## Synoptic Meteorology I: Finite Differences

**For Further Reading**

College-level Calculus texts contain extensive information regarding the mathematical definition of limits, partial derivatives, and Taylor functions and series. Sections 1.2.2 and 1.2.3 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provide similar information from the perspective of their applications to the atmospheric sciences.

**Partial Derivatives (or, Why Do We Care About Finite Differences?)**

Apart from the ideal gas law, the equations that govern the evolution of fundamental atmospheric properties such as wind, pressure, and temperature (the *primitive equations*) contain many terms with partial derivatives. Indeed, many thermodynamic and kinematic properties of the atmosphere are typically expressed in terms of partial derivatives. We will explore many specific examples of such equations throughout both this and next semester.

Mathematically speaking, the partial derivative of some generic field $f$ with respect to some generic variable $x$ can be expressed as:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \qquad\qquad (1)$$

In other words, $\frac{\partial f}{\partial x}$ is equal to the value of $\frac{\Delta f}{\Delta x}$ as $\Delta x$ approaches (but does not equal) zero. Thus, for small (or finite) values of $\Delta x$, we can approximate $\frac{\partial f}{\partial x}$ by $\frac{\Delta f}{\Delta x}$. That begs the question: how do we compute $\frac{\Delta f}{\Delta x}$ from available atmospheric data?

To do so, we use what are known as *finite differences* to <u>approximate</u> the value of $\Delta f$ over some finite $\Delta x$. Applied to isoplethed analyses of meteorological fields or gridded data, finite differences enable us to *compute* any quantity that depends upon one or more partial derivatives. Here, we wish to describe how finite difference approximations are obtained, the degree to which each is an approximation, and use examples to introduce how they can be applied to the atmosphere.

**Developing Finite Difference Approximations**

First, let us consider a generic continuous function $f(x)$, a graphical example of which is depicted below in Figure 1. This function doesn't necessarily represent a meteorological field, but it doesn't not necessarily represent one either; it is simply a generic function. Along the curve given by $f(x)$, there are three points of interest: $x_a$, $x_{a+1}$, and $x_{a-1}$. The function $f(x)$ has the values $f(x_a)$, $f(x_{a+1})$,

and $f(x_{a-1})$ at these three points, respectively. The distance between $x_a$ and $x_{a-1}$ is equal to the distance between $x_a$ and $x_{a+1}$, and we can denote this distance as $\Delta x$.

We derive finite difference approximations from Taylor series, or mathematical representations of functions as infinite sums of terms that are calculated from the values of the functions' derivatives at any given point. The Taylor series expansion of $f(x)$ about $x = b$, where $b$ is some generic point, is given by:

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3 + ... \tag{2}$$

This expansion states that $f(x)$ is equal to the value of $f(x)$ at $x = b$ plus an infinite series of higher-order terms, each of which contains an increasingly large partial derivative (primes; a single prime denotes the first partial derivative, two primes denote the second partial derivative, etc.), exponent on $x - b$, and factorial (!) order.
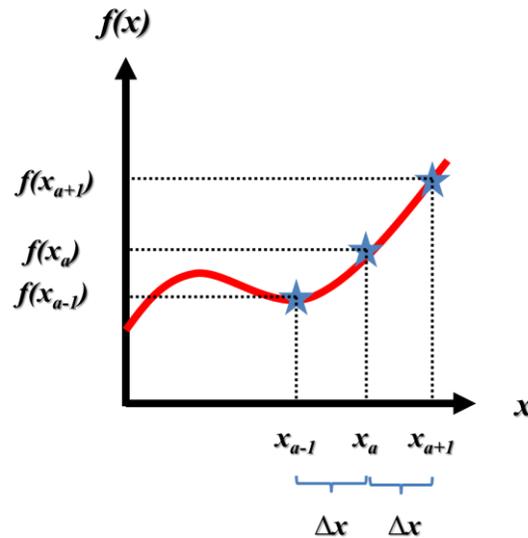


**Figure 1**. Graphical depiction of a generic function $f(x)$ evaluated at three points. Please see the text for further details.

Let us consider two cases: one where $x = x_{a+1}$ and $b = x_a$ and one where $x = x_{a-1}$ and $b = x_a$. In the first case, the distance $x - b$, or $x_{a+1} - x_a$, is equal to $\Delta x$; in the second case, the distance $x - b$, or $x_{a-1} - x_a$, is equal to $-\Delta x$. Making use of this information, we can expand (2) for these two cases:

$$f(x_{a+1}) = f(x_a) + f'(x_a)\Delta x + \frac{f''(x_a)}{2!}(\Delta x)^2 + \frac{f'''(x_a)}{3!}(\Delta x)^3 + ... \tag{3}$$

$$f(x_{a-1}) = f(x_a) - f'(x_a)\Delta x + \frac{f''(x_a)}{2!}(\Delta x)^2 - \frac{f'''(x_a)}{3!}(\Delta x)^3 + \dots \tag{4}$$

Note the similar appearance of (3) and (4) apart from the leading negative signs on the first and third order terms in (4). These arise because $x - b = -\Delta x$ for this case, as noted above.

From (3) and (4), we are interested in the value of $f'(x_a)$, or $\dfrac{\partial f}{\partial x}$ evaluated at $x_a$. We can use (3) and (4) to obtain an expression for this term; we simply need to subtract (4) from (3). Doing so, we obtain the following:

$$f(x_{a+1}) - f(x_{a-1}) = 2f'(x_a)\Delta x + \frac{2f'''(x_a)}{3!}(\Delta x)^3 + \dots \text{(odd order terms)} \dots \tag{5}$$

Note how the zeroth and second order terms in (3) and (4) cancel out in this operation. If we rearrange (5) and solve for $f'(x_a)$, we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_{a-1})}{2\Delta x} - \frac{f'''(x_a)}{3!}(\Delta x)^2 + \dots \tag{6}$$

At this point, we wish to neglect all terms higher than the first-order term from (6). Doing so, we are left with:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_{a-1})}{2\Delta x} \tag{7}$$

Equation (7) is what is known as a *centered finite difference*. It provides a means of calculating $\dfrac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of $f$ at $x = x_{a+1}$, subtracting from it the value of $f$ at $x = x_{a-1}$, and dividing the result by the distance between the two points ($2\Delta x$). Note that $x$ here and in later examples can be any variable; it does not have to represent the $x$-axis or the east-west direction. Equation (7) is equivalent if $x$ is replaced by $y$, $z$, $p$, or any number of other variables.

There exist other ways for us to use (3) and (4) to get expressions for $f'(x_a)$. For instance, we can solve (3) for this term. If we do so, we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_a)}{\Delta x} - \frac{f''(x_a)}{2!}(\Delta x) - \frac{f'''(x_a)}{3!}(\Delta x)^2 - \dots \tag{8}$$

Neglecting all terms higher than the first order term in (8), we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_a)}{\Delta x} \tag{9}$$

Equation (9) is what is known as a *forward finite difference*. It provides a means of calculating $\frac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of $f$ at $x = x_{a+1}$, subtracting from it the value of $f$ at $x = x_a$, and dividing the result by the distance between the two points ($\Delta x$).

Alternatively, we can solve (4) for $f'(x_a)$. If we do so, we obtain:

$$f'(x_a) = \frac{f(x_a) - f(x_{a-1})}{\Delta x} + \frac{f''(x_a)}{2!}(\Delta x) - \frac{f'''(x_a)}{3!}(\Delta x)^2 + \ldots \tag{10}$$

Neglecting all terms higher than the first order term in (10), we obtain:

$$f'(x_a) = \frac{f(x_a) - f(x_{a-1})}{\Delta x} \tag{11}$$

Equation (11) is what is known as a *backward finite difference*. It provides a means of calculating $\frac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of $f$ at $x = x_a$, subtracting from it the value of $f$ at $x = x_{a-1}$, and dividing the result by the distance between the two points ($\Delta x$).

**Finite Differences as Approximations**

We do not need to neglect the higher-order terms in obtaining any of the above expressions for $f'(x_a)$; we have done so here primarily for simplicity. If we were to retain the higher-order terms, we would obtain more accurate approximations for $f'(x_a)$. This highlights a key point: all finite differences are *approximations*. All finite differences are associated with what is known as *truncation error*, which is determined by the power of $\Delta x$ on the first term that is neglected in obtaining the finite difference approximation.

For instance, consider our centered finite difference given by Equation (7). In obtaining (7), the first term that we neglected in (6) included a $(\Delta x)^2$ term. As a result, we say this finite difference is "second-order–accurate." In contrast, consider our forward and backward finite differences, given by Equations (9) and (11), respectively. In obtaining each equation, the first terms that we neglected in (8) and (10) included a $(\Delta x)$ term. As a result, we say that these finite differences are "first-order–accurate." The higher the order of accuracy, the more accurate the finite difference.

In synoptic meteorology, where exact values for partial derivatives are often not necessary, we typically utilize the centered finite difference. Forward and backward finite differences are rarely utilized except along the edges of the data, where the -1 and +1 points may not exist. Higher-order finite differences, typically fourth- or higher-order–accurate, are necessary for numerical weather

prediction models given chaos theory, which states that very small differences in data can lead to very large forecast differences.

## A Finite Difference Approximation for Second Derivatives

While the first partial derivative of some field provides a measure of its *slope*, sometimes we are interested in evaluating the second partial derivative of some field. Recall from calculus that the second partial derivative of a field provides a measure of its *concavity*; positive second partial derivatives infer that a field is concave up (or convex), while negative second partial derivatives infer that a field is concave down. Applied to meteorology, a field that is convex represents a local minimum, whereas a field that is concave down represents a local maximum; i.e., the second partial derivative has the opposite sign of the field itself.

We can obtain a finite difference approximation for the second partial derivative by adding (3) and (4). Doing so, we obtain:

$$f(x_{a+1}) + f(x_{a-1}) = 2f(x_a) + 2\frac{f''(x_a)}{2!}(\Delta x)^2 + \ldots \tag{12}$$

If we solve (12) for $f''(x_a)$ and truncate the higher-order terms, we obtain:

$$f''(x_a) = \frac{f(x_{a+1}) + f(x_{a-1}) - 2f(x_a)}{(\Delta x)^2} \tag{13}$$

Equation (13) provides a fourth-order–accurate means of evaluating $\frac{\partial^2 f}{\partial x^2}$, or $f''(x_a)$, by adding the value of $f$ at $x_{a+1}$ to the value of $f$ at $x_{a-1}$, subtracting two times the value of $f$ at $x_a$, and dividing the result by the square of the distance between points $(\Delta x)^2$.

Just as for the finite difference approximation for the first partial derivative, (13) is equivalent if $x$ is replaced by $y$, $z$, $p$, or any number of other variables. Likewise, just as for the finite difference approximate for the first partial derivative, higher-order accurate finite difference approximations for the second partial derivative can also be obtained.

## Applying Finite Differences: Advection Examples

One of the most important attributes of the wind is its ability to transport. The transport of some quantity by the wind is known as *advection*. We are most often interested in its horizontal transport, or *horizontal advection*, where the horizontal surface can be taken to be Earth's surface, a constant height surface, an isobaric surface, or even an isentropic surface. For convenience, we sometimes refer to horizontal advection simply as advection.

In synoptic meteorology, we are particularly interested in *temperature advection*, referring to the horizontal transport of energy (recall that temperature is simply a measure of the average kinetic energy of the air) by the wind. Patterns of cold air advection and warm air advection reflect the (horizontal) motion of air masses and, as we will see next semester, play a crucial role in forcing vertical motions, can bring about changes in the amplitude of troughs and ridges, and can influence cyclone and anticyclone development.

Mathematically, temperature advection is expressed as the product of the appropriate component of the wind – whether east-west ($u$; also known as the zonal wind component) or north-south ($v$; also known as the meridional wind component) – and the local change of temperature in some direction – east-west ($x$) or north-south ($y$) – where:

$$advection = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} \tag{14}$$

In vector notation, (14) can be written as:

$$advection = -\vec{\mathbf{v}} \cdot \nabla T \tag{15}$$

The units of temperature advection are the units of wind – m s$^{-1}$ – multiplied by the units of temperature – generally either °C or K – divided by distance units – m. As a result, temperature advection has units of °C s$^{-1}$ or K s$^{-1}$; in other words, how temperature is changing *locally* over some finite amount of time $\Delta t$. We can evaluate (14) from charts of weather data using our centered finite difference approximation developed above.

Consider the hypothetical analysis presented in Figure 2. We are interested in computing the horizontal temperature advection at the point marked by the closed circle and wind observation. We have already completed an isotherm analysis using temperature data from this point as well as the other locations that surround it. We thus have everything we need for our calculation.

To compute horizontal temperature advection, we must first set up our $x$- and $y$-axes. Fortunately, since we are told in the Figure 2 caption that the data are plotted on a Mercator map projection, the positive $x$-axis points to the right, or due east, while the positive $y$-axis points up, or due north. Since our centered finite difference approximation is only valid over finite distances – here, $\Delta x$ and $\Delta y$ – we must set up a *small*, uniform grid centered on the wind observation location. This is done so that we can estimate the temperature at points *x+1*, *x-1*, *y+1*, and *y-1* – in other words, the terms that enter the numerator of our centered finite differences. The result of doing so is given in Figure 3.
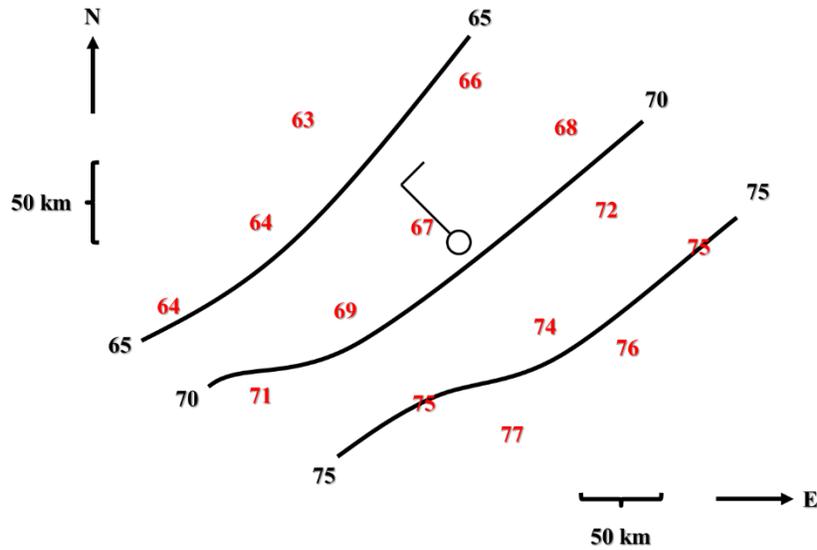
**Figure 2**. Hypothetical surface temperature observations (°F, red numbers), isotherm analysis (every 5°F, black lines), and a single wind observation (10 kt = 5.15 m s$^{-1}$ out of the northwest). Depicted for reference are horizontal scales and the north and east cardinal directions. Data are plotted on a map constructed using the Mercator map projection.
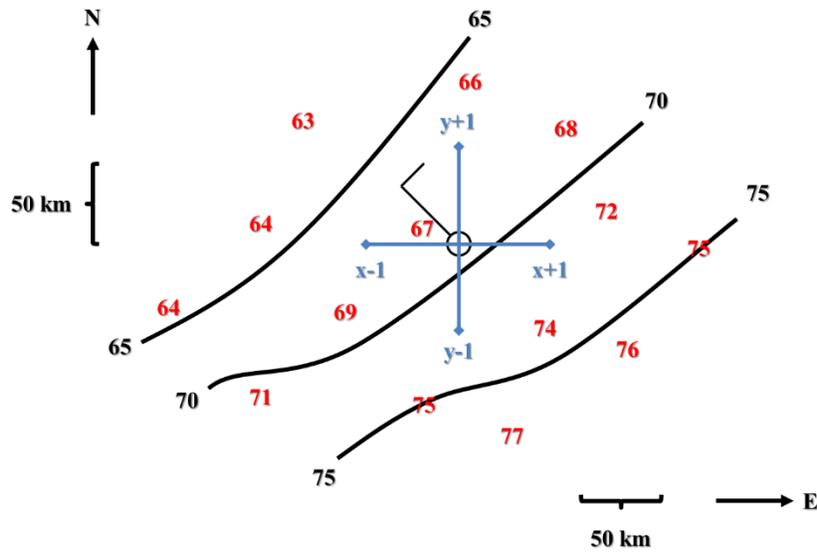


**Figure 3**. As in Figure 2, except with a finite grid drawn in centered on our wind observation. Both in this example and in practice, the distance $\Delta x$ is taken to be equal to the distance $\Delta y$. In this case, using the distance references on the edges of the map, both $\Delta x$ and $\Delta y$ are 50 km (or 50,000 m).

Next, we use our isotherm analysis to estimate the temperature at points *x+1*, *x-1*, *y+1*, and *y-1*. We must do so because we do not have an exact temperature observation at any of these locations. Visually doing so, we estimate that the temperature at *x+1* is 72°F, at *x-1* is 67°F, at *y+1* is 67°F, and at *y-1* is 73°F. This enables us to compute the finite difference approximations to our partial derivatives, where:

$$advection = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} = -u\frac{T_{x+1} - T_{x-1}}{2\Delta x} - v\frac{T_{y+1} - T_{y-1}}{2\Delta y} = -u\frac{72°F - 67°F}{100000m} - v\frac{67°F - 73°F}{100000m} \quad (16)$$

Note that, per Figure 3's caption, we know that $\Delta x = \Delta y = 50{,}000$ m, such that $2\Delta x = 2\Delta y = 100{,}000$ m. Now, we need to know the values of *u* and *v*, the zonal (east-west) and meridional (north-south) wind components, respectively. To obtain these values, we need to use a bit of trigonometry. Recall that in meteorological convention, from the north = 0°/360°, from the east = 90°, from the south = 180°, and from the west = 270°. If the wind direction (in degrees) is known, then the *u* and *v* components of the wind can be obtained using the following equations:

$$u = -\|\mathbf{v}\|\sin\left(\frac{\pi}{180} * wdir\right) \quad (17)$$

$$v = -\|\mathbf{v}\|\cos\left(\frac{\pi}{180} * wdir\right) \quad (18)$$

In both (17) and (18), $\|\mathbf{v}\|$ is the magnitude of the wind vector $\mathbf{v}$. In applied terms, $\|\mathbf{v}\|$ is simply equal to the wind speed. The *wdir* variable is the wind direction in degrees, and the $\pi/180$ factor in both statements converts wind direction from degrees to radians for trigonometric calculations.

Returning to our example in Figure 2, we know that the wind speed is equal to 10 kt = 5.15 m s$^{-1}$. We also know that the wind is from the northwest. Expressed in degrees, from the northwest = 315° (halfway between 270°/west and 360°/north). If we substitute these values into (17) and (18), we obtain:

$$u = -5.15ms^{-1}\sin\left(\frac{\pi}{180} * 315\right) = 3.64ms^{-1} \quad (19)$$

$$v = -5.15ms^{-1}\cos\left(\frac{\pi}{180} * 315\right) = -3.64ms^{-1} \quad (20)$$

A bit of a sanity check is in order before proceeding. The positive *x*-axis is to the east, while the positive *y*-axis is to the north. The wind is blowing ***from*** the north and west and, thus, ***to*** the south and east. Our wind thus blows in the positive *x* but negative *y* directions. Since *u* is along the *x*-

axis (east-west) and $v$ is along the $y$-axis (north-south), we would expect that $u$ should be positive and $v$ should be negative for a northwest wind – and, indeed, we find that this is true.

If we plug (19) and (20) into (16) and run through the calculations, we obtain:

$$advection = -3.64 ms^{-1} \frac{72°F - 67°F}{100000m} - \left(-3.64 ms^{-1}\right) \frac{67°F - 73°F}{100000m} = -0.0004 °Fs^{-1} \tag{21}$$

In other words, due solely to horizontal advection, the temperature at the location of our wind observation is cooling by 0.0004°F every second. If we multiply this by 3,600 (the number of seconds in one hour) or 84,600 (the number of seconds in one day), we can convert this to °F h$^{-1}$ or °F day$^{-1}$, respectively. Doing so, we obtain values of -1.44°F h$^{-1}$ and -34.56°F day$^{-1}$. In other words, due solely to horizontal advection, the temperature at the location of our wind observation is cooling by 1.44°F every hour and 34.56°F every day. Of course, horizontal advection is far from the only thing controlling the temperature at a given location – for instance, changes in cloud cover or time of day exert a significant influence on temperature – such that the estimated temperature change due to advection is usually much larger than the actual temperature change.

Before we proceed further, it is again time for another sanity check. In Figure 2, we see that the wind is blowing toward the station from where it is colder. As a result, we would expect the wind to be advecting (or transporting) colder air toward the observation station. Our calculation suggests that this is true – due to advection, the temperature at the observation station is cooling.

The above calculation process is a complex means of evaluating horizontal temperature advection. By contrast, our sanity check hints at another, far less complex means of doing so. Instead of using Cartesian ($x$,$y$) coordinates, as we did before, we may use a *natural coordinate system* to assess horizontal temperature advection.

Recall that in the natural coordinate system, the appropriate coordinates become $s$, or along (**streamwise**) the wind, and $n$, or **normal** to the wind. For the example given in Figure 2, the positive $s$-axis points to the southeast, in the direction that the wind is blowing, and the positive $n$-axis points to the northeast, or 90° to the left of the positive $s$-axis. Figure 4 below provides a graphical depiction of the natural coordinate system applied to the example from Figures 2 and 3.
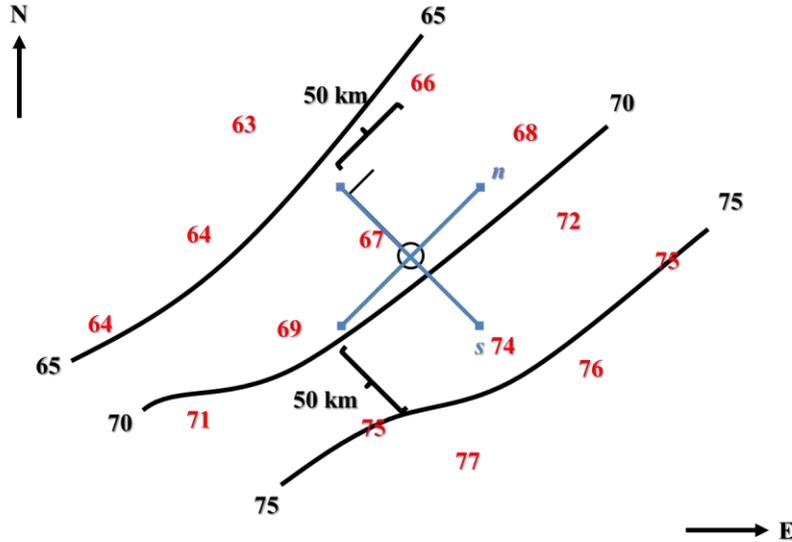
**Figure 4**. As in Figure 3, except with the finite grid drawn in the natural (rather than Cartesian) coordinate system. Both in this example and in practice, the distance $\Delta s$ is taken to be equal to the distance $\Delta n$. In this case, using the distance references on the finite grid, both $\Delta s$ and $\Delta n$ are 50 km (or 50,000 m).

In the natural coordinate system, advection is expressed mathematically as:

$$advection = -\|\mathbf{v}\|\frac{\partial T}{\partial s} = -V\frac{\partial T}{\partial s} = -V\frac{T_{s+1} - T_{s-1}}{2\Delta s} \tag{22}$$

In the above, $V$ is the *wind speed*, equivalent to the magnitude of the velocity vector ($\|\mathbf{v}\|$). There is no term in the normal direction since the wind is only directed (by definition) in the streamwise direction; this also allows for the wind speed to be used directly rather than breaking it into zonal and meridional components. Thus, (22) is a second-order–accurate finite difference approximation for horizontal temperature advection. We only need to know the wind speed – 10 kt or 5.15 m s$^{-1}$ in this example – and the change in temperature along the $s$-axis to compute the advection.

Evaluating from Figure 4, we estimate that $T_{s+1}$, or the temperature at the grid point along the positive $s$-axis, is 73.5°F and that $T_{s-1}$, or the temperature at the grid point along the negative $s$-axis, is 66°F. Plugging these values into (22), we obtain:

$$advection = -\left(5.15ms^{-1}\right)\frac{73.5°F - 66°F}{2\left(50000m\right)} = -0.000386°Fs^{-1} \tag{23}$$

Note that this result is very nearly identical to that in Equation (21), as we expect using the same data. That this is true provides a sanity check upon our result. The two are not exactly equal because

of the inherent approximate nature to each of our two analyses, namely in obtaining the values of *T* at each of our grid points.

Though this example only demonstrates two ways of computing horizontal temperature advection for a single set of data, we can nevertheless build on the concepts developed through the example to state several general rules for temperature advection in specific and advection more generally:

- Cold air advection occurs when the wind blows from cold toward warm air. Conversely, warm air advection occurs when the wind blows from warm toward cold air. In the general case, negative advection occurs when the wind blows from smaller toward larger values, whereas positive advection occurs when the wind blows from larger toward smaller values.

- When the change in advected quantity over a fixed distance is large, the magnitude of the advection is large. When the change in advected quantity over a fixed distance is small, the magnitude of the advection is small. For example, if the temperature at the *s+1* and *s-1* points in Figure 4 were 77.5°F and 62°F, respectively, instead of 73.5°F and 66°F, then the change in temperature over the 2Δ*s* interval would be 15.5°F and the advection would be -0.000798°F s$^{-1}$.

- When the wind is *parallel* to the isolines, no horizontal advection occurs. For example, if the wind in Figure 4 were from the southwest instead of from the northwest, the temperature at the *s+1* and *s-1* points would both be 68°F, and the advection would be 0°F s$^{-1}$. Conversely, when the wind is *perpendicular* to the isotherms/isohypses, horizontal advection is maximized. This is true for the examples in Figures 3 and 4.

- When the wind component blowing perpendicular to the isolines is larger, horizontal advection is larger. For example, if the wind in Figure 4 were 20 kt (10.3 m s$^{-1}$) instead of 10 kt (5.15 m s$^{-1}$), the advection would be doubled (-0.000773°F s$^{-1}$). Conversely, when the wind component blowing perpendicular to the isolines is smaller, horizontal advection is smaller.

Let us now demonstrate a qualitative application of these rules to horizontal temperature advection, colloquially known as "cherries and berries." Consider the 700 hPa observations, including wind and temperature, given in Figure 5. We can use our principles of isoplething to create a temperature (isotherm) and geopotential height (isohypse) analysis, as is depicted in Figure 6. In Figure 6, we can see that the wind generally blows parallel to the isohypses, with lower geopotential heights to the left of the wind. This information allows us to conduct our "cherries and berries" analysis at each intersection of an isohypse with an isotherm, and the result of doing so is given by the blue (for cold air advection) and red (for warm air advection) dots on Figure 6.

**Figure 5**. 700 hPa temperature (upper left), dew point temperature (lower left), geopotential height (upper right; leading 3 omitted), and wind (half-barb: 5 kt, barb: 10 kt, pennant: 50 kt) observations valid 1200 UT 18 September 2014. Image obtained from Plymouth State University.



**Figure 6**. As in Figure 5, except with isotherms (red lines every 3°C), isohypses (black lines every 30 m), and cherries (red dots; warm air advection) and berries (blue dots; cold air advection). Refer to the text for further details.

In Figure 6, note that the cherries and berries are not random but instead are organized in a distinct pattern: primarily cold air advection along the west and east coasts, primarily warm air advection in the central United States. Dots are more densely packed where the isotherms and/or isohypses are more densely packed, representing regions in which temperature changes are large over a small horizontal distance and where the wind speed is large, respectively; in this sense, dot density gives an estimate of the advection magnitude (larger where more densely packed, smaller where less so).

**A Refresher on Vector Notation**

As partial derivatives are found in nearly all aspects of synoptic meteorology, it is useful to close by reminding ourselves of some of their basic properties, particularly as it relates to vectors. In the last section, we noted that horizontal temperature advection can be written as:

$$advection = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} = -\vec{\mathbf{v}} \cdot \nabla T \tag{24}$$

The first representation is written in what is known as *component* notation, whereas the second is in what is known as *vector* notation (here, assuming that the gradient operator applies only in the horizontal direction). The two are equivalent to each other because of the definitions of the gradient operator and dot product, i.e.,

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \qquad\qquad \mathbf{a}\cdot\mathbf{b} = \left(a_x\mathbf{i} + a_y\mathbf{j}\right)\cdot\left(b_x\mathbf{i} + b_y\mathbf{j}\right) = a_x b_x + a_y b_y$$

where we have again considered only the horizontal directions in this analysis. For the dot product, $\mathbf{a}$ is simply $-\vec{\mathbf{v}} = -u\mathbf{i} - v\mathbf{j}$ and $\mathbf{b}$ is simply $\frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j}$.

There are other quantities that can be written in component or vector notation that we will consider in more detail later in the semester. For example, consider the divergence of the horizontal wind:

$$divergence = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \vec{\mathbf{v}} \tag{25}$$

This is a relatively straightforward application of the gradient operator and dot product, the proof of which is left to the student. We can also consider the vertical component of the vorticity of the horizontal wind:

$$vorticity = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \mathbf{k} \cdot \left(\nabla \times \vec{\mathbf{v}}\right) \tag{26}$$

This involves application of the gradient operator, dot product, and cross-product, where the cross-product of two vectors is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & 0 \\ b_x & b_y & 0 \end{vmatrix}$$

For this cross-product, $\mathbf{a}$ is simply $\nabla$ and $\mathbf{b}$ is simply $-\vec{\mathbf{v}} = -u\mathbf{i} - v\mathbf{j}$. Thus,

$$\nabla \times \vec{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} \dfrac{\partial}{\partial y} & 0 \\ v & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} \dfrac{\partial}{\partial x} & 0 \\ u & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ u & v \end{vmatrix}$$

$$= \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right)$$

Taking the dot product of this result with $\mathbf{k}$ retains only the $\mathbf{k}$-component of the vector, giving us the vorticity as defined in (26).

Together, the gradient operator, dot product, and cross-product – and, specifically, the divergence, advection, and vorticity applications thereof – represent the most commonly used vector properties in synoptic meteorology. We will introduce others as warranted later in this and next semesters.

Finally, we will often consider total derivatives, or terms like $\dfrac{D(\ )}{Dt}$. In their component forms, in Cartesian coordinates and excluding the vertical dimension, the total derivative is given by:

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y}$$

The first right-hand–side term represents the local change in the variable, whereas the remaining right-hand-side terms are equal to the negative of the advection of the variable. In some cases, the total derivative in an equation is written in component form, with the local change term remaining on one side of the equality and the advection terms being moved to the other side (acquiring their leading negative sign in so doing). Together, the total derivative is meteorologically interpreted as the change in the variable *following the motion* (i.e., tracking along with the air, not at a specific location).