Equatorial Waves

Introduction

Just as midlatitude wave modes such as Rossby waves can be obtained, given specific assumptions, from the linearized shallow-water equations, equatorially trapped, approximately geostrophic tropical wave modes can also be obtained, given specific assumptions, from the shallow-water equations. These wave modes, or equatorial waves, can be forced by localized diabatic heating or may be altogether unforced (i.e., free). To best understand these waves, it is fruitful to step through the derivation of the shallow-water equations for the set of approximations applicable to the tropics and each wave. Herein, we first introduce the relevant equations and the relevant physical assumptions that we make in order to simply this equation set. Subsequently, we linearize these equations about base and perturbation states. We then assume wave-like solutions for the remaining free variables of the linearized system and obtain the general dispersion relation for the shallow-water equation system. Solutions to the shallow-water system for each wave type are obtained and described. We close our discussion of equatorial waves with a brief discussion of how these waves are monitored and how they are manifest in larger-scale modes of tropical variability.

Key Questions

- What are Kelvin waves and why are they important?
- What are equatorial Rossby waves and why are they important?
- What are inertia-gravity and mixed Rossby-gravity waves and why are they important?
- How do these waves modulate convective activity and large-scale tropical phenomena?

An Introduction to Equatorial Waves

There are four primary types of equatorial waves that we are concerned with in this course. These are the *Kelvin*, *equatorial Rossby*, *mixed Rossby-gravity*, and, to lesser extent, *inertia-gravity* waves. Each of these waves represent specific solutions to the shallow-water equation system, the assumptions inherent to which are described as the solution for each wave type is detailed. We first describe each wave type, focusing almost exclusively upon the atmospheric manifestations of these waves. We then step through the derivation of each of these wave types, focusing upon their common mathematical basis and the inherent physical differences between them. We close by discussing how these waves modulate and are modulated by deep, moist convection, and how we can monitor and detect such waves.

Equatorial waves can be unforced (dry dynamics) or forced by diabatic heating (moist dynamics). Latent heat release in deep, moist convection (convective heating) is the most common diabatic driver of equatorial waves. The vertical motions resulting from diabatic heating help to locally counterbalance the heating and maintain the weak horizontal temperature gradients characteristic of the tropics. Later in this lecture, we will derive solutions of the shallow-water equations for the predominant equatorial waves in an unforced (dry) framework. This enables us to consider their basic structures. We will later consider how deep, moist convection and associated latent heat release modify this structure (e.g., the slowing of wave propagation described earlier).

An Introduction to the Predominant Equatorial Wave Modes

Kelvin Waves

Kelvin waves are large-scale waves that propagate along a physical boundary such as a mountain range or coastline. In the tropics, the northern and southern hemispheres (with non-zero Coriolis parameter) each act as a trapping barrier, such that equatorial Kelvin waves are said to be "equatorially trapped" waves. The Coriolis force (in the context of it not being zero away from the equator) and buoyancy are the restoring mechanisms for Kelvin waves. These waves are thought to be important to the El Niño-Southern Oscillation (ENSO), Madden-Julian Oscillation (MJO), and Quasi-Biennial Oscillation (QBO). They exert a significant influence on deep, moist convection within 10° latitude of the equator, with the greatest impact typically seen in the eastern Indian and central Pacific Ocean. Their impacts on deep, moist convection over Africa, South America, and the western Indian Ocean vary strongly with the seasons. The westerly wind bursts that often accompany Kelvin waves are occasionally caused by, or can cause the formation of, tropical cyclones.

Kelvin waves have a horizontal length scale of approximately 2,000 km, and it is diabatic heating centered on the equator with this length scale that is the most common forcing mechanism for Kelvin wave initiation. As will be demonstrated later, there is no meridional component of velocity with Kelvin waves. At any given location, Kelvin waves have a return period of about 6-7 days. Convectively coupled Kelvin waves, or those associated with and linked to deep, moist convection, propagate eastward with a phase speed between 12-25 m s^{-1}. Dry Kelvin waves propagate eastward at a somewhat faster velocity. Note that in general, for all types of equatorial waves, deep, moist convection results in a slowing of the propagation speed of the wave.

Equatorial Rossby Waves

In the mid-latitudes, Rossby waves arise out of – and thus have as their restoring mechanism – meridional variability in potential vorticity. In the tropics, this can be generalized to simply the meridional variability in the Coriolis parameter (or planetary vorticity). Equatorial Rossby waves are associated with twin vortices on either side of the equator; such vortices occur most often in the Indian Ocean and western Pacific Ocean. The direct impacts of equatorial Rossby waves are strongest over Asia and the West Pacific warm pool. These vortices have a horizontal length scale of approximately 1,000 km, and it is diabatic heating symmetric across the equator with this length scale that is the most common forcing mechanism for equatorial Rossby wave formation. Equatorial Rossby waves can repeat over very long zonal distances of up to 10,000 km, and their duration is on the order of several days. Convectively coupled (dry) equatorial Rossby waves move westward with a phase speed between 5-7 m s^{-1} (10-20 m s^{-1}).

Mixed Rossby-Gravity Waves

These waves are forced by and, subsequently, force deep, moist convection through their ties to buoyancy. These waves have characteristics of both inertia-gravity waves (which are tied to buoyancy) and equatorial Rossby waves (which are tied to meridional variation in the Coriolis parameter); as such, the restoring mechanism for mixed Rossby-gravity waves are both buoyancy and the meridional variation of the Coriolis parameter. Due to distortion by deep, moist convection, these waves are generally tilted northwest-southeast across the equator. Mixed Rossby-gravity waves occur most frequently across the equatorial western and central Pacific and during summer and autumn in the Northern Hemisphere. The horizontal length scale of mixed Rossby-gravity waves is approximately 1,000 km, or of similar magnitude to equatorial Rossby waves, and it is diabatic heating asymmetric across the equator with this length scale that is the most common forcing mechanism for mixed Rossby-gravity wave formation. They have a period of 4-5 days and move westward at a phase speed of approximately 8-10 m s^{-1}.

Derivation of the Equatorial Wave Modes: The Shallow-Water Equations

For the shallow-water system, there are three relevant equations. The first, the equation of motion, describes the two-dimensional *(x,y)* motion of the system. The second, the hydrostatic equation, describes the nature of vertical motions (here free of vertical parcel accelerations) within the system. Together, these equations describe the conservation of momentum (not to be confused with our earlier discussion on the conservation of absolute angular momentum). The third, the continuity equation, describes the conservation of mass within the system. These equations take the form:

(1) $$\frac{D\mathbf{v}}{Dt} + f\hat{k} \times \mathbf{v} = -\frac{1}{\rho}\nabla p$$

(2) $$\frac{\partial p}{\partial z} = -\rho g$$

(3) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

where *D*/*Dt* is the total derivative, boldface variables denote vectors, and all variables adhere to typical meteorological conventions.

There are many approximations inherent to the shallow-water equation system, some of which are reflected in the above equations and others of which are implicit. First and foremost, the concept of "shallow water" means that we assume that the vertical scale is much smaller than the horizontal scale. For equatorial waves, this is a fair assumption: vertical wavelengths are on the order of 15-40 km whereas horizontal wavelengths are on the order of 1,000 km or more. Further, the medium along which the waves propagate is even smaller: on the order of tens to hundreds of meters. We then assume that the atmosphere can be approximated by two distinct layers, each with constant density (ρ_1 in the lower layer, ρ_2 in the upper layer); the medium along which the wave propagate is the interface between these layers. Given that we invoked the hydrostatic approximation in our system of equations above, this enables us to say that the horizontal pressure gradient in each layer is independent of height. This can be demonstrated by taking a horizontal derivative (e.g., with respect to *x*) of (2) and commuting the order of the partial derivatives for *p*. Waves in the shallow-water system are said to be of finite amplitude (i.e., quasi-linear). We assume that the equations are incompressible, wherein the density is conserved following the motion (i.e., the total derivative is equal to zero); this assumption means that sound waves cannot be possible solutions. We neglect friction. We assume that the environment is stably stratified, such that the density in the lower layer is greater than that in the upper layer (i.e., $\rho_1 > \rho_2$, such that $p_1 > p_2$). Finally, we assume that there is no horizontal pressure gradient in the *upper* layer.

As we proceed, please refer to the slide in the "Equatorial Waves" lecture materials depicting the basic structure of a two-dimensional shallow-water system. In order to simplify our set of equations above, particularly (1), we desire an expression for the horizontal pressure gradient in the *lower* layer. In other words, we want to know how pressure varies between points B and A within the model depicted within the lecture materials. With no horizontal pressure gradient in the upper layer, the pressure along the interface between the two layers (above point B) is equivalent to that within the upper layer (above point A). We first assume that differences in pressure along and near the interface between the upper and lower layers are small (i.e., finite and infinitesimal). Pressure at point A is a function of a displacement in pressure associated with a downward-forced upper layer and pressure at point B is a function of a displacement in pressure associated with an upward-forced lower layer. We will refer to these displacements as δp_2 and δp_1, respectively. Thus, the pressure at points A and B can be expressed as:

(4a) $$A: p + \delta p_2$$

(4b) $$B: p + \delta p_1$$

We can use (2) to re-write (4) in terms of the density within each layer and the displacement in height associated with the wave, such that:

(5a) $$p + \delta p_2 = p + \rho_2 g \delta z$$

(5b) $$p + \delta p_1 = p + \rho_1 g \delta z$$

It should be noted in (5) that the leading negative associated with the hydrostatic approximation is folded into the vertical displacement variable ($\partial z = z_1 - z_2$).

Next, let the height of the interface at the point above point B be equal to h_2. Similarly, let the height of the interface at point A be equal to h_1. In this case, δz is merely equal to $h_2 - h_1$. However, let us consider the case where the distance between points B and A, δx, is infinitesimally small ($\delta x \approx 0$). In this case,

(6) $$\delta z = \frac{h_2 - h_1}{\partial x} \delta x = \frac{\partial h}{\partial x} \delta x$$

If we substitute (6) into (5), divide by δx, and subtract the expression for point A (5a) from that for point B (5b), we obtain the following:

(7) $$\lim_{\delta x \to 0} \left[\frac{(p + \delta p_1) - (p + \delta p_2)}{\delta x} \right] = g \delta \rho \frac{\partial h}{\partial x}$$

where $\delta \rho$ is equal to ρ_1-ρ_2. The limit notation arises simply because of the assumption that $\delta x \approx 0$. The corresponding expression for the meridional pressure (and height) gradients can be obtained in a similar manner and takes on an identical form (except in terms of *y*). These expressions are akin to saying that horizontal pressure gradients are equivalent to horizontal fluid depth gradients.

If we substitute the right-hand side of (7) and accompanying expression for $\partial h/\partial y$ into (1) and expand into the full *u*-momentum and *v*-momentum equations, we obtain:

(8a) $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}= fv-g\frac{\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

(8b) $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-fu-g\frac{\delta\rho}{\rho_1}\frac{\partial h}{\partial y}$$

We wish to make one further simplification to (8) before manipulating the continuity equation given by (3). Let us express the fluid depth *h*, which is a function of *(x,y,t)*, in terms of a constant height plus a perturbation height, i.e.,

(9) $$h(x,y,t)=H+h'(x,y,t)$$

In the above, *H* is defined as the equivalent depth and is proportional to the stability. It impacts the vertical wavenumber and thus the vertical structure and depth of the wave. In the tropics, *H* generally ranges between 10-500 m (smaller for dry dynamics, larger for moist dynamics), as associated with large vertical wavelengths (5-50 km). Thus, the fluid depth for our equatorial waves can be quite large with vertical wave structure found throughout the depth of that fluid! This becomes important when we discuss the Quasi-Biennial Oscillation in subsequent lectures.

If we substitute (9) into (8), the constant term drops out (since it is constant in *x* and *y*), such that:

(10a) $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}= fv-g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}$$

(10b) $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-fu-g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial y}$$

Next, we wish to manipulate the continuity equation (3). We wish to integrate both sides of (3) from the ground, where the vertical motion must be equal to zero, to *h*, the fluid height. Since the pressure gradient expressed in (7) is independent of height (i.e., not a function of *z*), (10) makes it clear that both *u* and *v* are independent of height (and thus are not functions of *z*) presuming that they were not functions of height at the initial time *t* = 0. As a result, our integration is simplified. First, the left-hand side of (3):

(11a) $$\int_{z'=0}^{z'=h}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz'=h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Next, the right-hand side of (3):

(11b) $$-\int_{z'=0}^{z'=h}\left(\frac{\partial w}{\partial z}\right)dz'=-\left[w(z=h)-w(z=0)\right]$$

Since our lower boundary is flat and rigid, *w*(*z* = 0) is equal to 0. Meanwhile, *w* at the fluid interface is merely a reflection of the vertical movement of the fluid-interface itself and can be expressed as:

(12) $$w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}$$

If we simplify (11b), substitute (12) in to (11b), and set the resulting expression equal to (11a), we obtain:

(13) $$-h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}$$

In the process of obtaining (13), we multiplied both sides by -1 to bring the leading negative on (11b) over to the left-hand side of the continuity equation.

Finally, substitute (9) into (13) to obtain:

(14) $$-(H + h')\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\partial h'}{\partial t} + u\frac{\partial h'}{\partial x} + v\frac{\partial h'}{\partial y}$$

This equation states that fluid-interface motion (right-hand side of (14), manifest only in the perturbation height) is equal to the depth of the lower fluid times the convergence in the lower fluid. Either greater or deeper convergence will lead to greater fluid-interface motion.

Thus, equations (10) and (14) describe our shallow-water equations.

Next, we wish to simplify the Coriolis term in (10). We introduce the concept of a Beta plane, where the range of latitudes under consideration is said to be sufficiently small as to enable the meridional variation in the Coriolis parameter to be treated as a linear, rather than non-linear (e.g., sin ϕ), function of y. This simplifies the solving of the shallow-water equation set. This approximation results from performing a Taylor series expansion on f and keeping only the first two terms, such that:

(15) $$f = f_0 + \beta y, \beta = \frac{\partial f}{\partial y} \text{ (assuming } y = 0 \text{ at the equator)}$$

Recalling that $f = 2\Omega\sin\phi$, where ϕ is latitude, β is thus given by $(2\Omega\cos\phi)/a$, where a is the radius of the Earth and generally taken constant at 6.37 x 10^6 m. If we restrict (15) to the equator, such that f_0 is zero, then $f = \beta y$. If we employ the small angle approximation (i.e., ϕ small), then $\cos\phi \approx 1$ and $\beta = 2\Omega/a$ = 2.3 x 10^{-11} m^{-1} s^{-1}. Rather than input this numerical value into (10) and (14), however, we substitute $f = \beta y$ into these equations to obtain:

(16a) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \beta y v - g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}$$

(16b) $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\beta y u - g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial y}$$

(17) $$-(H+h')\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial h'}{\partial t}+u\frac{\partial h'}{\partial x}+v\frac{\partial h'}{\partial y}$$

where (17) is identical to (14) as there are no Coriolis terms in (14).

Finally, we linearize (16) and (17) about a background state that is initially at rest. In this process, we assume that the three variables represented within the system given by (16) and (17) – *u*, *v*, and *h* – can each be partitioned into mean (or background flow) and perturbation (or wave flow) components. These take the form:

(18a) $$u(x,y,t)=\overline{u}(x,y,t)+u'(x,y,t)$$

(18b) $$v(x,y,t)=\overline{v}(x,y,t)+v'(x,y,t)$$

(18c) $$h(x,y,t)=H+\overline{h}(x,y,t)+h'(x,y,t)$$

Equation (18c) has three terms as our initial definition of *h* (given by (9)) did not allow for spatial or temporal variance in the base-state *H*. Note that in linearization theory, the perturbation fields *u'*, *v'*, and *h'* must be small. The mean terms in (18) can all be taken as equal to zero if we assume a background state with no horizontal or vertical motion, which we will do here.

It is worth noting that equatorial waves in the shallow-water system are expressed here in terms of three dimensions: *x*, *y*, and *t*. However, it is possible to formulate a vertical structure equation that describes the vertical structure of these waves. This equation is a second order partial differential equation and is a function of the vertical wavenumber *m*, itself largely dependent upon stability (e.g., as enters through the equivalent depth *H*). As noted above, the vertical wavelength of these equatorial waves is typically on the order of 25-40 km. The vertical structure equation can be used to provide detail of a wave's structure over that wavelength. For our purposes, however, it is most important to know that equatorial waves do contain vertical structure.

If we substitute (18) into (16) and (17) and simplify, we obtain:

(19a) $$\frac{\partial u'}{\partial t}-\beta yv'=-g'\frac{\partial h'}{\partial x}$$

(19b) $$\frac{\partial v'}{\partial t}+\beta yu'=-g'\frac{\partial h'}{\partial y}$$

(20) $$\frac{\partial h'}{\partial t}+H\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)=0$$

where we have moved the β terms to the left-hand side of the equations. We have also substituted an "effective gravity" term *g'* into the right-hand side of (16), as defined by:

(21) $$g' = g\frac{\delta\rho}{\rho_1}$$

Note that the terms involving products of linearized variables – namely, the advection terms – in (16) and (17) vanish. Substitution of (18) into these terms results in four terms. Three of these terms involve the mean fields in some way. Since we said above that these terms are zero, the three mean terms vanish. The remaining term involves the product of two perturbation fields. As perturbations are necessarily small, the product of perturbations is assumed to be negligible. Thus, each of these four terms vanish.

We are now ready to explore solutions to the shallow-water equations given by (19) and (20). This set has three free variables: *u'*, *v'*, and *h'*. We assume wave-like solutions for each of these three variables as follows:

(23a) $$u'(x, y, t) = U(y) * \exp(i(kx - \omega t))$$

(23b) $$v'(x, y, t) = V(y) * \exp(i(kx - \omega t))$$

(23c) $$h'(x, y, t) = H_w(y) * \exp(i(kx - \omega t))$$

The functions given by *U*, *V*, and H_w are the amplitudes of the wave function for each variable. These amplitudes vary only in the north-south direction (i.e., as functions of *y*). *k* is the zonal wavenumber, ω is the frequency of the wave (equal to the number of times the wave passes a given point per second, and thus related to its propagation), and *i* is equal to the square root of -1. Before substituting (23) into (19) and (20), it is helpful to note that derivatives of (23) with respect to *x* and *t* have special forms given by:

(24a) $$\frac{\partial}{\partial x}(\ ) = ik(\ )$$

(24b) $$\frac{\partial}{\partial t}(\ ) = -i\omega(\ )$$

These arise because *U*(*y*), *V*(*y*), and $H_w(y)$ are all not functions of *x* or *t* and because the derivative of an exponential function is equal to the derivative of the exponential multiplied by the exponential function.

Substituting (23) into (19) and (20), making use of the definitions in (24), and dividing through by a common factor of the exponential wave function, we obtain:

(25a) $$-i\omega U - \beta y V = -ikg' H_W$$

(25b) $$-i\omega V + \beta y U = -g'\frac{\partial H_W}{\partial y}$$

(26) $$-i\omega H_w + H\left(ikU + \frac{\partial V}{\partial y}\right) = 0$$

where we have dropped the (y) notation on U, V, and H_W for simplicity.

The set of equations given by (25) and (26) has three unique variables, U, V, and H_w. We now wish to simplify this system into one of two equations for two variables, after which we will simplify further into a single equation for a single variable. First, we solve (25a) for U:

(27) $$U = -\frac{\beta y V}{i\omega} + \frac{kg' H_W}{\omega}$$

We then substitute (27) into (25b), multiply all terms by -$i\omega$, group like terms (specifically, the V terms), and rearrange slightly to obtain:

(28) $$(B^2 y^2 - \omega^2)V - ikg'\beta y H_W - i\omega g' \frac{\partial H_W}{\partial y} = 0$$

Similarly, we then substitute (27) into (26), multiply all terms by $i\omega$, group the like V and H_w terms, and rearrange slightly to obtain:

(29) $$(\omega^2 - Hk^2 g')H_W + iH\omega(\frac{\partial V}{\partial y} - \frac{k\beta y}{\omega} V) = 0$$

We follow a similar procedure to reduce (28) and (29) into a single equation, first solving (29) for H_w and substituting the result into (28) to obtain a single equation for V. Note that from (29), H_w and its derivative with respect to y are given by the following:

(30a) $$H_W = \frac{-i\omega H \frac{\partial V}{\partial y} + ikH\beta y V}{\omega^2 - g' Hk^2}$$

(30b) $$\frac{\partial H_W}{\partial y} = \frac{-i\omega H \frac{\partial^2 V}{\partial y^2} + ikH\beta y \frac{\partial V}{\partial y} + ikH\beta V}{\omega^2 - g' Hk^2}$$

where we have made use of the product rule for derivatives in obtaining (30b) from (30a). Substituting (30) into (28) enables us to obtain a second order partial differential equation for V. This substitution leaves us with three sets of terms: V, its first derivative with respect to y, and its second derivative with respect to y. Specifically, these terms take the form:

(31a) $$V\left[\left(\beta^2 y^2 - \omega^2\right) + \frac{k\omega g' h\beta}{\omega^2 - Hk^2 g'} + \frac{k^2 g' \beta^2 y^2 H}{\omega^2 - Hk^2 g'}\right] +$$

(31b) $$\frac{\partial V}{\partial y}\left[\frac{-kg' H\omega\beta y}{\omega^2 - Hk^2 g'} + \frac{kg' H\omega\beta y}{\omega^2 - Hk^2 g'}\right] +$$

(31c) $$\frac{\partial^2 V}{\partial y^2}\left[\frac{-\omega^2 g' H}{\omega^2 - Hk^2 g'}\right]$$

all of which equals zero. Note that (31b) equals zero, as the terms inside the brackets cancel each other out.

If you multiply (31a) by $\omega^2 - Hk^2 g'$, cancel opposing terms, subsequently divide by $-\omega^2 g' H$, and perform minor rearrangement, the final result – a second-order PDE for *V* – is obtained, given by:

(32) $$\frac{\partial^2 V}{\partial y^2} + \left(\frac{\omega^2}{g' H} - \frac{\beta^2 y^2}{g' H} - k^2 - \frac{\beta k}{\omega}\right) V = 0$$

If solution(s) for *V* are obtained from (32), they can be used with (30a) to obtain solution(s) for H_w and, subsequently, with (27) for *U*.

We are now ready to consider the full complexity of (32). We require solutions where *V* approaches zero as *y*, the distance from the equator, grows increasingly large (i.e., approaches ±∞). This statement aids in constraining our solutions to have maximum amplitude near the equator and to decay north and south from there. In his seminal work on equatorial waves, Matsuno (1966) demonstrated that the solutions for (32) only satisfy this condition if there are a finite odd integer number of waves present in the meridional direction, i.e.,

(33a) $$\frac{\sqrt{g' H}}{\beta}\left(\frac{\omega^2}{g' H} - k^2 - \frac{\beta k}{\omega}\right) = 2n + 1$$

Or, put in a way that is perhaps more apparent from a consideration of the coefficient on *V* in (32),

(33b) $$\frac{\omega^2}{g' H} - \frac{\beta}{\sqrt{g' H}} - k^2 - \frac{\beta k}{\omega} = \frac{2n\beta}{\sqrt{g' H}}$$

(33a) is obtained from a consideration of the coefficient on *V* in (32) in the context of *y* approaching ±∞ and how it modulates the solution to the second-order PDE in *V* given by (32). The value *n* is a general wavenumber and is equal to 0, 1, 2, 3, and so on.

Equation (33a) relates the frequency ω and zonal wavenumber *k* for all possible wave solutions (n = 0, 1, 2, 3…) to the shallow-water system and thus provides the basis for the generic *dispersion relation* of the system. The dispersion relation is cubic, i.e., dependent upon ω^3 (if multiplied through by ω to eliminate the $1/\omega$ term) such that there are at most three unique solutions to (33a). These solutions illuminate three of our four wave types: equatorial Rossby ($n \geq 1$), mixed Rossby-gravity ($n = 0$), and inertia-gravity waves ($n \geq 1$). As Kelvin waves have no meridional structure, these solutions in *V* do not directly describe Kelvin waves; instead, they are accounted for separately, as will be demonstrated in subsequent sections. However, it should be noted that Kelvin waves can also be obtained from (33a) for the special case where n = -1.

The solutions described here represent the case of no external forcing (diabatic heating) to drive the shallow-water system. When a generic external forcing is included, a combination of the equatorial Rossby and Kelvin wave solutions, representing the waves with the smallest frequencies, is most commonly obtained. We will demonstrate this in a later lecture where we consider solutions to the shallow-water equations for in the presence of external heat forcing. As discussed by Matsuno (1966), the lower frequency wave modes dominate the externally forced solution because they are more responsive to a given external forcing than are the higher frequency wave modes.

From (33b), we see that there are two primary physical forcings on *V*. The first is buoyancy, analogous to *g*'*H* and thus akin to potential energy, wherein the equivalent depth *H* is a function of the divergence within the lower layer. The second is the meridional planetary vorticity gradient *β*. The structure of *V* is modulated by the frequency *ω*, zonal wavenumber *k*, and the distance along the meridional axis *y*.

Physical Description of the Equatorial Wave Solutions

Solutions to (32) take the form of solutions to the Schrodinger equation for an oscillator (e.g., an oscillatory wave mode such as given by the shallow-water system). The general form of these solutions is given by:

(34)
$$V(Y) = A\exp(-\frac{Y^2}{2})H_N(Y)$$

In equation (34), *A* is an amplitude function, $H_N(Y)$ are Hermite polynomials of order *N* and are integers and/or some multiple or power of *Y*, and *Y* is defined by:

(35)
$$Y = \left(\frac{\sqrt{g'H}}{\beta}\right)^{1/2} y$$

Note that (35) is slightly different than that contained within Appendix C of Chapter 4 in *An Introduction to Tropical Meteorology*. *Y* can be related to the Rossby radius of deformation and is defined by the ratio of buoyancy to the meridional planetary vorticity gradient. Solutions for *V* can be found for individual values of *N*. These solutions, as noted above, can be used with (30a) to obtain solutions for H_w and, subsequently, with (27) to obtain solutions for *U*. These give the meridional amplitudes of solutions for *u'*, *v'*, and *h'*, each with wave-like structure in *x* and *t* (and a vertical structure that is unspecified herein).

As an aside, the Rossby radius of deformation L_R is typically on the order of 6,000 km in the tropics. As described above, however, equatorial waves have horizontal length scales on the order of 1,000-2,000 km. Following the principle of geostrophic adjustment, remembering that these waves are assumed to be at least approximately geostrophic, the mass fields (such as temperature and pressure) adjust in response to evolutions within the wind fields associated with the equatorial wave "perturbations" to the shallow-water system. This leads to fairly broad, relatively weak atmospheric perturbations that align with expectations of linearity and finite-amplitude waves inherent to the shallow-water system. Indeed, the magnitude of the wind and pressure perturbations associated with each of the equatorial wave modes is relatively small (e.g., 1-2 m s^{-1} or hPa, of similar order to observational uncertainty).

At this point, we now turn to examining the solutions for each equatorial wave mode, focusing on the physical manifestations and propagation characteristics inherent to each. In this manner, we will have used the shallow-water system and appropriate assumptions to not only obtain dispersion relations for each wave type but to also understand their direction of propagation and how their solutions are manifest in the near-equatorial wind and pressure fields.

Equatorial Rossby Waves

Of our equatorial wave modes, equatorial Rossby waves evolve relatively slowly, similar to their mid-latitude counterparts. The period (i.e., duration) of an equatorial Rossby wave is approximately 14-21 days. As the frequency of a wave is related to the inverse of its period, equatorial Rossby waves have a relatively small frequency. This implies that ω is small. For equation (33a), small ω ($\omega << 1$) enables us to neglect the ω^2 term and enables us to write the dispersion equation for equatorial Rossby waves as:

(36) $$-\frac{\sqrt{g'H}}{\beta}\left(k^2+\frac{\beta k}{\omega}\right)=2n+1$$

If we rearrange and solve for ω, we obtain:

(37) $$\omega=-\frac{\beta k}{\left(k^2+\frac{\beta(2n+1)}{\sqrt{g'H}}\right)}$$

As with (35), note that (37) is slightly different from, with a minor correction incorporated here, what is contained within Appendix C of Chapter 4 in *An Introduction to Tropical Meteorology* and Matsuno (1966).

For any wave, the phase speed c_p is defined by $c_p = \omega/k$. Given (37), the phase speed is simply:

(38) $$c_p=-\frac{\beta}{\left(k^2+\frac{\beta(2n+1)}{\sqrt{g'H}}\right)}$$

As each of the values of (38) are positive-definite, $c_p < 0$ and thus equatorial Rossby waves propagate *westward*. The speed at which they do so is determined chiefly by buoyancy and the meridional planetary vorticity gradient.

For $n = 1$, the theoretical solution for equatorial Rossby waves is depicted within the lecture materials. It is characterized by westward-propagating high and low pressure centers symmetrically displaced north and south of the equator. Wind field maxima are located along the Equator. Deep, moist convection preferentially forms where convergence associated with the equatorial Rossby waves is maximized east (west) of locally lower (higher) pressures. Note that there is also confluence near the Equator to the west (east) of locally lower (higher) pressures. However, this confluence is weak and is largely mitigated by the presence of speed divergence. Deep, moist convection acts to lead to the diabatic redistribution of positive potential vorticity from higher to lower altitudes, acting to reduce the continued westward propagation of the downstream lower pressure centers (and thus the entire wave as a whole).

Inertia-Gravity Waves

These buoyancy-dependent waves have high frequency (i.e., large ω) and short longevity. As a result, the $-\beta k/\omega$ term in (33a) becomes negligibly small. This enables us to express the dispersion relation as the following:

$$\frac{\sqrt{g'H}}{\beta}\left(\frac{\omega^2}{g'H}-k^2\right)=2n+1 \qquad (39)$$

Solving for ω by re-arranging terms, multiplying both sides by $\beta\sqrt{g'H}$, and taking the square root:

$$\omega=\pm\sqrt{g'Hk^2+\sqrt{g'H}\beta(2n+1)} \qquad (40)$$

The phase speed of such waves is given by:

$$c_p=\pm\frac{\sqrt{g'Hk^2+\sqrt{g'H}\beta(2n+1)}}{k} \qquad (41)$$

We see that there are two modes of propagation for inertia-gravity waves, one eastward and one westward, that correspond to the positive and negative roots of (41).

Mixed Rossby-Gravity Waves

For the special case of mixed Rossby-gravity waves, we let $n = 0$ in (33a) such that:

$$\left(\frac{\omega^2}{g'H}-k^2-\frac{\beta k}{\omega}\right)=\frac{\beta}{\sqrt{g'H}} \qquad (42)$$

Or, expressed in terms of ω^3,

$$\frac{\omega^3}{g'H}-\omega\left(k^2+\frac{\beta}{\sqrt{g'H}}\right)-\beta k=0 \qquad (43)$$

There are three possible roots to the cubic equation given by (43). It is possible to solve for these using mathematical techniques designed to solve cubic equations; however, as this is a tedious, time-consuming process, we will instead focus on the solutions themselves. Two of these solutions (or roots) are akin to the inertia-gravity wave roots described by (40). However, the westward-moving inertia-gravity wave solution is not an allowable solution for $n = 0$ as it amplifies away from the Equator (not shown), reducing the number of allowable solutions to (43) from three to two. The third root of (43) gives the dispersion relation for mixed Rossby-gravity waves. For convenience, we simply write it as:

(44) $$\omega = \frac{k\sqrt{g'H}}{2}\left(1 - \left(1 + \frac{4\beta}{k^2\sqrt{g'H}}\right)^{1/2}\right)$$

And, as before, we can express the phase speed of these waves as:

(45) $$c_p = \frac{\sqrt{g'H}}{2}\left(1 - \left(1 + \frac{4\beta}{k^2\sqrt{g'H}}\right)^{1/2}\right)$$

The $\frac{4\beta}{k^2\sqrt{g'H}}$ term in (44) and (45) is positive-definite since each of its constituents are all positive-definite. Thus, $1 + \frac{4\beta}{k^2\sqrt{g'H}} > 1$, such that its square root is also greater than 1. This means that the phase speed for mixed Rossby-gravity waves is negative, describing *westward* propagation of these waves.

For small *k*, the $k\sqrt{g'H}$ term in (44) is relatively small but the $\frac{4\beta}{k^2\sqrt{g'H}}$ term is relatively large. Given the influence of k^2 (rather than *k*) in the latter term, ω is relatively large and the wave behaves more like an inertia-gravity wave. For large *k*, the inverse is true, ω is relatively small, and the wave behaves more like an equatorial Rossby wave.

The theoretical solutions for a mixed Rossby-gravity wave are depicted within the lecture materials. In a dry atmosphere, the mixed Rossby-gravity wave is not horizontally tilted. It is characterized by clockwise and/or counterclockwise flow within ±10° latitude of the Equator with maximum circulation magnitudes centered on the Equator. Higher (lower) pressures are found to the north (south) of the Equator for clockwise flow, but note that the pressure fields are not in exact geostrophic balance with the wind fields to which they adjust. Conversely, for counterclockwise flow, lower (higher) pressures are found to the north (south) of the Equator. Winds are nearly meridional and at their peak magnitude along the Equator; they decay rapidly to the north and south away from there, as constrained by our assumption used to obtain (33a). In a moist environment, deep, moist convection predominantly forms in regions of speed convergence, on the eastern (western) flank of locally lower (higher) pressures. The diabatically driven vertical redistribution of positive potential vorticity from higher to lower altitudes in regions of deep, moist convection acts as a brake on the westward movement of the wave. As the wave's pressure and convection fields are not symmetric about the Equator, this also leads to the waves being tilted horizontally toward areas of speed convergence and locally lower pressures.

Kelvin Waves

As noted before, Kelvin waves have no meridional motion. Thus, equations (32) and (33a) are not exactly valid for these waves. We thus need to derive the dispersion relation for this system in another manner. To do so, we make use of the shallow-water equations given by (19) and (20). These equations in u', v', and h' represent the basis of the unforced shallow-water system after the bulk of our assumptions (namely the β-plane approximation and linearization) have been made to the system. If we set $v' = 0$ in these equations, we obtain:

(46a) $$\frac{\partial u'}{\partial t} = -g'\frac{\partial h'}{\partial x}$$

(46b) $$\beta y u' = -g'\frac{\partial h'}{\partial y}$$

(47) $$\frac{\partial h'}{\partial t} + H\frac{\partial u'}{\partial x} = 0$$

Equations (46) and (47) give us a set of three equations for two variables (u' and h'). Now, we want to assume wave-form (wave-like) solutions for u' and h', as given by (23a) and (23c). We substitute these solutions in to (47), making use of (24) to simplify the resultant expression:

(48) $$-i\omega H_w + HikU = 0$$

Solving for U, we obtain:

(49) $$U = \frac{\omega H_w}{kH}$$

If we substitute (49), (23a), and (23c) into (46a), we obtain:

(50) $$\frac{i\omega^2 H_w}{kH} = g' ik H_w$$

Solving (50) for ω, we obtain:

(51) $$\omega = \pm k\sqrt{g'H}$$

This is identical to the dispersion relation of a pure gravity wave and can be obtained for small *n* and large *k* in (40), such that the equatorial Kelvin wave can be viewed as a special case of a gravity wave.

As the negative root for ω in (51) does not decay away from the Equator (and, in fact, leads to amplification of the wave away from the Equator; not shown), we discard it as an allowable solution. Thus, the dispersion relation for Kelvin waves is given by the positive root of (51). Note that for small *n* and large *k*, (40) approaches this expression, which is the dispersion relation for pure gravity waves. The phase speed for Kelvin waves can thus be expressed as:

(52) $$c_p = \sqrt{g'H}$$

As the radical in (52) is positive-definite, $c_p > 0$ and thus Kelvin waves propagate *eastward*. With no dependence on *k* in (52), Kelvin waves are also said to be non-dispersive.

The mathematical solution for a Kelvin wave is obtained in a similar manner to that for the other equatorial wave modes, wherein U and u' can be used to obtain H_w and h'. The resultant solutions are depicted within the lecture materials. Zonal velocity is strongest along the Equator and decays poleward of there. Velocity is strongest at the heart (or core) of the mass/pressure field responses centered along the equator and decays east and west from there. The signs of the pressure fields can be viewed in terms of shear vorticity arguments: for westerly flow, shear vorticity to the north and south of the Equator is cyclonic. Thus, along the Equator, pressures should be correspondingly higher. Similarly, for easterly flow, shear vorticity is anticyclonic. Thus, along the Equator, pressures should be correspondingly low. Deep, moist convection preferentially forms along the Equator where convergence is maximized between pressure maxima and minima. For Kelvin waves, this is to the west (east) of lower (higher) pressure. The diabatically driven vertical redistribution of positive potential vorticity from higher to lower altitudes within the deep, moist convection slows, as it does for the other equatorial waves, the wave's eastward progression.

Equatorial Wave Monitoring and Forecasting

Each of the equatorial wave modes described above propagates at a unique velocity and in a unique direction. Each equatorial wave mode is also associated with a unique kinematic and, in particular, convective structure and typically lasts for a unique length of time. As a consequence, monitoring and forecasting of equatorial waves is fairly straightforward given appropriate spatial and temporal filtering of anomalous outgoing longwave radiation (modulated by clouds) and upper/lower tropospheric wind fields. This allows for the isolation of a given wave mode from a set of observations or forecast fields. Specific insights into this process are provided by Wheeler and Kiladis (1999), among other references.

Unfortunately, however, there is limited skill associated with forecasts of equatorial wave activity whether such forecasts are derived from persistence (e.g., extrapolation of ongoing conditions into the future) or numerical model forecasts. Typical forecast skill extends out to 1-5 days, or up to half of a given equatorial wave's life span (or period). As a result, improving the skill of forecasts of equatorial waves and associated meteorological phenomena is an active, vibrant area of research in tropical meteorology. Section 4.1.5.2 of *An Introduction to Tropical Meteorology* contains a number of links to resources for the real-time monitoring and prediction of equatorial waves and is a recommend resource.

For Further Reading

- Chapter 4, *An Introduction to Tropical Meteorology, 2nd Edition*, A. Laing and J.-L. Evans, 2016.
- Chapter 7, An Introduction to Dynamic Meteorology, 3rd Edition, J. R. Holton, 1992.
- Gill, A. E., 1980: Some simple solutions for heat-induced tropical circulation. *Quart. J. Roy. Meteor. Soc.*, **106**, 447-462.
- Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, **44**, 25–43.
- Wheeler, M., and G. N. Kiladis, 1999: Convectively coupled equatorial waves: analysis of clouds and temperature in the wavenumber-frequency domain. *J. Atmos. Sci.*, **56**, 374-399.