Synoptic Meteorology I: Kinematic Properties

For Further Reading

The distinction between streamlines and trajectories is illustrated by Section 4.5 of *Mid-Latitude Atmospheric Dynamics* by J. Martin and Sections 3-6 and 3-8 of *Weather Analysis* by D. Djurić. Kinematic properties of the wind field are discussed in Section 1.4 of *Mid-Latitude Atmospheric Dynamics* and Section 4-9 of *Weather Analysis*. Practical kinematics are introduced in Section 4-3 of *Weather Analysis*.

What is the Wind?

Qualitatively, we understand the wind reflecting the air's (three-dimensional) motion. Before we proceed, however, it is worthwhile to quantitatively define the wind. In the Cartesian coordinate system, an air parcel's location in two dimensions is defined as (x,y). Depending upon our choice of vertical coordinate, an air parcel's vertical position may be defined as z or p. Wind, therefore, is simply defined as the change in the air parcel's location with time, i.e.,

$$\vec{v} = \frac{D\vec{x}}{Dt} \qquad (1)$$

In (1), for z as the vertical coordinate, $\vec{x} = (x, y, z)$ and $\vec{v} = (u, v, w)$. With p as the vertical coordinate, $\vec{x} = (x, y, p)$ and $\vec{v} = (u, v, \omega)$.

Streamlines and Trajectories

A *streamline* represents a line that is tangent (or parallel) to the wind at a given location. Recall that under the constraint of geostrophic balance, height contours on middle-to-upper tropospheric isobaric charts are parallel to the wind with greater packing for faster wind speeds. The same basic idea applies to streamlines. Although streamlines do not have numerical values ascribed to them, velocity magnitude (or wind speed) is inferred from streamline spacing. Streamlines that are more densely packed together imply a faster wind speed, while streamlines that are less densely packed together imply a slower wind speed. Unlike isopleths, streamlines *can* stop and start on a chart. This occurs most commonly where a discontinuity in the wind exists, such as along a front. While streamlines do not intersect, they may diverge from or converge to a point on the chart. This occurs most commonly with areas of high and low pressure near the surface, respectively.

Conversely, a *trajectory* represents the path that an air parcel follows through time. As an air parcel moves from one location to another, not only does its location change, so too does the wind itself (as documented by the equations of motion). This highlights an important distinction between streamlines and trajectories: streamlines are the path that an air parcel would follow if the wind did not change with time, while trajectories are the path that an air parcel follows while accounting for the fact that the wind does change with time. Note also that while streamlines are often analyzed only using the horizontal wind, trajectories are typically analyzed using the fully three-dimensional wind. A *forward trajectory* depicts where the air parcel moves to as time moves forward, while a *backward trajectory* depicts where an air parcel came from at previous times. While streamlines can be manually analyzed on synoptic charts, trajectories are typically analyzed using computer-based methods (e.g., the HYSPLIT trajectory model).

A generic equation for a trajectory is given by:

$$\vec{x}(t_1) = \vec{x}(t_0) + \int_{t_0}^{t_1} \vec{v}(x, y, p, t)\,dt \quad (2)$$

In other words, the position of the air parcel at time t_1 is equal to the position of the air parcel at time t_0 plus the integrated motion of the air parcel over the time interval between t_0 and t_1. For a forward trajectory, $t_1 > t_0$; for a backward trajectory, $t_1 < t_0$. Note that $\vec{v}$ is a function of time – in other words, it changes with time. Thus, knowledge of how $\vec{v}$ changes with time over the motion of the air parcel is necessary to calculate the air parcel's trajectory. But, we don't know what the wind along the trajectory will be like until we know the path of the trajectory, nor do we know the path of the trajectory until we know the wind along the trajectory! As a result, (2) is typically solved using iterative (i.e., successive guesses) means, and this is the chief reason why trajectories are typically only analyzed using computer-based methods.

In Figure 1, streamlines and forward trajectories are presented for an eastward-moving trough. We consider three separate scenarios:

- **Case 1:** An air parcel moves slower than the trough. Starting at A, the air parcel initially moves due east. However, because the trough moves east faster than does the air parcel, the air parcel falls behind and ends up in the northwesterly flow behind the base of the trough. As a result, its motion turns toward the east-southeast. This results in the trajectory given by A->D.

- **Case 2:** An air parcel moves at the same speed as the trough. Starting at A, the air parcel initially moves due east. Because the trough and air parcel move to the east at the same rate of speed, the air parcel is always located in the base of the trough where the wind is directed due east. As a result, its motion remains toward the east. This results in the trajectory given by A->C.

- **Case 3:** An air parcel moves faster than the trough. Starting at A, the air parcel initially moves due east. However, because it does so faster than does the trough, the air parcel ends up in the southwesterly flow ahead of the base of the trough. As a result, its motion turns toward the east-northeast. This results in the trajectory given by A->B.

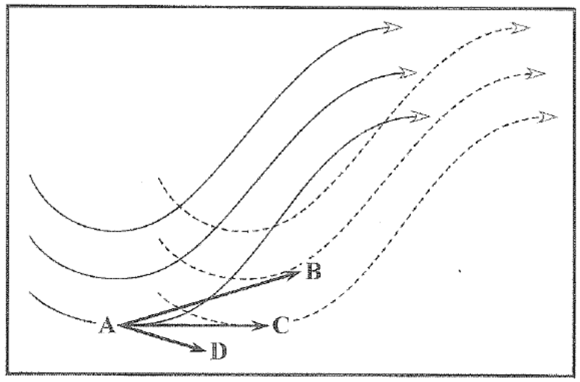

Figure 1. Streamlines (curved lines) and forward trajectories (bold vectors) in an eastward-moving trough. Streamlines at the initial time are given by the solid curved lines while streamlines at the later time are given by the dashed curved lines. Three forward trajectories are considered: A->B, A->C, and A->D. Please refer to the text above for further details. Figure reproduced from *Mid-Latitude Atmospheric Dynamics* by J. Martin, their Figure 4.22.

When we considered atmospheric force balances in natural coordinates, we often encountered an *R* term, which we identified as the radius of curvature. It should be noted that this R explicitly represents the radius of curvature of a trajectory rather than the radius of curvature of a streamline. The two radii can be quite different. For example, the streamlines in Figure 1 are cyclonically curved, such that $R > 0$. The forward trajectory A->B is cyclonically curved, but less so than the streamline. Meanwhile, the forward trajectory A->C is not curved, whereas the forward trajectory A->D is anticyclonically curved (such that $R < 0$). The two radii are only equal when the wind does not change with time. Thus, when evaluating force balances, bear in mind that where possible *R* should represent the curvature of the air parcel's motion – that given by its trajectory.

Kinematics of the Wind Field

Let us consider the horizontal wind, as given by the components *u*, representing motion in the east-west direction, and *v*, representing motion in the north-south direction. Intuitively, we know that both *u* and *v* are not constant from one location to another – in other words, they change in both the east-west (or *x*) and north-south (or *y*) directions. Representing these changes in terms of partial derivatives, we have:

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

Let us now consider combinations of two of these four terms. In our combinations, we require that one term include a partial derivative of *u* with respect to one position variable and a partial derivative of *v* with respect to the other position variable. This requirement means that there are four such unique combinations:

$$\text{(a) } \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}, \text{ (b) } \frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}, \text{ (c) } \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}, \text{ (d) } \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

We define each of the terms (a) through (d) as follows:

- **Divergence**, or δ, is given by term (a). In vector form, it can be expressed as $\nabla \cdot \vec{v}$.
- **Stretching deformation**, or STD, is given by term (b).
- **Shearing deformation**, or SHD, is given by term (c).
- **Vorticity**, or ζ, is given by term (d). Formally, this is the vertical vorticity, and only that part which does not include the effects of Earth's rotation; the full vorticity includes other terms that we are not considering here. In vector form, vertical vorticity can be expressed as $\hat{k} \cdot (\nabla \times \vec{v})$.

How do each of these kinematic properties contribute to the wind? Let us first consider a mathematical expression for both *u* and *v*. We know that each vary in the *x* and *y* directions, such that *u = u(x,y)* and *v = v(x,y)*. Because both are continuous functions in space – in other words, *u* and *v* are defined everywhere – we can express each in terms of a Taylor series expansion. The generic form of a Taylor series expansion of a variable *f* that is a function of *x* and *y* is given by:

$$f(x,y) = f_0 + \sum_{n=1}^{N}\left[\left(\frac{\partial^n f}{\partial x^n}\right)_0 \frac{x^n}{n!}\right] + \sum_{n=1}^{N}\left[\left(\frac{\partial^n f}{\partial y^n}\right)_0 \frac{y^n}{n!}\right] + \sum_{m=1}^{M}\sum_{n=1}^{N}\left[\frac{\partial^m}{\partial y^m}\left(\frac{\partial^n f}{\partial x^n}\right)_0 \frac{x^n y^m}{n!m!}\right] \tag{3}$$

In (3), subscripts of 0 are meant to evaluate a quantity evaluated at the air parcel's origin. The term at the end of (3), involving partial derivatives of both *x* and *y*, is typically neglected. If we apply (3) to *u* and *v* and keep only the *n* = 1 terms – thus neglecting terms involving second- and higher-order partial derivatives as a result of being small – from the first two summation terms, we obtain:

$$u(x,y) = u_0 + \left(\frac{\partial u}{\partial x}\right)_0 x + \left(\frac{\partial u}{\partial y}\right)_0 y \tag{4a}$$

$$v(x,y) = v_0 + \left(\frac{\partial v}{\partial x}\right)_0 x + \left(\frac{\partial v}{\partial y}\right)_0 y \tag{4b}$$

Equation (4) states that the velocity at any location *x,y* is equal to the velocity at the origin (0,0) plus measures of how the velocity changes in space – the partial derivative terms – multiplied by the distance from the origin.

If we rewrite (4) in terms of (a) through (d) above, we obtain:

$$u(x,y) = u_0 + \frac{1}{2}\left(-\zeta_0 y + \delta_0 x + STD_0 x + SHD_0 y\right) \quad (5a)$$

$$v(x,y) = v_0 + \frac{1}{2}\left(\zeta_0 x + \delta_0 y - STD_0 y + SHD_0 x\right) \quad (5b)$$

Equation (5) demonstrates that the wind field can be expressed in terms of a linear combination of the four kinematic elements – divergence, stretching deformation, shearing deformation, and vertical vorticity – described above as well as a translational component given by u_0 and v_0.

We now wish to describe the wind that results from each kinematic element in isolation.

Translation Flow

Translation flow is defined where $u(x,y) = u_0$ and $v(x,y) = v_0$. In this case, an air parcel moves at a constant velocity, as given by (u_0,v_0). Because the velocity is constant everywhere, streamlines of the flow are given by straight lines. Wind vectors representing solutions to this equation are presented in Figure 2.

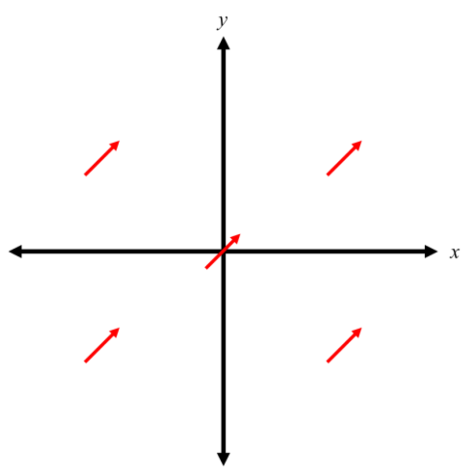

Figure 2. Example of pure translation flow, given by $u(x,y) = u_0$ and $v(x,y) = v_0$, where u_0 and y_0 are arbitrary constant values. Note how an object placed in this flow would simply move with the flow – it would not rotate, nor would it change shape or size.

Pure Divergence

Pure divergence is defined where u_0, v_0, ζ_0, STD_0, and SHD_0 are all equal to 0 while δ_0 is equal to 1. In this case, (5) becomes:

$$u(x,y) = \frac{1}{2}x \quad (6a)$$

$$v(x,y) = \frac{1}{2}y \quad (6b)$$

Wind vectors representing solutions to (6) are presented in Figure 3.

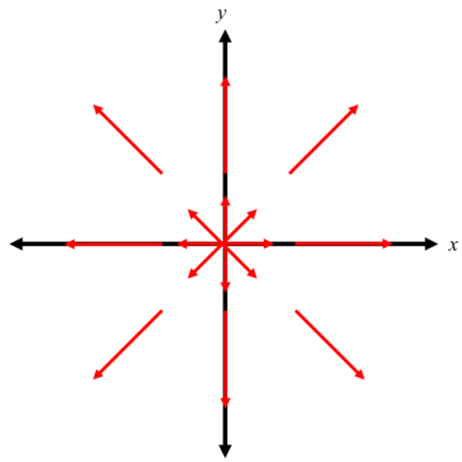

Figure 3. Example of pure divergence flow. Note how the wind is directed in all directions away from the origin – i.e., air parcels diverge from the origin. If we let δ_0 be equal to -1, then the wind vectors would be directed in all directions toward the origin, denoting convergence. Adapted from *Mid-Latitude Atmospheric Dynamics* by J. Martin, their Figure 1.8.

Pure Vertical Vorticity

Pure vertical vorticity is defined where u_0, v_0, δ_0, STD_0, and SHD_0 are all equal to 0 while ζ_0 is equal to 1. In this case, (5) becomes:

$$u(x,y) = -\frac{1}{2}y \quad (7a)$$

$$v(x,y) = \frac{1}{2}x \quad (7b)$$

Wind vectors representing solutions to (7) are presented in Figure 4.

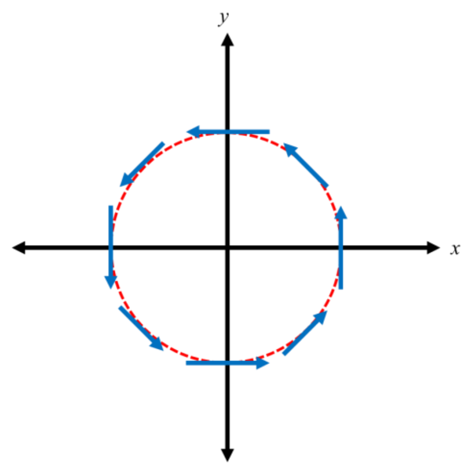

Figure 4. Example of pure vertical vorticity, or rotational, flow. Note how the wind is directed in a counterclockwise fashion around the origin. The case of ζ_0 = 1 defines cyclonic (in the Northern Hemisphere) or counterclockwise flow, whereas the case of ζ_0 = -1 defines anticyclonic (in the Northern Hemisphere) or clockwise flow. Adapted from *Mid-Latitude Atmospheric Dynamics* by J. Martin, their Figure 1.9.

Pure Stretching Deformation

Pure stretching deformation is defined where u_0, v_0, δ_0, ζ_0, and SHD_0 are all equal to 0 while STD_0 is equal to 1. In this case, (5) becomes:

$$u(x,y) = \frac{1}{2}x \quad (8a)$$

$$v(x,y) = -\frac{1}{2}y \quad (8b)$$

Wind vectors representing solutions to (8) are presented in Figure 5. The solution presented in Figure 5 implies convergence along the *y*-axis and divergence along the *x*-axis. Deformation and divergence are not the same, however! Consider a square embedded within the pure divergence solution in Figure 3. The wind changes the size (and thus area) of the square but not its shape. By contrast, consider a square embedded within the pure stretching deformation solution in Figure 5. The wind changes the shape of the square, as it becomes rectangular as it is compressed along one axis and stretched along another, though it can be shown that the area of the feature does not change. We instead refer to the compression of the flow as *confluent* flow and the stretching of the flow as *diffluent* flow.

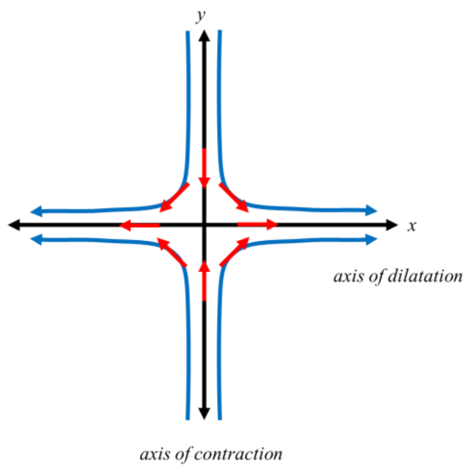

Figure 5. Example of pure stretching deformation flow. We find that the flow is stretched outward along the *x*-axis and compressed inward along the *y*-axis. The axis along which the flow is stretched (here, the *x*-axis) is known as the *axis of dilatation* and the axis along which the flow is compressed (here, the *y*-axis) is known as the *axis of contraction*. Adapted from *Mid-Latitude Atmospheric Dynamics* by J. Martin, their Figure 1.10.

Pure Shearing Deformation

Finally, pure shearing deformation is defined where u_0, v_0, δ_0, ζ_0, and STD_0 are all equal to 0 while SHD_0 is equal to 1. In this case, (5) becomes:

$$u(x,y) = \frac{1}{2}y \quad (9a)$$

$$v(x,y) = \frac{1}{2}x \quad (9b)$$

Wind vectors representing solutions to (9) are presented in Figure 6. Note that typically, we are not concerned with stretching deformation versus shearing deformation but rather with the total deformation, which we can define as:

$$D = \sqrt{STD^2 + SHD^2} \quad (10)$$

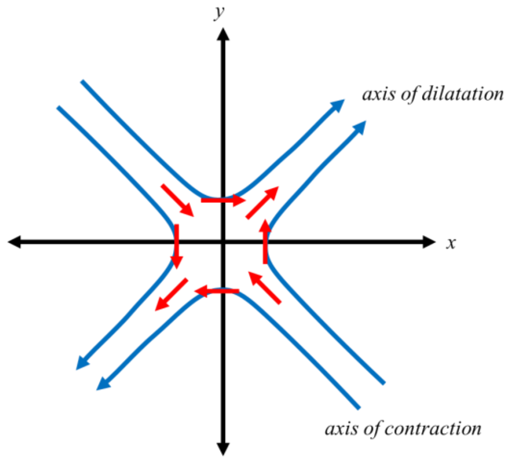

Figure 6. Example of pure shearing deformation flow. The solution is identical to that for pure stretching deformation, except rotated 45° to the left. Adapted from *Mid-Latitude Atmospheric Dynamics* by J. Martin, their Figure 1.12.

The Full Wind

In observations, the full wind cannot be represented by just one of the basic solutions presented in Figures 2-6. Rather, the full wind represents a combination of these solutions, as implied from (5). We nevertheless can analyze wind observations on synoptic charts and qualitatively characterize the flow's characteristics – whether divergent, rotational, deformative, or translation in nature.

Practical Kinematics

To facilitate qualitative evaluation of divergence and vorticity from synoptic data, we now wish to consider how each may be expressed in natural coordinates. Recall that in a natural coordinate system, there exist two horizontal coordinates: *s*, following the flow, and *n*, perpendicular to the flow. The positive *s*-axis is defined along the wind in the direction in which it is blowing, while the positive *n*-axis is perpendicular and to the left of the *s*-axis.

Presented without derivation, the natural coordinate forms of divergence and vorticity are:

$$\delta = \frac{\partial V}{\partial s} - V\frac{\partial \alpha}{\partial n} \tag{11a}$$

$$\zeta = \frac{V}{R} - \frac{\partial V}{\partial n} \tag{11b}$$

In (11), V is the wind speed (formally, in m s^{-1}), R is the radius of curvature (where $R > 0$ indicates cyclonic flow; formally, in m), and α is the wind direction (formally, in radians). A full derivation of (11) is provided in Appendix H of *Weather Analysis* by D. Djurić.

Divergence in Natural Coordinates

The terms on the right-hand side of (11a) represent *speed divergence* and *diffluence*, respectively. Let us consider each individually.

The speed divergence term, or $\frac{\partial V}{\partial s}$, represents the change in wind speed along a streamline. If the wind speed increases along a streamline, this term is positive, indicating speed divergence. Conversely, if the wind speed decreases following the flow, this term is negative, indicating speed convergence. Examples of both are provided in Figure 7.

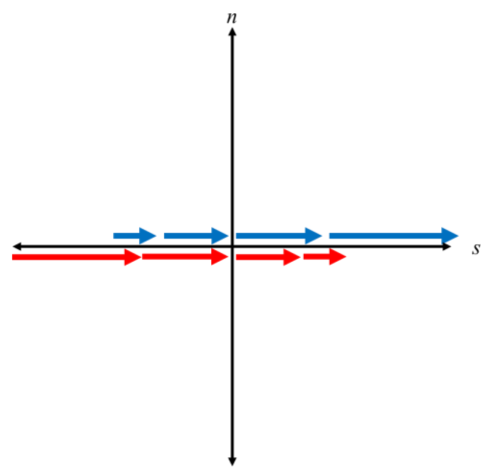

Figure 7. Conceptual examples of speed divergence (blue), where the wind speed (given by the length of the arrow) increases following the flow, and speed convergence (red), where the wind speed decreases following the flow. Note that the orientation of the *n*- and *s*-axes will change for other flow configurations.

The diffluence term, or $-V\frac{\partial \alpha}{\partial n}$, is a function of the change in wind direction perpendicular to the flow. As we noted before, α is in radians. The following equation can be used to convert from degrees to radians in meteorological convention:

$$\alpha_{rad} = \left(\frac{\pi}{180}\right)\alpha_{\deg} \qquad (12)$$

where $\alpha_{\deg}$ = 0°/360° for wind out of the north, 90° for wind out of the east, 180° for wind out of the south, and 270° for wind out of the west. In radians, these values correspond to 0 (or 2π), 0.5π, π, and 1.5π, respectively.

In the case where wind angle increases along the positive *n*-axis (i.e., the horizontal wind direction turns clockwise along the positive *n*-axis), $\frac{\partial \alpha}{\partial n}$ is positive. Since *V* is positive at the location being considered, the leading negative on this term indicates that the diffluence term is negative. This is identified with *confluence*. Conversely, in the case where wind angle decreases along the positive *n*-axis (i.e., the horizontal wind direction turns counterclockwise along the positive *n*-axis), $\frac{\partial \alpha}{\partial n}$ is negative. Given the convention on *V*, the diffluence term is positive, which we identify with *diffluence*. Examples of both confluence and diffluence are depicted in Figure 8.

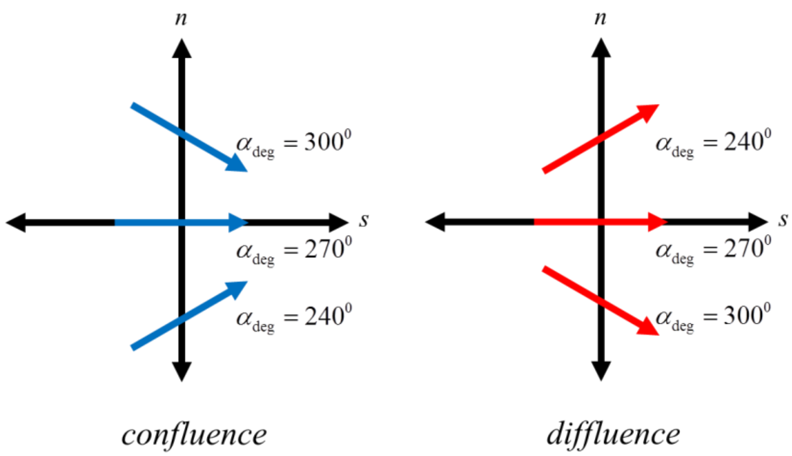

Figure 8. Conceptual examples of confluence (left) and diffluence (right). The wind is directed toward a point with confluence and spreads apart from a point with diffluence. Note that the orientation of the *n*- and *s*-axes will change for other flow configurations.

It is possible to have speed divergence coincident with confluence, as is illustrated in the example given by Figure 11. Likewise, it is possible to have speed convergence coincident with diffluence. Thus, care must be taken when evaluating divergence on a weather map to not conflate diffluence with divergence or confluence with convergence.

Vorticity in Natural Coordinates

The terms on the right-hand side of (11b) represent *curvature* and *normal shear*, respectively. As with divergence, let us consider each individually.

The curvature term, or $\frac{V}{R}$, is a function of the wind speed and the radius of curvature. For cyclonically curved flow, where $R > 0$, this term is positive. For anticyclonically curved flow, where $R < 0$, this term is negative. The magnitude of this term is largest when the wind speed is large and/or the magnitude of the radius of curvature is small. Small positive, large positive, small negative, and large negative values of this term are illustrated in Figure 9.

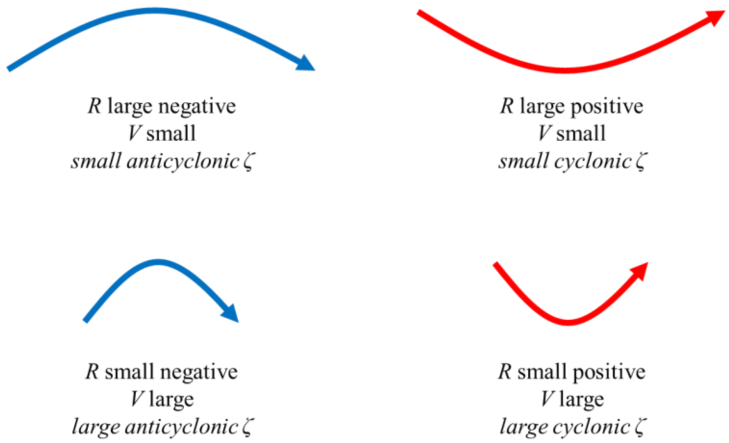

Figure 9. Conceptual examples of anticyclonic curvature (left) and cyclonic curvature (right) for large magnitudes of *R* and small *V* (top) and small magnitudes of *R* and large *V* (bottom). Recall that *R* is defined by the distance of the curved air flow from the central point around which it rotates. Large *V* and small *R* (and vice versa) are often, but not necessarily always, found in conjunction with one another.

The normal shear term, or $-\frac{\partial V}{\partial n}$, is a function of the change in wind speed perpendicular to the flow. If the wind speed decreases along the positive *n*-axis, $\frac{\partial V}{\partial n}$ is negative. The leading negative sign results in the normal shear term being positive, representing cyclonic shear. Conversely, if the wind speed increases along the positive *n*-axis, $\frac{\partial V}{\partial n}$ is positive. The leading negative sign results in the normal shear term being negative, representing anticyclonic shear. Examples of each are provided in Figure 10.

Recall that we first defined vertical vorticity as being associated with rotation. Conceptually, we tend to view rotational flow from the concept of the flow being curved, such as in Figure 9. However, from Figure 10, it is evident that vertical vorticity can exist even when the flow is straight if there is horizontal – or, more exactly, normal – wind shear.

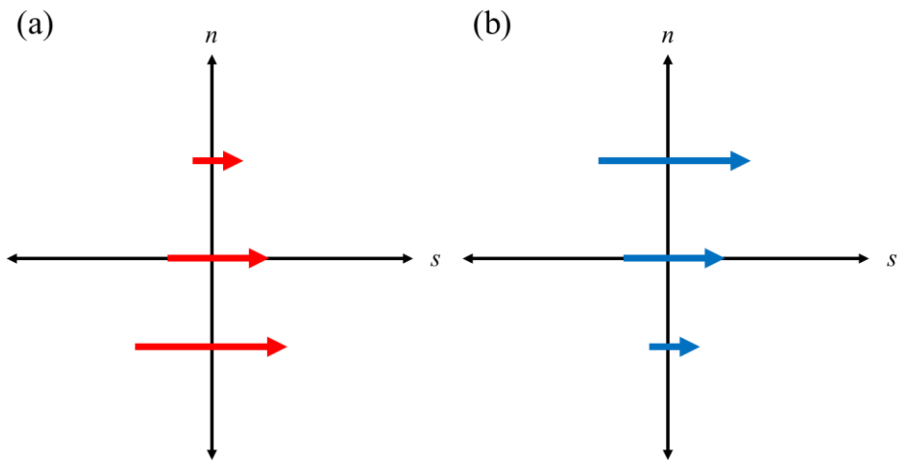

Figure 10. Conceptual examples of (a) positive and (b) negative normal shear terms. In the former, an object placed in the flow will acquire cyclonic (counterclockwise) rotation, whereas in the latter, an object placed in the flow will acquire anticyclonic (clockwise) rotation. Note that the orientation of the *n*- and *s*-axes will change for other flow configurations.

Putting it All Together: A Real-World Example

As in Cartesian coordinates, the total wind is not represented exclusively by divergence, vorticity, or any single part of either kinematic field. Rather, the total wind represents a combination of each of the four terms examined in the above. An example is given in Figure 11.

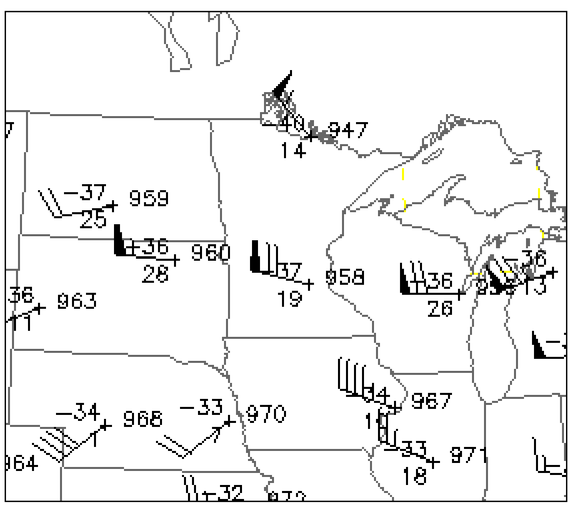

Figure 11. Observations of temperature (upper left), dewpoint temperature (lower left), geopotential height (dam), and wind speed and direction (barbs; half flag: 5 kt, full flag: 10 kt, pennant: 50 kt) on the 300 hPa isobaric surface valid at 1200 UTC 27 August 2014 across the upper Midwest of the United States. Data obtained from http://weather.ral.ucar.edu/upper/.

Let us consider Minneapolis, MN, roughly located in the center of the diagram. At Minneapolis, the wind is out of the west-northwest at 70 kt. If we draw the *s*- and *n*-axes at Minneapolis, we find that International Falls, MN is nearly located along the positive *n*-axis, Omaha, NE is nearly located along the negative *n*-axis, Green Bay, WI is nearly located along the positive *s*-axis, and Aberdeen, SD is located nearly along the negative *s*-axis, thereby aiding our analysis.

We first wish to evaluate the speed divergence term at Minneapolis. At Green Bay, the wind speed along the *s*-axis is roughly 70 kt, the same as at Minneapolis. At Aberdeen, the wind speed along the *s*-axis is roughly 50 kt, lower than at both Green Bay and Minneapolis. Thus, at Minneapolis, there exists *speed divergence* because wind speed increases in the positive *s*-direction.

We next evaluate the diffluence term at Minneapolis. At International Falls, the wind is out of the northwest (315°, or 1.75π). At Omaha, the wind is out of the southwest (225°, or 1.25π). Thus, the wind angle increases along the positive *n*-axis. Given the sign convention on this term, this indicates *confluence* at Minneapolis.

We now evaluate the curvature term at Minneapolis. However, the flow at Minneapolis itself is relatively straight, at least given the data available to us in Figure 11. If we look along both the positive and negative *n*-axis, we find that the flow is *cyclonically curved* along the positive *n*-axis (toward International Falls) and *anticyclonically curved* along the negative *n*-axis (toward Omaha).

Finally, we evaluate the normal shear term at Minneapolis. Using trigonometry, we could find the wind speed perpendicular to the *n*-axis at International Falls and Omaha. For our purposes, however, it is sufficient to qualitatively state that the wind speed perpendicular to the *n*-axis is less than or equal to the observed wind speed at each location. Between Minneapolis and International Falls, *V* decreases along the positive *n*-axis, which we previously identified with *cyclonic normal shear*. Conversely, between Omaha and Minneapolis, *V* increases along the positive *n*-axis, which we previously identified with *anticyclonic normal shear*.

Divergence and Vorticity of the Geostrophic Wind

In the above, we considered divergence and vorticity in terms of the full wind. We now desire to consider the divergence and vorticity of the geostrophic wind. First, recall the definition of the geostrophic wind expressed in isobaric coordinates:

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x} = \frac{1}{f}\frac{\partial \Phi}{\partial x} \tag{13a}$$

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y} = -\frac{1}{f}\frac{\partial \Phi}{\partial y} \tag{13b}$$

Let us first consider the vorticity of the geostrophic wind. The *geostrophic relative vorticity* is simply equal to the relative vorticity with u_g and v_g substituted for *u* and *v*, i.e.,

$$\zeta_g = \hat{k} \cdot \left(\nabla \times \vec{v}_g \right) = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\partial}{\partial x}\left(\frac{1}{f}\frac{\partial \Phi}{\partial x} \right) - \frac{\partial}{\partial y}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y} \right) \tag{14}$$

Note that the Coriolis parameter *f* does not change in the *x*-direction, such that the partial derivative of 1/*f* with respect to *x* is exactly zero. The Coriolis parameter does change in the *y*-direction, such that the partial derivative of 1/*f* with respect to *y* is not zero. However, the value of this term over finite meridional distances is small enough to be neglected. Thus, assuming that *f* is constant with respect to both *x* and *y*, (14) becomes:

$$\zeta_g = \frac{1}{f}\left(\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial y} \right) \right) = \frac{1}{f}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) = \frac{1}{f}\nabla^2 \Phi \tag{15}$$

In (15), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and is known as the *Laplacian* operator. Recall that the second partial derivative of a field returns a measure of its concavity; the second partial derivative is positive for fields that are concave up (or convex) whereas it is negative for fields that are concave down. Consider, then, a trough and a ridge. Isopleths of geopotential height in association with a trough are convex, whereas isopleths of geopotential height in association with a ridge are concave down. *Consequently, the geostrophic relative vorticity is positive (or cyclonic) with a trough, maximized in its base, and negative (or anticyclonic) with a ridge, most negatively so at its apex.* This makes sense: the flow around a trough is cyclonic, whereas the flow around a ridge is anticyclonic. When the magnitude of the local Φ minimum or maximum is large, the geostrophic relative vorticity also has large magnitude (and vice versa). This makes sense: a large magnitude to the local Φ minimum or maximum implies large horizontal Φ gradients and thus large geostrophic winds.

Now let us consider the divergence of the geostrophic wind. This is simply equal to the divergence of the full wind with u_g and v_g substituted for *u* and *v*, i.e.,

$$\delta_g = \nabla \cdot \vec{v}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y} \right) + \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial \Phi}{\partial x} \right) \tag{16}$$

Again, note that while the Coriolis parameter *f* is constant in the *x*-direction, it is not constant in the *y*-direction. But, if we approximate *f* as being constant in the *y*-direction, (16) becomes:

$$\delta_g = -\frac{1}{f}\left(\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial y} \right) - \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial x} \right) \right) \tag{17}$$

If we commute the order of one set of the partial derivatives in (17), we find that the term inside the outer parentheses is zero, such that $\delta_g = 0$. In other words, *the geostrophic wind is very nearly non-divergent*!

Since $\vec{v} = \vec{v}_g + \vec{v}_{ag}$, $\nabla \cdot \vec{v} = \nabla \cdot \left(\vec{v}_g + \vec{v}_{ag}\right) = \nabla \cdot \vec{v}_g + \nabla \cdot \vec{v}_{ag} = \nabla \cdot \vec{v}_{ag}$. In other words, the divergence of the full wind can be represented (to first approximation, at least) by the divergence of the ageostrophic wind. As we will soon see, this has important implications for vertical motions within the troposphere.