Model Formulation and Resolved versus Unresolved Scales

The Equations

Modern NWP models solve, in their full complexity, what are known as the **primitive equations**. The primitive equations describe atmospheric motions as well as the conservation of mass, energy, and water vapor. Written in Cartesian coordinates with *z* (height) as the vertical coordinate, the primitive equations can be written as:

$$\frac{\partial u}{\partial t} = -\mathbf{v}\cdot\nabla u + \frac{uv\tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega\left(w\cos\phi - v\sin\phi\right) + Fr_x \quad (1)$$

$$\frac{\partial v}{\partial t} = -\mathbf{v}\cdot\nabla v - \frac{u^2\tan\phi}{a} - \frac{vw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + Fr_y \quad (2)$$

$$\frac{\partial w}{\partial t} = -\mathbf{v}\cdot\nabla w + \frac{u^2+v^2}{a} - g - \frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi + Fr_z \quad (3)$$

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} + w\left(\Gamma - \Gamma_d\right) + \frac{1}{c_p}\frac{dH}{dt} \quad (4)$$

$$\frac{\partial \rho}{\partial t} = -\mathbf{v}\cdot\nabla\rho - \rho\left(\nabla\cdot\mathbf{v}\right) \quad (5)$$

$$\frac{\partial q_v}{\partial t} = -\mathbf{v}\cdot\nabla q_v + Q_v \quad (6)$$

$$p = \rho RT \quad (7)$$

In the above, note that **v** is the three-dimensional velocity vector.

Equations (1) through (3) are the momentum equations, themselves being an application of Newton's second law of motion. The left-hand side terms are local time rate of change terms. The terms on the right-hand side of these equations include advection terms, curvature terms (those involving *a*, the radius of the Earth), pressure gradient terms (those involving *p*), Coriolis terms (those involving Ω), frictional terms (the *Fr* terms), and gravity (those involving *g*).

Equation (4) is a statement of the conservation of energy, manifest as the thermodynamic equation; e.g., energy is not created or destroyed. The left-hand side of this equation is the local time rate of change of temperature, while terms on the right-hand side represent horizontal advection, vertical advection (for $\Gamma = -\frac{\partial T}{\partial z}$), adiabatic cooling/warming (involving Γ_d), and diabatic heating (where *dH/dt* is the diabatic heating rate).

Equation (5) is a statement of the conservation of mass, manifest as the continuity equation; e.g., mass is not created or destroyed. The left-hand side of this equation is the local time rate of change of density, defined as mass per unit volume. On the right-hand side of this equation are advection and divergence terms. Oftentimes, (5) will be written in **flux form**, where:

$$-\mathbf{v}\cdot\nabla\rho - \rho\left(\nabla\cdot\mathbf{v}\right) = -\nabla\cdot\left(\rho\mathbf{v}\right)$$

We will revisit this distinction in the context of the WRF-ARW model governing equations shortly.

Equation (6) is a statement of the conservation of water vapor. The left-hand side of this equation is the local time rate of change of water vapor mixing ratio. On the right-hand side of this equation is an advection term and a source/sink term Q_v that includes conversions between microphysical species (e.g., rain, cloud water, cloud ice, snow, graupel, and hail).

Full physics models, however, contain **many** additional equations akin to (6), with one for each microphysical variable that is treated prognostically by the chosen microphysical parameterization used by the model simulation. For example, a model simulation conducted using the WSM6 microphysical parameterization, which predicts mixing ratio for water vapor q_v, rain q_r, cloud water q_c, cloud ice q_i, snow q_s, and graupel/hail q_g, will have six such equations. A model simulation conducted using the WDM6 microphysical parameterization, which also predicts number concentration for rain N_r, cloud water N_c, and cloud condensation nuclei N_{CCN}, will have nine such equations.

The source and sink term in (6), or its counterpart for other microphysical variables, includes both grid-resolved and, to much greater extent, sub-grid processes. As we will see later this semester, there are *many* processes that must be accounted for by such terms, making equations such as (6) far more complex than they would otherwise seem.

Equation (7) is the ideal gas law.

In the form presented in (1) – (7), we have seven equations with seven unknowns: u, v, w, T, p, ρ, and q_v. We have **prognostic** (or predictive) equations for six of these unknowns, with pressure obtained from the **diagnostic** ideal gas law. Thus, with appropriate numerical methods and some means of representing $Fr_{x,y,z}$, dH/dt, and Q_v (plus other microphysical source/sink terms), we could solve the primitive equations so as to obtain a forecast valid at some future time.

The Equations Manifest in the WRF-ARW Model

Let us now contrast these equations with their manifestation in the WRF-ARW model (Skamarock et al. 2019, Section 2.2). First, however, a few notes and definitions.

The WRF-ARW model uses a terrain-following vertical coordinate η that is defined primarily as a function of dry hydrostatic pressure (i.e., the pressure of dry air under hydrostatic balance):

$$\eta = \frac{p_{dh} - p_{dht}}{p_0 - p_{dht}} + B(\eta)\left[1 - \frac{p_{dhs} - p_{dht}}{p_0 - p_{dht}}\right] \text{ (where } 0 \le \eta \le 1)$$

In the above, p_{dh} is the dry hydrostatic pressure, p_{dht} is the dry hydrostatic pressure at the top of the model (generally a user-defined parameter), p_{dhs} is the dry hydrostatic pressure at the surface, p_0 is a reference sea-level pressure, and $B(\eta)$ defines the relative weighting between a purely isobaric vertical coordinate and a terrain-following vertical coordinate. At locations where $p_{dhs} \sim p_0$ (where the surface is near sea-level; i.e., flat terrain), the vertical coordinate reduces to the first right-hand side term in the equation above. In all cases, η is 0 at the top of the model (where $p_{dh} = p_{dht}$ and $B(\eta) = 0$) and 1 at the surface (where $p_{dh} = p_{dhs}$ and $B(\eta) = 1$).

The general form of the coordinate transformation between the height and terrain-following vertical coordinates is given by:

$$\nabla_\eta(\ ) = \nabla_z(\ ) + \frac{\partial(\ )}{\partial z}\nabla_\eta z \qquad \text{(for horizontal coordinate transforms)}$$

$$\frac{\partial(\ )}{\partial \eta} = \frac{\partial z}{\partial \eta}\frac{\partial(\ )}{\partial z} \qquad \text{(for vertical coordinate transforms)}$$

Subscripts on the gradient operator denote the vertical coordinate surface on which it is applied.

For example, converting the *x*-direction pressure gradient term in (1) into the terrain-following vertical coordinate:

$$\frac{\partial p}{\partial x}_\eta = \frac{\partial p}{\partial x}_z + \frac{\partial p}{\partial z}\frac{\partial z}{\partial x}_\eta$$

By definition, the total derivative is equal to the local rate of change plus a series of advection terms. The vertical coordinate surface transformation for advection terms is encapsulated by the varying definition of the total derivative between vertical coordinates. In particular, the vertical advection term varies between coordinate systems as a function of the coordinate-specific vertical velocity, defined generally as the total derivative of the vertical position vector.

The WRF-ARW model is what is known as **coupled** to the dry air mass field. This means that all variables are multiplied by the hydrostatic dry air mass per unit area in the column. This hydrostatic dry air mass is defined as the change in pressure between two vertical levels; i.e.,

$$\mu_d = \frac{\partial p_{dh}}{\partial \eta}$$

In coupled form, the three-dimensional velocity vector, moist potential temperature, and all mixing ratios take the form:

$$\mathbf{V} = \mu_d \mathbf{v} \qquad \Theta_m = \mu_d \theta_m \qquad Q = \mu_d q$$

As a consequence, the continuity equation for WRF-ARW is written in terms of this hydrostatic dry air mass. Formally, the model conserves hydrostatic dry air mass (rather than total mass), such that the continuity equation is equivalently the hydrostatic dry air mass conservation equation.

We start our discussion of the WRF-ARW model equations with the continuity equation, as it is most analogous to its form given by (5). This equation is given by equation (2.12) of Skamarock et al. (2019):

$$\frac{\partial \mu_d}{\partial t} + \nabla \cdot (\mu_d \mathbf{v}) = 0 \quad \text{(E)}$$

Equation (E) is equivalent to (5), except that it is written in terms of μ_d (hydrostatic dry air mass) rather than ρ (mass per unit volume), in flux form rather than advective form, and with all terms on the left-hand side of the equation to more clearly express the conservative nature of the equation. Here, the local rate of change of hydrostatic dry air mass is equal to the flux convergence of hydrostatic dry air mass. Note that (E) can equivalently be written as:

$$\frac{\partial \mu_d}{\partial t} + \nabla \cdot \mathbf{V} = 0 \quad \text{(E.a)}$$

In (E.a), the definition of the coupled velocity vector was used to rewrite the divergence term.

We next consider the *u*-momentum equation, given by equation (2.8) of Skamarock et al. (2019):

$$\frac{\partial U}{\partial t} + \nabla \cdot (\mathbf{V}u) + \mu_d \alpha \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d} \frac{\partial p}{\partial \eta} \frac{\partial \Phi}{\partial x} = F_u \quad \text{(A)}$$

In (A), the first term is the local time rate of change term, the second term is the flux divergence term, the third and fourth terms are pressure gradient terms, and the forcing term on the right-hand side of the equation includes Coriolis, curvature, and frictional terms. $\Phi = gz$ is the geopotential, $\alpha = \rho^{-1}$ is the inverse density, $\alpha_d = \rho_d^{-1}$ is the inverse dry air density, and α is related to α_d by $\alpha = \alpha_d(1+q)^{-1}$, where q refers to the sum of all microphysical species' mixing ratios.

We do not consider the exact forms of the Coriolis and curvature terms at this time because they depend upon the chosen **map projection**. We will cover map projections in more detail in our next lecture.

The most evident way in which (A) differs from (1) is that it is written in terms of the coupled form of u, U. Thus, to obtain (A), (1) was multiplied by the hydrostatic dry air mass. Further, we note that (A) contains a flux term whereas (1) contains an advection term. Recall the definition of the flux term, here written in terms of the relevant variables:

$$-\mathbf{V} \cdot \nabla u - u(\nabla \cdot \mathbf{V}) = -\nabla \cdot (\mathbf{V}u)$$

Thus, it is natural to ask: where did the divergence term go? Consider equation (E.a). Solving that equation for the divergence term, we find that it is equal to $-\frac{\partial \mu_d}{\partial t}$. Multiplying this result by u, as it appears above, results in $-u\frac{\partial \mu_d}{\partial t}$. Next, by making use of the definition of the coupled velocity vector and the product rule, we can expand $\frac{\partial U}{\partial t}$ as follows:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t}(\mu_d u) = u\frac{\partial \mu_d}{\partial t} + \mu_d \frac{\partial u}{\partial t}$$

Note the presence of a $+u\frac{\partial \mu_d}{\partial t}$ term in this expansion. This term exactly balances that from the expansion of the flux divergence term. Thus, if we substitute the above two expansions into (A), we obtain:

$$\mu_d \frac{\partial u}{\partial t} + \mathbf{V}\cdot\nabla u + \mu_d \alpha \frac{\partial p}{\partial x} + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial \eta}\frac{\partial \Phi}{\partial x} = F_u \qquad \text{(A.a)}$$

In (A.a), as in (1), there is a local time rate of change term and an advection term. The only difference in these terms between (A.a) and (1) is that the former is coupled to the hydrostatic dry air mass. As a result, we state that (A) is equivalent to (1) for these terms.

Note that the *v*-momentum, *w*-momentum, thermodynamic, and mixing ratio conservation equations all involve similar cancellation of a divergence term. We will discuss the forms of these equations shortly.

Continuing with (A), note that (A) contains two pressure gradient terms whereas (1) contains only one such term. This arises as a result of the vertical coordinate transformation from the *z* coordinate to the η coordinate. To demonstrate this, we start with the form of the pressure gradient term in (1), applicable on constant height surfaces, after substituting $\alpha = \rho^{-1}$:

$$-\alpha \left.\frac{\partial p}{\partial x}\right|_z$$

Apply the definition of the vertical coordinate transformation to the partial derivative in this term:

$$\left.\frac{\partial p}{\partial x}\right|_z = \left.\frac{\partial p}{\partial x}\right|_\eta - \frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\eta \text{, such that } -\alpha\left.\frac{\partial p}{\partial x}\right|_z = -\alpha\left.\frac{\partial p}{\partial x}\right|_\eta + \alpha\frac{\partial p}{\partial z}\left.\frac{\partial z}{\partial x}\right|_\eta$$

Next, substitute for ∂z with the definition of the geopotential, Φ, where $\partial\Phi = g\partial z$:

$$-\alpha\left.\frac{\partial p}{\partial x}\right|_\eta + \alpha g\frac{\partial p}{\partial \Phi}\frac{1}{g}\left.\frac{\partial \Phi}{\partial x}\right|_\eta \text{, or simply } -\alpha\left.\frac{\partial p}{\partial x}\right|_\eta + \alpha\frac{\partial p}{\partial \Phi}\left.\frac{\partial \Phi}{\partial x}\right|_\eta$$

The hydrostatic equation applicable on the model vertical coordinate is given by equation (2.15) of Skamarock et al. (2019):

$$\frac{\partial \Phi}{\partial \eta} = -\alpha_d \mu_d \qquad \text{(H)}$$

Equation (H) can be obtained from the hydrostatic equation applicable on constant height surfaces by transforming the vertical coordinate using the appropriate coordinate transform from page three, applying the definitions of the geopotential Φ and hydrostatic dry air mass μ_d, and simplifying the resulting equation.

If we rearrange (H) to solve for $\partial\Phi$ and plug the result from doing so into the $\frac{\partial p}{\partial\Phi}$ term in the *u*-momentum equation, we obtain:

$$\partial\Phi = -\alpha_d \mu_d \partial\eta \text{, such that we obtain } -\alpha \frac{\partial p}{\partial x}_\eta - \frac{\alpha}{\alpha_d}\frac{1}{\mu_d}\frac{\partial p}{\partial\eta}\frac{\partial\Phi}{\partial x}_\eta$$

Finally, multiply this equation by μ_d to couple it to the hydrostatic dry air mass to obtain:

$$-\alpha\mu_d \frac{\partial p}{\partial x}_\eta - \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial\eta}\frac{\partial\Phi}{\partial x}_\eta$$

Moving these terms from the right-hand side of (1) to the left-hand side gives us the pressure gradient terms as they appear in (A). Thus, we have demonstrated that (1) and (A) are functionally equivalent.

The same principles apply to the *v*- and *w*-momentum equations, given by equations (2.9) and (2.10) of Skamarock et al. (2008):

$$\frac{\partial V}{\partial t} + \nabla\cdot(\mathbf{V}v) + \mu_d\alpha\frac{\partial p}{\partial y} + \frac{\alpha}{\alpha_d}\frac{\partial p}{\partial\eta}\frac{\partial\Phi}{\partial y} = F_v \quad \text{(B)}$$

$$\frac{\partial W}{\partial t} + \nabla\cdot(\mathbf{V}w) - g\frac{\alpha}{\alpha_d}\frac{\partial p}{\partial\eta} + g\mu_d = F_u \quad \text{(C)}$$

Note that only one pressure gradient term appears in (C). This is because the vertical coordinate transformation for ∂z only results in one term, whereas those for ∂x and ∂y result in two terms. Specifically,

$$-\alpha\frac{\partial p}{\partial z} = -\alpha\frac{\partial\eta}{\partial z}\frac{\partial p}{\partial\eta} = -g\alpha\frac{\partial\eta}{\partial\Phi}\frac{\partial p}{\partial\eta} = \frac{g\alpha}{\mu_d\alpha_d}\frac{\partial p}{\partial\eta}$$

In the above, we first transformed the vertical coordinate, then applied the definition of the geopotential to the result, then substituted from the hydrostatic equation (H). Multiplying this result by the hydrostatic dry air mass μ_d and moving it to the left-hand side of the equation results in the pressure gradient term as it appears in (C).

In WRF-ARW, the thermodynamic equation is written in terms of moist potential temperature Θ_m,

$$\Theta_m = \mu_d\theta_m \text{, where } \theta_m = \theta\left(1 + \frac{R_v}{R_d}q_v\right) = \theta(1 + 1.61q_v)$$

where q_v (the water vapor mixing ratio) is dimensionless. For a typical maximum q_v = 40 g kg^{-1} = 0.04, $\theta_m = 1.0644\theta$; in other words, differences between θ and θ_m are generally small. As is θ, θm is a conserved quantity for dry adiabatic motions (i.e., conditions in which q_v is constant following the flow). The resulting equation is given by equation (2.11) of Skamarock et al. (2019):

$$\frac{\partial \Theta_m}{\partial t} + \nabla \cdot \left(\mathbf{V} \theta_m \right) = F_{\Theta_m} \qquad \text{(D)}$$

Note that Θ_m, in the first and last terms of (D), is the coupled moist potential temperature, whereas θ_m in the second term of (D) is the uncoupled moist potential temperature. The right-hand side of (D) reflects diabatic processes. Thus, absent diabatic processes, (D) simplifies to the conservation statement for θ_m, as coupled to the hydrostatic dry air mass.

The conservation equation for the various microphysical species is nearly identical to (6) and is given by equation (2.14) of Skamarock et al. (2019):

$$\frac{\partial Q_m}{\partial t} + \nabla \cdot \left(\mathbf{V} q_m \right) = F_{Q_m} \qquad \text{(F)}$$

In (F), Q_m is the coupled mixing ratio and q_m is the uncoupled mixing ratio. Here, m is taken to be one of the allowable microphysical species, and there is one equation like (F) for each. The right-hand side of (F) reflects the source/sink term for the given microphysical specie.

There also exists within WRF-ARW a prognostic equation for the geopotential, given by equation (2.13) of Skamarock et al. (2019). Unlike the other equations, this equation is written in advective form. This equation is obtained by simply taking the total derivative of the geopotential:

$$\Phi = gz \text{, such that } \frac{D\Phi}{Dt} = g\frac{Dz}{Dt}$$

The right-hand side of this equation is equal to gw, since Dz/Dt is defined as w. In terms of the coupled variables, this can be written as:

$$\frac{g}{\mu_d} W$$

The left-hand side of the definition of the geopotential can be expanded into local time rate of change and advection terms by using the definition of the total derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \qquad \text{(in terms of uncoupled fields)}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{1}{\mu_d} \mathbf{V} \cdot \nabla \qquad \text{(in terms of coupled fields)}$$

Making this substitution, in terms of coupled fields, and moving the right-hand side above to the left-hand side, we obtain the prognostic equation for the geopotential:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{\mu_d} \left(\mathbf{V} \cdot \nabla \Phi - gW \right) = 0 \qquad \text{(I)}$$

Note that the geopotential is not a conserved quantity.

The final equation is a diagnostic equation for p given by the ideal gas law. We start with the ideal gas law for dry air, such that $R = R_d$ and $\rho = \alpha^{-1} = \alpha_d^{-1}$. For now, we do not substitute for temperature with virtual temperature, such that we obtain:

$$p\alpha_d = R_d T$$

Poisson's equation allows us to rewrite T in terms of θ:

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}}, \text{ such that } T = \theta\left(\frac{p}{p_0}\right)^{\frac{R_d}{c_p}} = \theta p^{\frac{R_d}{c_p}} p_0^{-\frac{R_d}{c_p}}$$

Substituting this expression into the ideal gas law, we obtain:

$$p\alpha_d = R_d \theta p^{\frac{R_d}{c_p}} p_0^{-\frac{R_d}{c_p}} \text{ or, grouping } p \text{ terms, } pp^{-\frac{R_d}{c_p}} = \frac{R_d}{\alpha_d}\theta p_0^{-\frac{R_d}{c_p}}$$

The left-hand side of this equation can equivalently be written as $p^{1-\frac{R_d}{c_p}}$.

Note that $R_d = c_p - c_v$ (1005.7 J kg^{-1} K^{-1} – 719 J kg^{-1} K^{-1} = 286.7 J kg^{-1} K^{-1}), such that $1-\frac{R_d}{c_p} = 1-\frac{c_p - c_v}{c_p} = 1-1+\frac{c_v}{c_p} = \frac{c_v}{c_p}$ and $-\frac{R_d}{c_p} = -\frac{c_p - c_v}{c_p} = -1+\frac{c_v}{c_p}$. If we let $\gamma = \frac{c_p}{c_v}$, such that $\frac{1}{\gamma} = \frac{c_v}{c_p}$, then $1-\frac{R_d}{c_p} = \frac{1}{\gamma}$ and $-\frac{R_d}{c_p} = -1+\frac{1}{\gamma} = \frac{1-\gamma}{\gamma}$.

If we substitute this into the ideal gas law, we obtain:

$$p^{\frac{1}{\gamma}} = \frac{R_d}{\alpha_d}\theta p_0^{\frac{1-\gamma}{\gamma}}$$

If we then raise both sides of this expression to the power of γ, we obtain:

$$p = \left(\frac{R_d}{\alpha_d}\theta\right)^{\gamma}\left(p_0^{\frac{1-\gamma}{\gamma}}\right)^{\gamma}$$

By definition, we can write:

$$\left(p_0^{\frac{1-\gamma}{\gamma}}\right)^{\gamma} = p_0^{\gamma\frac{1-\gamma}{\gamma}} = p_0^{1-\gamma} = p_0 p_0^{-\gamma}$$

Thus, we obtain:

$$p = \left(\frac{R_d}{\alpha_d} \theta \right)^{\gamma} p_0 p_0^{-\gamma} = p_0 \left(\frac{R_d \theta}{\alpha_d p_0} \right)^{\gamma}$$

To account for the effects of moisture upon pressure, we replace θ with θ_m, or the moist potential temperature, as defined earlier. Earlier, we did not substitute a moisture-based temperature variable in the ideal gas law when we substituted R_d for R. We now make that substitution, simply replacing θ with θ_m. This results in our final form of the ideal gas law, given by equation (2.16) of Skamarock et al. (2019):

$$p = p_0 \left(\frac{R_d \theta_m}{\alpha_d p_0} \right)^{\gamma} \tag{G}$$

We thus have a closed set of nine equations, (A) through (I), with nine unknowns: U, V, W, Θ, μ_d, Φ, Q_m, α_d, and p. The first seven of these unknowns are prognosed; α_d is diagnosed from Φ and μ_d, while p is diagnosed from Θ, Q_v, α_d, and μ_d. To first order, these are the equations solved by WRF-ARW so as to obtain a forecast. We will revisit these equations in later lectures.

Resolved versus Unresolved Scales

Formally, these equations are valid for all scales of motion. Without loss of generality, however, it is helpful to formulate these equations in a way such that they are valid on the scales of the model grid and larger. Thus, we wish to perform a *scale separation* on these equations, separating the resolved from the unresolved scales. For any scalar dependent (or prognostic) variable, the variable can be represented as the sum of mean and perturbation terms, where *mean* refers to the resolved scales (e.g., grid-scale; average over a grid box) and *perturbation* refers to the unresolved scales (e.g., sub-grid-scale):

$$x = \overline{x} + x'$$

The prognostic variable and its perturbation component typically vary in all four dimensions (x, y, z, and t). The mean component is often, but not always, a function of three or fewer dimensions; e.g., a horizontally homogeneous, temporally constant base state is only a function of z.

We now wish to perform this scale separation on the primitive equations. Let us do so in the context of how they appear in (1) through (7), though we note that the same scale separation can be applied to (A) through (I) as well. The course text illustrates this scale separation for the u-momentum equation (1); let us instead do so for the v-momentum equation (2).

First, however, we wish to expand the frictional term. This term can be written in terms of frictional stresses τ, or viscous forces (in the form of molecular diffusion) that arise from molecular motions. Frictional stresses transfer quantities down-gradient, from high toward low values, and thus act to homogenize a field.

Representing the frictional term Fr_y in (2) in terms of these stresses, we obtain:

$$Fr_y = \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)$$

A given frictional stress $\tau_{\_y}$ represents friction that is exerted on the flow in the *y*-direction – the meridional wind *v* – by the fluid (or molecules) on one side of a constant _-plane as they flow along or across the fluid (or molecules) on the other side of a constant _-plane. This is depicted in Figure 1 below for τ_{xy}.

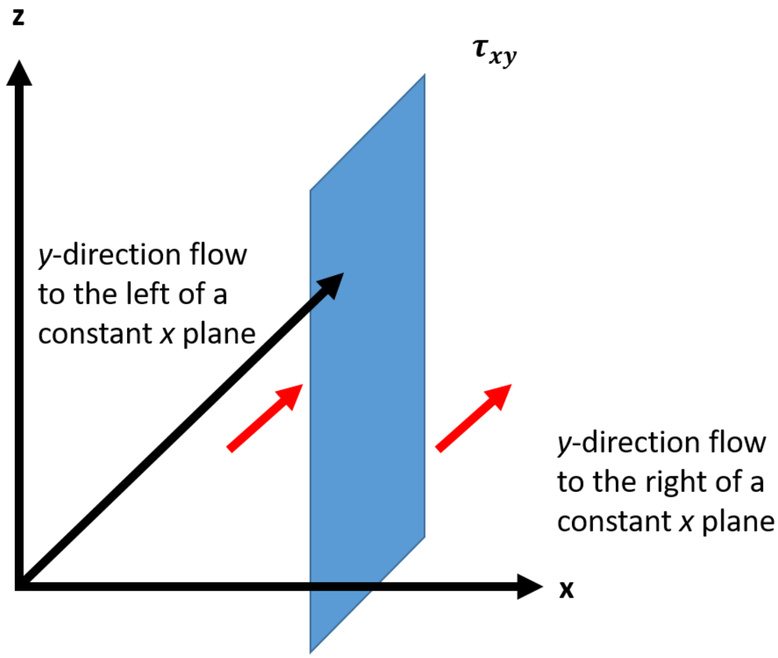

Figure 1. Conceptual illustration of flow on opposite sides of a constant *x* plane (blue; flow vectors in red) comprising the frictional stress τ_{xy}. Frictional stresses relative to constant *y* and *z* planes are similarly construed, except with the constant plane rotated accordingly.

These frictional stresses can be expressed as a function of wind shear and a frictional (or dynamic viscosity) coefficient *k*, i.e.,

$$\tau_{xy} = k\frac{\partial v}{\partial x} \qquad \tau_{yy} = k\frac{\partial v}{\partial y} \qquad \tau_{zy} = k\frac{\partial v}{\partial z}$$

Substituting, we obtain:

$$Fr_y = \frac{k}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \frac{k}{\rho}\nabla^2 v$$

In the discussion that follows, however, we will utilize the formulation for Fr_y in terms of τ. We will discuss friction in greater detail when we cover planetary boundary layer, surface layer, and land-surface parameterizations.

Substituting into (2), we obtain:

$$\frac{\partial v}{\partial t} = -\mathbf{v} \cdot \nabla v - \frac{u^2 \tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi + \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \quad \text{(a)}$$

The dependent variables in (a) are u, v, w, ρ, p, τ_{xy}, τ_{yy}, and τ_{zy}. If we decompose these into mean (grid-scale-resolved) and perturbation (sub-grid-scale) components, we obtain the following:

$$\frac{\partial(\overline{v}+v')}{\partial t} = -(\overline{\mathbf{v}}+\mathbf{v}')\cdot\nabla(\overline{v}+v') - \frac{(\overline{u}+u')^2\tan\phi}{a} - \frac{(\overline{u}+u')(\overline{w}+w')}{a} - \frac{1}{(\overline{\rho}+\rho')}\frac{\partial(\overline{p}+p')}{\partial y} - 2\Omega(\overline{u}+u')\sin\phi$$
$$+\frac{1}{(\overline{\rho}+\rho')}\left(\frac{\partial(\overline{\tau_{xy}}+\tau_{xy}')}{\partial x} + \frac{\partial(\overline{\tau_{yy}}+\tau_{yy}')}{\partial y} + \frac{\partial(\overline{\tau_{zy}}+\tau_{zy}')}{\partial z}\right)$$

We make the assumption that $\rho' << \overline{\rho}$ (i.e., density perturbations are small relative to the base-state or grid-scale density). We also note that $f = 2\Omega\sin\phi$. If we expand the above with this in mind, we obtain:

$$\begin{aligned}
\frac{\partial \overline{v}}{\partial t} + \frac{\partial v'}{\partial t} = & -\overline{u}\frac{\partial \overline{v}}{\partial x} - \overline{u}\frac{\partial v'}{\partial x} - u'\frac{\partial \overline{v}}{\partial x} - u'\frac{\partial v'}{\partial x} - \overline{v}\frac{\partial \overline{v}}{\partial y} - \overline{v}\frac{\partial v'}{\partial y} - v'\frac{\partial \overline{v}}{\partial y} - v'\frac{\partial v'}{\partial y} \\
& -\overline{w}\frac{\partial \overline{v}}{\partial z} - \overline{w}\frac{\partial v'}{\partial z} - w'\frac{\partial \overline{v}}{\partial z} - w'\frac{\partial v'}{\partial z} \\
& -\frac{\tan\phi}{a}\left(\overline{u}\overline{u} + 2\overline{u}u' + u'u'\right) - \frac{1}{a}\left(\overline{u}\overline{w} + u'\overline{w} + \overline{u}w' + u'w'\right) \\
& -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y} - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial y} - f\overline{u} - fu' \\
& +\frac{1}{\overline{\rho}}\left(\frac{\partial \overline{\tau_{xy}}}{\partial x} + \frac{\partial \tau_{xy}'}{\partial x} + \frac{\partial \overline{\tau_{yy}}}{\partial y} + \frac{\partial \tau_{yy}'}{\partial y} + \frac{\partial \overline{\tau_{zy}}}{\partial z} + \frac{\partial \tau_{zy}'}{\partial z}\right)
\end{aligned} \quad \text{(b)}$$

To make (b) applicable only on the grid-scales of motion and larger, we take what is known as the **Reynolds average** of the equation. This involves taking the *mean* of the entire equation. Note that this should *not* be confused with linearizing the equation, as is frequently done when studying wave solutions in dynamic meteorology. While in many ways similar, we want to retain the non-linearity inherent to the primitive equations. Before we take the Reynolds average of this equation, however, it is useful to first state Reynolds' postulates:

- $\overline{a'} = 0$ (grid-scale mean of all sub-grid perturbations is zero)

- $\overline{\overline{a}} = \overline{a}$ (mean of a mean is equivalent to the mean)
- $\overline{\overline{a}b} = \overline{\overline{a}}\,\overline{b} = \overline{a}\,\overline{b}$ (similar to the previous postulate, except including a second variable)
- $\overline{\overline{a}b'} = \overline{a}\,\overline{b'} = \overline{a}0 = 0$ (because of the first postulate)

Of these, the first and last postulates are the most important, as they result in many terms in (b) becoming zero when taking the Reynolds average. However, note that $\overline{a'b'} \neq 0$. Terms such as this are **covariance** terms and are only zero if the two perturbation fields are not correlated (nominally, not physically correlated) with each other. Generally speaking, this is not the case, nor do we have *a priori* knowledge of the nature of such correlations.

Making use of the Reynolds postulates to simplify the averaged form of (b), we obtain:

$$\begin{aligned}\frac{\partial \overline{v}}{\partial t} = &-\overline{u}\frac{\partial \overline{v}}{\partial x} - \overline{u'\frac{\partial v'}{\partial x}} - \overline{v}\frac{\partial \overline{v}}{\partial y} - \overline{v'\frac{\partial v'}{\partial y}} - \overline{w}\frac{\partial \overline{v}}{\partial z} - \overline{w'\frac{\partial v'}{\partial z}} \\ &- \frac{\tan\phi}{a}\left(\overline{u}\,\overline{u} + \overline{u'u'}\right) - \frac{1}{a}\left(\overline{u}\,\overline{w} + \overline{u'w'}\right) - \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y} - f\overline{u} \\ &+ \frac{1}{\overline{\rho}}\left(\frac{\partial \overline{\tau_{xy}}}{\partial x} + \frac{\partial \overline{\tau_{yy}}}{\partial y} + \frac{\partial \overline{\tau_{zy}}}{\partial z}\right)\end{aligned} \qquad \text{(c)}$$

We can rewrite the second, fourth, and sixth terms on the right-hand side of (c) in flux form. On the sub-grid (i.e., turbulence) scale, on a constant height surface, the following continuity equation applies:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = \nabla \cdot \mathbf{v}' = 0$$

Multiply this equation by *–v'* and take the Reynolds average to obtain:

$$-\overline{v'\frac{\partial u'}{\partial x}} - \overline{v'\frac{\partial v'}{\partial y}} - \overline{v'\frac{\partial w'}{\partial z}} = 0, \text{ which can be written } \overline{-v'\left(\nabla \cdot \mathbf{v}'\right)} = 0$$

This is a divergence term, whereas the second, fourth, and sixth terms on the right-hand side of (c) comprise an advection term of the form $-\overline{\mathbf{v}' \cdot \nabla v'}$. Recalling that a flux term is equal to the advection term plus the divergence term (which, here, is equal to zero, and thus can be added without changing the result), the advection terms can be written as:

$$-\frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y} - \frac{\partial \overline{w'v'}}{\partial z}$$

We can define *turbulent stresses* (i.e., the grid-scale-resolved effects of parameterized sub-grid processes; note the different definition from the frictional stresses defined earlier) as:

$$T_{xy} = -\overline{\rho}\,\overline{u'v'} \qquad T_{yy} = -\overline{\rho}\,\overline{v'v'} \qquad T_{zy} = -\overline{\rho}\,\overline{w'v'}$$

Substituting these for the flux formulation of the advection terms and combining the resulting terms with the frictional stresses, we obtain:

$$\begin{aligned}\frac{\partial \overline{v}}{\partial t} = -\overline{u}\frac{\partial \overline{v}}{\partial x} - \overline{v}\frac{\partial \overline{v}}{\partial y} - \overline{w}\frac{\partial \overline{v}}{\partial z} - \frac{\tan\phi}{a}\left(\overline{u}\,\overline{u} + \overline{u'u'}\right) - \frac{1}{a}\left(\overline{u}\,\overline{w} + \overline{u'w'}\right) - \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y} - f\overline{u} \\ + \frac{1}{\overline{\rho}}\left(\frac{\partial\left(T_{xy} + \overline{\tau_{xy}}\right)}{\partial x} + \frac{\partial\left(T_{yy} + \overline{\tau_{yy}}\right)}{\partial y} + \frac{\partial\left(T_{zy} + \overline{\tau_{zy}}\right)}{\partial z}\right)\end{aligned} \qquad \text{(d)}$$

From left to right, the terms of (d) are the local time rate of change, grid-scale-resolved advection, curvature terms, pressure gradient term, Coriolis term, and turbulent and frictional stresses. All terms of the form $\overline{a'b'}$ are parameterized. The $\overline{\tau_{\_y}}$ terms, which explicitly refer to sub-grid-scale friction that is a function of molecular motion, are also parameterized. Other terms not involving perturbations are resolved on the model grid.

Generally, (d) is written in a form that drops the resolved-scale notation, e.g.,

$$\frac{\partial v}{\partial t} = -\mathbf{v}\cdot\nabla v - \frac{u^2\tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_v \qquad \text{(e)}$$

where F_v represents all sub-grid-scale processes, similar to its notation in (B). Otherwise, (e) closely resembles (2). The remaining equations (1) and (3) through (7) can be transformed similarly.

The Trouble with Sound Waves

Even though we have now made the equations explicitly applicable only on the model grid scale and larger, they still permit a wide range of solutions – Rossby waves, gravity waves, and sound waves, for example. Sound waves are problematic: as discussed in our last lecture, they are rapidly moving (c ~ 340 m s^{-1}) non-meteorological waves.

Recall that we stated that for numerically stable solutions to the primitive equations (non-exponential growth of the solution) to exist, the CFL criterion must be met. Here, let us assume that this criterion is $C \leq 1$. Since U in the Courant number is equal to the speed of the fastest wave on the model grid, allowing horizontally propagating sound waves to be a solution to the primitive equations poses a strict upper limit on Δt for a given Δx. For example, for Δx = 10 km,

$\frac{(340 ms^{-1})\Delta t}{(10000m)} \leq 1$, such that $\Delta t \leq 29.4$ s, whereas if sound waves are not allowed…

$$\frac{(100ms^{-1})\Delta t}{(10000m)} \leq 1 \text{, such that } \Delta t \leq 100 \text{ s}$$

The problem is even worse if we consider stability in the vertical dimension, where the Courant number and generalized CFL condition can be expressed as:

$$C = \frac{W\Delta t}{\Delta z} \leq 1$$

Here, Δz is typically on the order of a few hundred meters, while W is typically of magnitude 10 m s^{-1} or less except in convective updrafts and downdrafts. For Δz = 300 m and W = 10 m s^{-1}, the maximum allowable Δ*t* is 30 s. But, if W is now the speed of sound (340 m s^{-1}), as associated with vertically propagating sound waves, the maximum allowable Δ*t* is 0.88 s – a full 34 times smaller than that if sound waves are not permitted. This poses a substantial computational challenge.

Thus, we must deal with sound waves in some fashion. As we will discuss in a future lecture, most modern models use what is known as **split time differencing**, where a large time step is used for everything but sound waves, which are treated separately with a shorter time step. This maintains accuracy in the equations while minimizing added computational expense.

Simpler models, however, are formulated in such a way so that the sound waves *are not* permitted solutions but that some or most other meteorological waves *are* permitted solutions. How is this done? You may recall from an atmospheric dynamics course that the restoring mechanism for sound waves is their **compressibility** – in other words, they propagate by means of adiabatic compression and expansion. To eliminate sound waves, therefore, we need to remove the ability for compression – i.e., to make the equations **incompressible**.

Before we consider ways in which this may be done, let us briefly revisit what is meant by adiabatic compression and expansion. Imagine an air parcel of constant unit mass 1 kg. It has an initial air pressure $p = p_0$. Consider the case of adiabatic compression, such as may be associated with large-scale subsidence. As the air parcel descends, the environmental air pressure p_{env} increases. The force exerted upon the air parcel by the higher environmental pressure causes it to compress: volume V decreases, so density ρ, mass divided by volume, increases.

The pressure of the air parcel adjusts instantaneously to the environmental pressure; the two come into equilibrium with one another. More specifically, as denoted by Wallace and Hobbs (1977, p. 68): “Any pressure differences between the parcel and its environment give rise to sound waves which produce a rapid adjustment.”

Generally speaking, compressibility is a measure of how density (and volume) change in response to changes in pressure, or vice versa. It implies that the two are linked; ergo, incompressibility implies that density changes are independent of pressure changes. We know this not to be the case in the atmosphere, but we nevertheless can represent many atmospheric processes reasonably using an incompressible approximation.

For adiabatic flow, a generalized incompressible approximation is given by:

$$\frac{1}{\rho}\frac{D\rho}{Dt} = 0$$

In other words, density does not change following the motion.

Consider the continuity equation (5). Divide all terms by ρ, move the advection term to the left-hand side of the equation, and apply the definition of the total derivative to obtain:

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -(\nabla \cdot \mathbf{v})$$

In other words, for $\rho > 0$, convergence results in an increase in density following the motion. This is what we saw with our adiabatic compression example. By contrast, for an incompressible fluid, this equation becomes:

$$(\nabla \cdot \mathbf{v}) = 0$$

In other words, the net divergence in three dimensions must be zero; convergence in one direction must be balanced by divergence in another direction, or vice versa, rather than be balanced by a change of density following the flow. Consider the example of a full tube of toothpaste with the lid on. You squeeze the tube at the middle – convergence – but the toothpaste diverges to the left and to the right. The volume of the tube is unchanged. As a result, we say that the tube of toothpaste is incompressible…or at least approximately so.

A scale analysis of the continuity equation – see Bluestein (2013, pgs. 34-35) for details – indicates that the atmosphere may approximated as incompressible under the following conditions:

- The maximum wind speed is one or more orders of magnitude smaller than the speed of sound ($c \sim 340$ m s^{-1}).
- The vertical scales of motion are one or more orders of magnitude smaller than the scale height ($H \sim 8$-10 km).

Generally speaking, the first criterion is often met, at least approximately; maximum wind speeds are often on the order of 100 m s^{-1} or smaller. The second criterion is also often met, particularly outside of positively or negatively buoyant vertical motions. We wish to consider three means by which both criteria can be satisfied, in whole or in part, to eliminate sound waves as solutions to the primitive equations. These are the hydrostatic, Boussinesq, and anelastic approximations.

Hydrostatic Approximation

The hydrostatic approximation arises out of a scale analysis of the vertical momentum equation for synoptic-scale motions, stating that vertical parcel accelerations (as may result from buoyancy) are zero and that, implicitly, vertical motions have small magnitudes. However, this approximation filters out gravity waves, which have buoyancy as their restoring mechanism, as possible solutions to the primitive equations. As a consequence, the hydrostatic approximation is not used in modern mesoscale models except possibly as a diagnostic relationship for another model variable.

To illustrate how the hydrostatic approximation filters vertically propagating sound waves, consider the hydrostatic relationship, here equivalent to the vertical momentum equation:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

Here, we have posed the hydrostatic relationship in the form that it appears from the vertical momentum equation, rather than multiplying both sides by ρ.

Let us linearize this equation about a base state given by $p(z,t) = \overline{p} + p'(z,t)$ and $\rho(z,t) = \overline{\rho} + \rho'(z,t)$, where barred quantities represent constant base state fields and primed quantities represent deviations or perturbations from that constant base state.

For $\rho' << \overline{\rho}$, a binomial expansion of $1/\rho$ results in the following approximation:

$$\frac{1}{\rho} = \frac{1}{\overline{\rho} + \rho'} \approx \frac{1}{\overline{\rho}}\left(1 - \frac{\rho'}{\overline{\rho}}\right)$$

Noting that products of two perturbation quantities are zero, such that non-linear terms are said to be small compared to the dominant linear forcing terms, we obtain:

$$\frac{\partial p'}{\partial z} = -\overline{\rho} g$$

Integrating this expression with respect to z indicates that the pressure at any location is simply a function of the constant base state weight of the air above that location. Though adiabatic compression and expansion are permitted – i.e., $\rho' \neq 0$ – the pressure field does not respond to such changes in density because it is tied exclusively to the constant base state density. In other words, changes in pressure are decoupled from, or not linked to, changes in density. This eliminates the ability for vertically propagating sound waves to exist as solutions.

Formally, the hydrostatic approximation only filters vertically propagating sound waves, which are particularly problematic with respect to numerical stability. However, the hydrostatic approximation does not filter a special type of horizontally propagating sound wave known as the Lamb wave, for which the vertical velocity is zero. Lamb waves have maximum amplitude at the lower boundary (i.e., ground) and decay away from the boundary while the pressure and density fields remain in hydrostatic balance. Lamb waves can be filtered, however, simply by applying the lower boundary condition $\omega = 0$.

<u>Boussinesq Approximation</u>

Under this approximation, density is treated as a constant following the motion *except* where it is coupled with gravity in the buoyancy term of the w-momentum equation (3; albeit with some manipulation). This retains gravity waves as possible solutions to the primitive equations. Where density is coupled to gravity, density variations are assumed to be small perturbations of the constant base state density field.

Thus, under the Boussinesq approximation, the continuity equation is equal to the incompressible form stated above, $(\nabla \cdot \mathbf{v}) = 0$, but the *w*-momentum equation is no longer given by the hydrostatic relationship. Thus, gravity waves are allowable solutions under the Boussinesq approximation. In the *w*-momentum equation, $\rho = \rho_0 + \rho'$, where ρ_0 is the constant base state density field. A full derivation can be found in the Holton or Holton and Hakim dynamics textbook.

Under the Boussinesq approximation, the effects of compressibility are retained in the vertical momentum equation for *shallow* vertical motions of magnitude 1-2 km and smaller, since $\rho' << \rho_0$. Vertical oscillations found with gravity waves generally meet this criteria. Otherwise, the atmosphere is said to be incompressible.

<u>Anelastic Approximation</u>

The anelastic approximation relaxes the second of our constraining simplifications slightly. Namely, the scale of vertical motions is now permitted to approach that of the scale height *H* ~ 8-10 km, making the anelastic approximation a reasonable approximation for motions that are troposphere-deep or shallower. In this case, the continuity equation takes the form:

$$\frac{\partial}{\partial x}\left(\overline{\rho}u\right)+\frac{\partial}{\partial y}\left(\overline{\rho}v\right)+\frac{\partial}{\partial z}\left(\overline{\rho}w\right)=0$$

In this equation, $\overline{\rho} = \overline{\rho}(z)$, such that the base state density does not vary in the horizontal but does vary in the vertical direction.

This equation, presented in flux form, can alternatively be expressed in advective form:

$$\mathbf{v}\cdot\nabla\overline{\rho}+\overline{\rho}(\nabla\cdot\mathbf{v})=0$$

Since $\overline{\rho} = \overline{\rho}(z)$ only, the advection term in this equation can be simplified:

$$w\frac{\partial\overline{\rho}}{\partial z}+\overline{\rho}(\nabla\cdot\mathbf{v})=0$$

Dividing both sides of the equation by $\overline{\rho}$, we obtain:

$$\frac{w}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z}+(\nabla\cdot\mathbf{v})=0$$

This continuity equation is somewhat analogous to that given by (5): rather than resulting in a change in density following the motion, however, divergence results in a vertical gradient in base-state density $\overline{\rho}$. The same cannot be said for the continuity equation in the hydrostatic or Boussinesq approximations, where net divergence must be zero for incompressibility.

Again, as earlier noted, we typically do *not* make these approximations in full physics models used for real-data simulations. They are often only used in idealized numerical models or in routines in which simplified dynamics can be used to extract meaningful insight into atmospheric phenomena. Instead, split time differencing methods are employed. We will discuss these methods in greater detail in a later lecture.