Linear Numerical Stability and Implicit Numerical Damping

A Primer on Waves, Damping, and Dispersion

In this and the next lecture, we will consider implicit numerical damping and numerical dispersion, each of which impact the modeled representation of wave-like features and result from the model's chosen finite difference methods. Before we do so, however, it is helpful to establish a framework through which we can better understand these concepts.

Consider a block that is eleven units tall. Let's assume that a westerly wind of 10 m s^{-1} advects the block to the east. In the real world, this eleven-unit-tall block will move eastward at 10 m s^{-1}.

Things are a bit different in the modeled world, however. A model views this block as the sum of blocks of all different sizes. For the sake of illustration, let us assume that a model views this block as the sum of blocks that are seven, three, and one unit(s) tall. Together, these still sum to eleven, so at the outset nothing is different from the real-world representation.

The chosen finite difference methods are approximations. Though the specific details vary between differencing methods, the two most common manifestations of these approximations are in implicit numerical damping (i.e., features lose amplitude over time) and numerical dispersion (i.e., features move at different phase speeds and group velocities). As we will see, both depend on wavelength, such that features of different wavelengths can be impacted by implicit damping and dispersion to a greater or lesser extent than those of other wavelengths.

Let's return to our eleven-unit-tall block, viewed by the model as a combination of seven-, three-, and one-unit blocks. Under *implicit damping* alone, these blocks move eastward at 10 m s^{-1} but their sizes are reduced, perhaps to 6.5, 2.5, and 0.5 units, respectively. Under *numerical dispersion* alone, these blocks remain as seven, three, and one unit(s) tall, but they move eastward at different velocities, perhaps at 9.75 m s^{-1}, 9 m s^{-1}, and 7 m s^{-1}, respectively.

It is the representation of atmospheric fields not as having a single wavelength but as a combination of wavelengths each with different amplitudes that is key to understanding linear stability, implicit numerical damping, and numerical dispersion. Keep our hypothetical example in mind as we dive into the next two lectures.

Introduction to Linear Stability

In the computational sense, **stability** is defined by the temporal evolution of the model solution. Does the model solution grow exponentially with time, leading to floating point overflow – one or more model variables becoming too large for the computer to represent – and the model crashing? If it does, the model is said to be computationally unstable. In general, we assess stability in the context of identifying the conditions under which this occurs.

The *CFL criterion*, in its most general form, is a stability criterion for a linear advection term: under what conditions (U, Δt, and Δx) does the model solution become unstable? In this lecture, we will formally derive the CFL criterion for (linear) advection terms for several combinations of temporal and spatial finite differencing schemes. We will find that the CFL criterion typically varies as a function of the differencing schemes used.

As in the atmosphere, such as for vertical parcel displacements, there are three types of numerical stability that may exist, listed roughly from least to most common:

- **Absolutely unstable**: no matter what values are chosen for the dependent parameters (e.g., U, Δt, and Δx), the model will always crash.
- **Absolutely stable**: no matter what values are chosen for the dependent parameters, the model will never crash due to floating point overflow.
- **Conditionally stable**: so long as the chosen values for the dependent parameters adhere to an appropriate stability criterion, the model solution will remain stable.

All terms in the primitive equations contribute to the numerical stability of the model solution. Most problematic are the advection terms, whether they are linear (e.g., the product of a velocity and a partial derivative of a mass-related field) or non-linear (e.g., the product of a velocity and the partial derivative of that velocity) in nature. Advection terms, and in particular linear advection terms, form the basis for our investigation in this lecture.

Consider a one-dimensional advection equation for a generic variable h that is advected by a mean velocity U:

$$\left.\frac{\partial h}{\partial t}\right|_j^\tau = -U\left.\frac{\partial h}{\partial x}\right|_j^\tau$$

Here, subscripts indicate that the terms are evaluated at a point j along the x-axis, while superscripts indicate that the terms are evaluated at a time step τ.

We wish to specify harmonic, or wave-like, solutions for h of the form:

$$h = \hat{h}e^{i(kx-\omega t)}$$

Here, $\hat{h}$ is amplitude, k is a zonal wavenumber equal to $2\pi/L$, L is wavelength, and ω is frequency (s^{-1}) and equal to Uk. In the above, the exponential function specifies wave-like structure through Euler's formula, where $e^{i\theta} = \cos\theta + i\sin\theta$.

Before proceeding, it is helpful for us to note that any feature on a model grid is truly represented by the summation of an infinite number of waves, with wavenumber k from 1 (one wavelength on the model grid) to infinity (an infinite number of waves, with infinitesimally small wavelength, on the model grid), of varying amplitude. Keep this in mind as we consider wavenumber dependence in the linear stability criteria we will soon derive, as some differencing schemes will be stable for some wavelengths and unstable for others. As we will see in our next lecture, this is also important for the artificial dispersion of waves that can result from the chosen differencing scheme.

We assume that frequency ω has both real and imaginary components, such that $\omega = \omega_R + i\omega_I$. Though ω_I itself is real-valued, the leading *i* makes it imaginary. If we substitute this definition for ω into the definition for *h*, we obtain:

$$h = \hat{h}e^{i(kx-\omega t)} = \hat{h}e^{i(kx-(\omega_R+i\omega_I)t)} = \hat{h}e^{\omega_I t}e^{i(kx-\omega_R t)}$$

It is clear that the amplitude of *h* is no longer equal to a constant value $\hat{h}$ but is now a function of time, through $e^{\omega_I t}$, such that ω_I determines whether the amplitude of *h* **grows exponentially** ($\left|e^{\omega_I t}\right| > 1$, such that $\omega_I > 0$), **remains constant** ($\left|e^{\omega_I t}\right| = 1$, such that $\omega_I = 0$), or **dampens** with time ($\left|e^{\omega_I t}\right| < 1$, such that $\omega_I < 0$). Consequently, to determine linear numerical stability, we need to determine the conditions under which $\left|e^{\omega_I t}\right| \le 1$ (for stability; values less than one indicate *implicit numerical damping*) and $\left|e^{\omega_I t}\right| > 1$ (for unstable solutions). This process is known as *linear stability analysis* or, in some references, *von Neumann stability analysis*, and is but one stability assessment method.

Linear Stability of Forward-in-time, Backward-in-space Finite Differences

Let us examine the stability of the forward-in-time, backward-in-space combination of finite difference schemes. Though this is a combination of differencing schemes that is associated with particularly large truncation error, it also provides a fairly direct evaluation of numerical stability. In this case, the one-dimensional advection equation becomes:

$$\frac{h_j^{\tau+1} - h_j^{\tau}}{\Delta t} = -U\frac{h_j^{\tau} - h_{j-1}^{\tau}}{\Delta x}$$

Or, equivalently:

$$h_j^{\tau+1} - h_j^{\tau} = -\frac{U\Delta t}{\Delta x}\left(h_j^{\tau} - h_{j-1}^{\tau}\right)$$

Note that $x = j\Delta x$, such that the location is equal to the grid point *j* multiplied by the grid spacing Δx, and $t = \tau\Delta t$, such that the time is equal to the time step # τ multiplied by the time step Δt. Given map projections, the Δx in the above is that on the Earth (Δx_e), which is often smaller than that on the model grid (Δx_g).

The wave-like solution for *h* can be rewritten in terms of *j* and τ:

$$h = \hat{h}e^{\omega_I t}e^{i(kx-\omega_R t)} = \hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)}$$

If we substitute this solution into the finite difference form of the 1-D equation above, we obtain:

$$\hat{h}e^{\omega_I(\tau+1)\Delta t}e^{i(kj\Delta x-\omega_R(\tau+1)\Delta t)} - \hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)} = -\frac{U\Delta t}{\Delta x}\left(\hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)} - \hat{h}e^{\omega_I \tau\Delta t}e^{i(k(j-1)\Delta x-\omega_R \tau\Delta t)}\right)$$

Divide by a common factor of $\hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)}$ to obtain:

$$e^{\omega_I \Delta t}e^{-i\omega_R \Delta t} - 1 = -\frac{U\Delta t}{\Delta x}\left(1 - e^{-ik\Delta x}\right)$$

Note the difference in this equation from that which is equation (3.38) in the course text; here, there is a leading negative sign in the last exponential, which is correct, whereas the course text lacks such a leading negative despite obtaining the correct solution at the end.

The exponentials involving i can be rewritten using Euler's formula, where $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. Doing so, we obtain:

$$e^{\omega_I \Delta t}\left(\cos(\omega_R \Delta t) - i\sin(\omega_R \Delta t)\right) - 1 = -\frac{U\Delta t}{\Delta x}\left(1 - \left(\cos(k\Delta x) - i\sin(k\Delta x)\right)\right)$$

If we separate this equation into its real (top) and imaginary (bottom) parts, we obtain:

$$e^{\omega_I \Delta t}\cos(\omega_R \Delta t) - 1 = -\frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)$$

$$-ie^{\omega_I \Delta t}\sin(\omega_R \Delta t) = -i\frac{U\Delta t}{\Delta x}\sin(k\Delta x)$$

Or, written equivalently,

$$e^{\omega_I \Delta t}\cos(\omega_R \Delta t) = 1 - \frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)$$

$$e^{\omega_I \Delta t}\sin(\omega_R \Delta t) = \frac{U\Delta t}{\Delta x}\sin(k\Delta x)$$

We wish to combine these equations so that we can obtain an equation for $e^{\omega_I \Delta t}$. To do so, we want to eliminate ω_R, which is associated with wave propagation and dispersion. This can be done by squaring each equation and adding them together, since $\cos^2\theta + \sin^2\theta = 1$. Doing so, we obtain:

$$e^{\omega_I \Delta t}e^{\omega_I \Delta t} = \left(1 - \frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)\right)^2 + \left(\frac{U\Delta t}{\Delta x}\sin(k\Delta x)\right)^2$$

Taking the square root of both sides of this equation, we obtain:

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{\left(1 - \frac{U\Delta t}{\Delta x}\left(1 - \cos\left(k\Delta x\right)\right)\right)^2 + \left(\frac{U\Delta t}{\Delta x}\sin\left(k\Delta x\right)\right)^2}$$

Note that we have taken the absolute value of the left-hand side to keep only the positive root. We are less interested in the sign of $e^{\omega_I \Delta t}$ as we are in whether its magnitude is greater than 1.

We can expand everything under the radical as follows:

$$\left(\frac{U\Delta t}{\Delta x}\sin\left(k\Delta x\right)\right)^2 = \frac{U^2\left(\Delta t\right)^2}{\left(\Delta x\right)^2}\sin^2\left(k\Delta x\right)$$

$$\begin{aligned}\left(1 - \frac{U\Delta t}{\Delta x}\left(1 - \cos\left(k\Delta x\right)\right)\right)^2 &= 1 - 2\frac{U\Delta t}{\Delta x}\left(1 - \cos\left(k\Delta x\right)\right) + \frac{U^2\left(\Delta t\right)^2}{\left(\Delta x\right)^2}\left(1 - \cos\left(k\Delta x\right)\right)^2 \\ &= 1 - 2\frac{U\Delta t}{\Delta x} + 2\frac{U\Delta t}{\Delta x}\cos\left(k\Delta x\right) + \frac{U^2\left(\Delta t\right)^2}{\left(\Delta x\right)^2}\left(1 - 2\cos\left(k\Delta x\right) + \cos^2\left(k\Delta x\right)\right)\end{aligned}$$

Adding these two equations, substituting for the resulting $\frac{U^2\left(\Delta t\right)^2}{\left(\Delta x\right)^2}\left(\sin^2\left(k\Delta x\right) + \cos^2\left(k\Delta x\right)\right)$ term, and combining like terms, we obtain:

$$\begin{aligned}\left|e^{\omega_I \Delta t}\right| &= \sqrt{1 - 2\frac{U\Delta t}{\Delta x} + 2\frac{U\Delta t}{\Delta x}\cos\left(k\Delta x\right) + 2\frac{U^2\left(\Delta t\right)^2}{\left(\Delta x\right)^2} - 2\frac{U^2\left(\Delta t\right)^2}{\left(\Delta x\right)^2}\cos\left(k\Delta x\right)} \\ &= \sqrt{1 + 2\frac{U\Delta t}{\Delta x}\left(\cos\left(k\Delta x\right) - 1 + \frac{U\Delta t}{\Delta x} - \frac{U\Delta t}{\Delta x}\cos\left(k\Delta x\right)\right)}\end{aligned}$$

For $(a+b)(c+d) = ab + ad + bc + bd$, if $a = \cos(k\Delta x)$, $b = -1$, $c = 1$, and $d = -\frac{U\Delta t}{\Delta x}$, the terms in the parentheses underneath the radical can be simplified:

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{1 + 2\frac{U\Delta t}{\Delta x}\left[\left(\cos\left(k\Delta x\right) - 1\right)\left(1 - \frac{U\Delta t}{\Delta x}\right)\right]}$$

Recall that the value of $e^{\omega_I t}$ determines whether the amplitude of *h* grows, decays, or remains constant in time. We previously defined $t = \tau\Delta t$, such that $e^{\omega_I t} = e^{\omega_I \tau \Delta t} = \left(e^{\omega_I \Delta t}\right)^{\tau}$. Thus, the value of $\left|e^{\omega_I \Delta t}\right|$ determines how the amplitude of *h* will change *over one time step*, which is then raised to the power of τ (i.e., this amplitude change grows exponentially from one time step to the next).

The stability criterion above is a function of both the Courant number and of $k\Delta x$, which for $k = 2\pi/L$ becomes $2\pi(\Delta x/L)$, and is thus a function of the ratio of the horizontal grid spacing to the wavelength.

Let us consider a simple case: $\frac{U\Delta t}{\Delta x} = 1$. In that case, everything under the radical collapses to 1, such that $\left|e^{\omega_I \Delta t}\right|$ = 1. This is **numerically stable**, with *no change in amplitude* with time.

What about when $\frac{U\Delta t}{\Delta x} \neq 1$? Note that the largest-possible value of Δx is $L/2$, defining a grid of three points to represent the $2\Delta x$ wave. There, $k\Delta x = \pi$, with $\cos(\pi) = -1$. The smallest-possible value of Δx is approximately zero, defining a grid of an infinite number of points to resolve all waves. As $k\Delta x$ approaches zero, $\cos(k\Delta x)$ approaches 1. Thus, $\cos(k\Delta x)$ has allowable values between -1 and 1, such that $\cos(k\Delta x) - 1$ has allowable values between -2 and 0; in other words, it is always negative.

For $\frac{U\Delta t}{\Delta x} > 1$, $\left(1 - \frac{U\Delta t}{\Delta x}\right) < 0$. Thus, for $\cos(k\Delta x) - 1 < 0$, the number under the radical in the equation for $\left|e^{\omega_I \Delta t}\right|$ above is always greater than 1. Consequently, $\left|e^{\omega_I \Delta t}\right| > 1$, which defines a **numerically unstable** solution with *exponential amplitude growth* over time.

Conversely, for $\frac{U\Delta t}{\Delta x} < 1$, $\left(1 - \frac{U\Delta t}{\Delta x}\right) > 0$. Thus, for $\cos(k\Delta x) - 1 < 0$, the number under the radical in the equation for $\left|e^{\omega_I \Delta t}\right|$ above is always less than 1. Consequently, $\left|e^{\omega_I \Delta t}\right| < 1$, which defines a **numerically stable** solution with (implicit) *exponential amplitude damping* over time.

Thus, for the backward-in-space, forward-in-time differencing scheme, the stability criterion is given by the generic form of the CFL criterion:

$$\frac{U\Delta t}{\Delta x} \leq 1$$

The precise degree to which the wave's amplitude grows or dampens over time is *a function of its wavelength* given the $\cos(k\Delta x)$ term that appears in the equation for $\left|e^{\omega_I \Delta t}\right|$ above. Recall: $k = 2\pi/L$, such that $k\Delta x$ is equal to $2\pi\Delta x/L$. Thus, a wave of wavelength $2\Delta x$ will have $k\Delta x = \pi$ (with $\cos \pi = -1$) while a wave of wavelength $20\Delta x$ will have $k\Delta x = \pi/10$ (with $\cos \pi/10 = 0.951$). The precise degree to which the wave's amplitude grows or dampens over time is also a function of the Courant number itself, given the relationship to $\frac{U\Delta t}{\Delta x}$ in the expression for $\left|e^{\omega_I \Delta t}\right|$.

Both of these dependencies are illustrated in Fig. 1, plotting the damping magnitude per time step for ten selected waves of wavelengths 2Δ*x* to 50Δ*x* over a range of stable Courant numbers. *For this specific combination of differencing schemes*, shorter wavelengths are damped to greater extent per time step than are longer wavelengths. Stated equivalently, as the horizontal grid spacing becomes small relative to the wavelength (e.g., more points to resolve the wave), the damping becomes smaller over the range of stable Courant numbers. There is no damping at Courant numbers of 0 or 1 with maximum damping for a Courant number of 0.5. Although not shown, exponential growth for Courant numbers greater than 1 is greatest for short wavelengths, just as is the damping for Courant numbers less than 1.

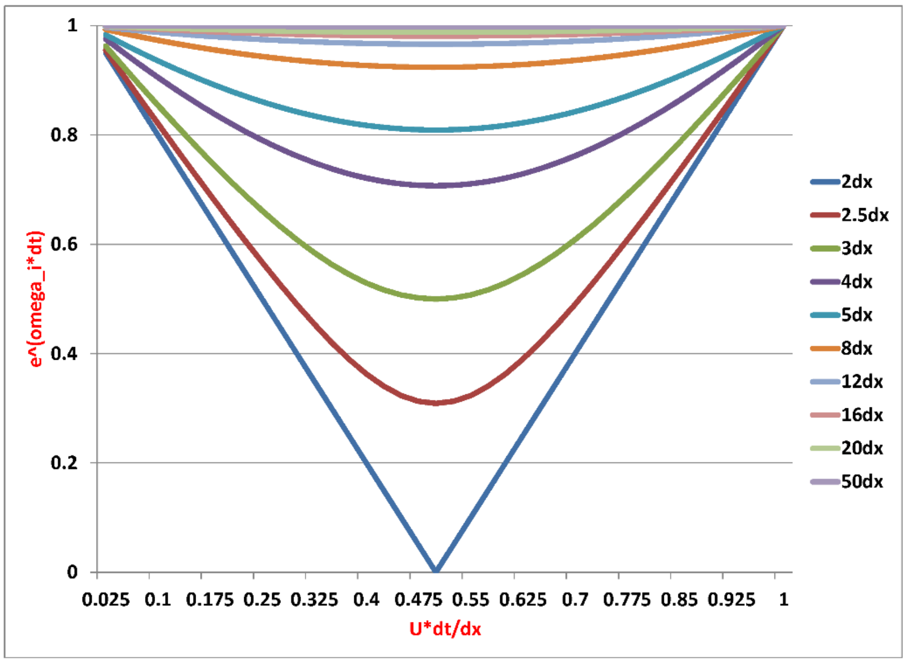

Figure 1. The value of $\left|e^{\omega_I \Delta t}\right|$ as a function of Courant number (numerically stable values only) for waves of wavelength between 2Δ*x* and 50Δ*x*. Please refer to the text for further details.

Linear Stability for Other Spatial and Temporal Differencing Schemes

The numerical stability of any combination of spatial and temporal differencing schemes can be assessed using the process outlined above. The course text describes this in some detail for the centered-in-time, centered-in-space scheme and states only the end results for the forward-in-time, second-order-accurate centered-in-space and centered-in-time, fourth-order-accurate centered-in-space differencing schemes. Here, we consider only basic insight for each; please refer to the course text, or consider conducting the derivations yourself, for more details.

(1) Forward-in-time, centered-in-space

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{1+\left(\frac{U\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x)}$$

Both $\left(\frac{U\Delta t}{\Delta x}\right)^2$ and $\sin^2(k\Delta x)$ are positive-definite, such that the value under the radical is greater than 1 for all values of $k\Delta x$ and $\frac{U\Delta t}{\Delta x}$. Consequently, no matter the time step, this combination of differencing schemes is **numerically unstable**. As a result, this scheme is only used to advance the model for the first model time step when a centered-in-time scheme is used with the centered-in-space scheme, with the amplitude growth between the two time steps being acceptably small for this single instance.

(2) Centered-in-time (leapfrog), second-order-accurate centered-in-space

$$\frac{U\Delta t}{\Delta x}\sin(k\Delta x) \le 1$$

As was stated above, the allowable values of Δx range from $L/2$ to ~0, such that the allowable values of $k\Delta x$ range from π to ~0. The sin function in both cases evaluates to 0. Between 0 and π, the maximum value of $\sin(k\Delta x)$ is 1, which occurs when $k\Delta x = \pi/2$ (for $\Delta x = L/4$). This allows us to more generally state the stability criterion as:

$$\frac{U\Delta t}{\Delta x} \le 1$$

No matter the value of the sin function (between 0 and 1), so long as this criterion is met, numerical stability will be ensured.

(3) Centered-in-time (leapfrog), fourth-order-accurate centered-in-space

$$\frac{U\Delta t}{\Delta x} \le 0.73$$

Note that it can be shown that $\left|e^{\omega_I \Delta t}\right| = 1$ for all stable values of the Courant number for both (2) and (3). In other words, those schemes are either numerically stable *without damping* or they are numerically unstable, and this is true of all centered even-order-accurate differencing schemes. In contrast, all odd-order-accurate time and space differencing schemes are associated with implicit damping for stable values of the Courant number, with the specific damping magnitude dependent on wavelength and the Courant number.

The WRF-ARW Technical Document, as reproduced from Wicker and Skamarock (2002), lists the stability criteria for a wide range of spatial and temporal differencing schemes relative to the Courant number, where an X indicates that it is always numerically unstable:

	3rd Order	**4th Order**	**5th Order**	**6th Order**
Leapfrog	X	0.72	X	0.62
Runge-Kutta 2	0.88	X	0.30	X
Runge-Kutta 3	1.61	1.26	1.42	1.08

Note that these stability criteria are for one-dimensional linear advection, as we have considered to this point. We do not consider wavelength dependence in the above. Note that the default choices for WRF-ARW – Runge-Kutta 3 in time, 5th order in space – strike an effective balance between accuracy (higher-order in time and space) and computational efficiency (high Courant number) as compared to other available differencing schemes. Based upon this, the general guidance for the model time step Δt in WRF-ARW is $6^*\Delta x$, where Δx is input in km and the resulting Δt is in s.

As noted earlier, the *U* in the Courant number is not determined by the meteorology – and is instead determined by rapidly moving sound and/or gravity waves – if these waves are not addressed in some fashion (e.g., semi-implicit or split-explicit temporal differencing). For split-explicit models, the shorter time step used to address sound waves is usually 3-4 times shorter than that used by the rest of the model given a speed of sound approximately 3-4 times faster than the meteorologically dependent *U*.

Further, vertical advection terms in the primitive equations also pose a constraint on numerical stability, where $U \rightarrow W$ and $\Delta x \rightarrow \Delta z$. Though both *W* and Δz are typically smaller than *U* and Δx, Δz is non-uniform over the model domain, with smaller values near the surface and tropopause and larger values in the middle troposphere. Fortunately, *W* is typically large where Δz is typically large, with the inverse being true as well. However, where *W* is large when Δz is small, such as may be observed with intense vertical circulations within the boundary layer or thunderstorms, the vertical advection term may limit stability more than horizontal advection terms. In practice, it is more often for vertical advection that the CFL criterion is violated in WRF-ARW simulations.

Physical and Mathematical Interpretations of Linear Stability

At its essence, a stability criterion based upon the Courant number indicates that there is a limit to the ratio between the maximum distance that can be traveled in one timestep and the grid spacing. This limit varies as a function of the chosen spatial and temporal differencing schemes used. For the general CFL condition, the maximum distance that can be traveled in one timestep cannot be larger than the grid spacing. For less restrictive differencing schemes, a greater maximum distance relative to the grid spacing can be traveled in one timestep; the opposite is true for more restrictive differencing schemes.

What does this mean, however? Mathematically, the CFL condition can be phrased in terms of the numerical and physical domains of dependence. The former is defined by the finite differencing scheme, while the latter is defined by the underlying meteorology. The numerical domain contains the grid points that contribute to the finite differencing approximated solution, while the physical domain contains the region that contributes to the physical solution. The CFL condition, therefore, states that the chosen differencing scheme will remain numerically stable so long as the numerical domain does not exceed the bounds of the physical domain.