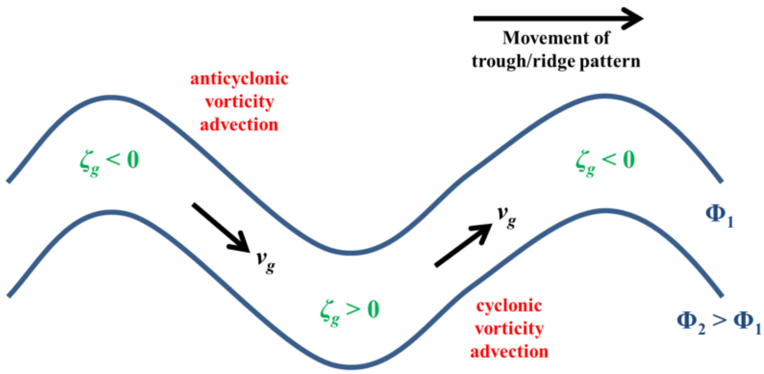

Figure 1. Conceptual view of the movement of middle-tropospheric ridges and troughs from the perspective of the geostrophic vorticity advection forcing term.

However, the geopotential height field – and thus the geostrophic wind and geostrophic relative vorticity – are rarely as simple as in the idealized example above. Section 2.4 of the Lackmann text, particularly Figs. 2.13-2.15 (pgs. 51-53), illustrate two examples in which the geostrophic vorticity advection term *can* result in the amplification or deamplification of troughs and ridges. In these examples, isohypse packing – and thus the magnitude of the geostrophic wind – is not uniform between the upstream and downstream sides of a trough or ridge. As Lackmann notes, however, other effects are often important, and in general the geostrophic vorticity advection term is chiefly responsible for trough and ridge motion rather than amplification.

Differential Potential-Temperature Advection

The contribution to the local geopotential height tendency exclusively due to differential potential-temperature advection can be expressed by:

$$\chi \propto f_0^2 \frac{\partial}{\partial p}\left[\frac{h}{\sigma}\left(-\vec{\mathbf{v}}_g \cdot \nabla\theta\right)\right] \tag{19}$$

The term inside the parentheses is the geostrophic advection of potential temperature. However, it is encapsulated within a partial derivative with respect to pressure – a vertical derivative – thus giving rise to the name "differential potential temperature advection." Thus, the local geopotential height tendency on the isobaric level on which the quasi-geostrophic height tendency equation is applied depends on the vertical structure of the geostrophic advection of potential temperature. In general, this requires evaluating potential-temperature advection at isobaric levels both above and below the level on which the equation is applied (e.g., we evaluate potential-temperature advection at 700 hPa and 300 hPa to diagnose the 500 hPa height tendency).

However, note the *h* inside the partial derivative. As given in (8), *h* is inversely related to pressure. This acts as a scaling parameter: it reduces the influence of the potential-temperature advection in the lower troposphere more than it does in the upper troposphere. For 700 hPa and 300 hPa, the potential-temperature advection is reduced by over twice as much at 700 hPa than it is at 300 hPa! Thus, though this term is a *differential* advection term, oftentimes it is dominated by the uppermost of the two isobaric surfaces on which potential-temperature advection is evaluated. Here, however, we consider the idealized case of equal contributions from the lower and upper isobaric surfaces on which we evaluate potential-temperature advection.

Before proceeding, note that warm (or, more accurately, positive) potential-temperature advection is given by $-\vec{\mathbf{v}}_g \cdot \nabla\theta > 0$. Conversely, cold (or, more accurately, negative) potential-temperature advection is given by $-\vec{\mathbf{v}}_g \cdot \nabla\theta < 0$. Since the geostrophic vorticity advection term generally only influences trough/ridge motion, the differential potential-temperature advection term must be non-zero for the amplitude of a middle-tropospheric trough or ridge to change.

We first consider the case where potential-temperature advection becomes more positive with increasing height (or, more accurately, decreasing pressure). This corresponds to where (a) warm advection increases in magnitude upward (b) cold advection decreases in magnitude upward. The change in potential-temperature advection is positive. The denominator, representing the change in pressure over this layer, is negative. From (19), this implies $\chi < 0$, signifying a local decrease in the geopotential height in time *on the isobaric surface on which the equation is applied*.

Likewise, consider the case where potential-temperature advection becomes more negative with increasing height (or, more accurately, decreasing pressure). This corresponds to where (c) warm advection decreases in magnitude upward or (d) cold advection increases in magnitude upward. The change in potential-temperature advection is negative. The denominator, representing the change in pressure over this layer, is negative. From (19), this implies $\chi > 0$, signifying a local increase in the geopotential height in time *on the isobaric surface on which the equation is applied*.

This term can be interpreted utilizing thickness arguments, where we recall that the thickness of a vertical layer is directly proportional to the temperature of that layer. Consider first the case where warm potential-temperature advection decreases in magnitude upward. Here, warm advection is largest *below* the isobaric surface on which we evaluate the height tendency, and thus the greatest increase in thickness due to this vertical structure of warm potential-temperature advection is in the layer *below* the isobaric surface on which we evaluate height tendency. Increasing the thickness of this layer forces the isobaric surface on which the height tendency is evaluated upward to a higher altitude. This results in a local *increase* in geopotential height on this isobaric surface with time, in agreement with our previous interpretation. This is illustrated in the left half of Fig. 2.

Conversely, consider the case where cold potential-temperature advection decreases in magnitude upward. In this case, cold advection is maximized *below* the isobaric surface on which we evaluate the height tendency. The greatest decrease in thickness due to this structure of cold potential-temperature advection is *below* the isobaric surface on which we evaluate the height tendency. Decreasing the thickness of this layer forces the isobaric level on which the height tendency is evaluated downward to a lower altitude. This results in a local *decrease* in geopotential height on this isobaric surface with time, in agreement with our previous interpretation. This is illustrated in the right half of Fig. 2.

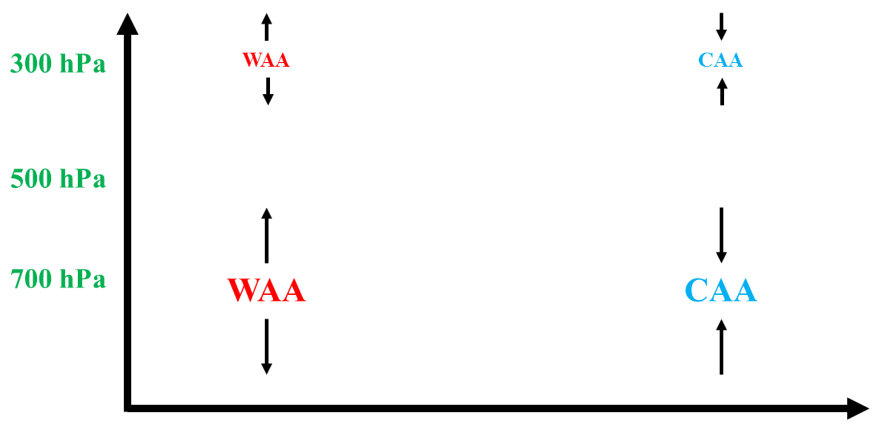

Figure 2. Idealized schematic of the thickness-based interpretation of the effects of (left) warm potential-temperature advection (or WAA) decreasing in magnitude upward and (right) cold potential-temperature advection (or CAA) decreasing in magnitude upward on height tendency at 500 hPa. The black arrows indicate the influence of potential temperature advection upon the local thickness, with larger arrows denoting a larger change in thickness (of either sign); arrows pointing in opposite directions indicating increasing thickness, and arrows pointing toward each other indicating decreasing thickness. At left, the 500 hPa height increases more due to the increased thickness below than it decreases due to the increased thickness above, and thus height tendency is positive. At right, the 500 hPa height decreases more due to the decreased thickness below than it decreases due to the decreased thickness above, and thus height tendency is negative.

Friction

The contribution to the local geopotential height tendency from friction is given by:

$$\chi \propto f_0 K \zeta_g \qquad (20)$$

In (20), K represents the effects of friction and is positive. K is non-zero only within the boundary layer, or close to the surface, where the frictional effects of the land-surface can be meaningfully communicated to the troposphere; it varies as a function of the land-surface's characteristics.

Because K is positive, χ has the same sign as ζ_g. Thus, in the base of a trough (minimum Φ) with cyclonic geostrophic relative vorticity ($\zeta_g > 0$ in the Northern Hemisphere), χ is positive, implying a local increase in geopotential height with time. Conversely, at the apex of a ridge (maximum Φ) with anticyclonic geostrophic relative vorticity ($\zeta_g < 0$ in the Northern Hemisphere), χ is negative, implying a local decrease in geopotential height with time.

Thus, in the lower troposphere where friction is important, friction acts as a 'brake' on the intensity of both lower-tropospheric troughs (cyclones) and ridges (anticyclones). However, in the middle troposphere, where the quasi-geostrophic height tendency equation is typically applied, friction is often negligible and thus has no direct effect on height tendency.

Differential Diabatic Heating

The contribution to the local geopotential height tendency exclusively due to differential diabatic heating can be expressed by:

$$\chi \propto \frac{f_0^2 R}{c_p} \frac{\partial}{\partial p}\left(\frac{1}{\sigma p}\frac{dQ}{dt}\right) \qquad (21)$$

Here, dQ/dt is the diabatic heating rate. Diabatic warming refers to the situation where $dQ/dt > 0$, while diabatic cooling refers to the situation where $dQ/dt < 0$. This term is non-zero only in the presence of diabatic heating, such as from radiation and latent heat release. On the synoptic-scale where motions are largely adiabatic in nature and the atmosphere is unsaturated, this term is often neglected.

Like with the differential potential-temperature advection term, diabatic heating is encapsulated within a partial derivative with respect to pressure. Thus, the local geopotential height tendency on the isobaric level on which the quasi-geostrophic height tendency equation is applied depends upon the vertical structure of diabatic heating. As for differential potential-temperature advection, there is a scaling parameter of $1/p$ here that reduces the influence of lower-tropospheric diabatic heating relative to that aloft; however, below, we consider the idealized case of equal contributions from the lower and upper isobaric surfaces on which we evaluate potential-temperature advection.

We first consider the case where diabatic warming ($dQ/dt > 0$) increases in magnitude upward (or diabatic cooling decreases in magnitude upward). This leads to a positive numerator on the right-hand side of (21). Since the change in pressure is negative, $\chi < 0$, resulting in a local decrease in geopotential height with time. Conversely, if diabatic warming decreases in magnitude upward or diabatic cooling increases in magnitude upward, the numerator on the right-hand side of (21) is negative. Since the change in pressure is negative, $\chi > 0$, resulting in a local increase in geopotential height with time.

As with the differential potential-temperature advection forcing term, thickness arguments can be utilized to interpret the differential diabatic-heating term as well. The arguments are identical to those posed above, albeit with "diabatic heating" replacing "potential temperature advection."

A Digression: Why Another Equation?

Because the geopotential height is linked to the geostrophic relative vorticity, we can use the quasi-geostrophic vorticity equation to describe the motion and evolution of the middle tropospheric mid-latitude trough/ridge pattern. Therefore, it is fruitful to ask: why do we need another equation to also describe the motion and evolution of the middle tropospheric pattern, especially if the first forcing term of each (geostrophic relative vorticity advection) is practically identical between the two equations?

The quasi-geostrophic height tendency equation provides an alternate means of expressing the $\partial\omega/\partial p$ term within the quasi-geostrophic vorticity equation. Rather than seeking to evaluate $\partial\omega/\partial p$ to describe the evolution of the mid-latitude trough/ridge pattern, however, we seek to evaluate differential potential-temperature advection and differential diabatic heating. As we will see when we derive the quasi-geostrophic omega equation, two of the forcings resulting in vertical motion are potential-temperature advection and diabatic heating on a given isobaric surface. Thus, if these fields are related to ω, it stands to follow that the derivative of each with respect to p (the "differential" part in the quasi-geostrophic height tendency equation) is related to $\partial\omega/\partial p$! In other words, the quasi-geostrophic height tendency equation incorporates the underlying physical forces that lead to changes in the geopotential height.

Additionally, when we described the quasi-geostrophic vorticity equation, we neglected friction. Conversely, when describing the quasi-geostrophic height tendency equation, we included friction. As a result, the quasi-geostrophic height tendency equation gives us a more complete view of the physical processes contributing to the evolution of the midlatitude, synoptic-scale trough/ridge pattern. Furthermore, given the above, it is entirely consistent with the quasi-geostrophic vorticity equation! These explain why the quasi-geostrophic height tendency equation is used to describe the motion and evolution of troughs and ridges rather than the quasi-geostrophic vorticity equation.

Evaluation of the Quasi-Geostrophic Height Tendency Equation

The quasi-geostrophic height tendency equation contains a partial derivative of the geopotential height with respect to time. As a result, this equation may be used to *predict* the evolution of the geopotential height field on a given isobaric surface. However, this is often not done. Why is this the case? The evolution of the geopotential height field with respect to time, as given by χ, depends upon the second derivative of χ with respect to x and y (as manifest through the Laplacian operator) as well as p. In other words, the local geopotential height tendency depends upon its value at adjacent locations in the horizontal and vertical. Thus, to solve this system requires an iterative approach, one that can be difficult to code and is computationally expensive to execute. Similarly, it is difficult to accurately compute the frictional and diabatic heating forcing terms that make up part of the right-hand side of (16).

Conversely, geostrophic vorticity advection and differential potential-temperature advection can be computed or readily estimated from any available atmospheric data source, such as a numerical model analysis or forecast. This, in concert with the general proportionality stated in (17), enables us to *diagnose* the likely temporal evolution of the middle tropospheric geopotential height field.