1 The Principal of Uniform Boundedness

Many of the most important theorems in analysis assert that pointwise hypotheses imply uniform conclusions. Perhaps the simplest example is the theorem that a continuous function on a compact set is uniformly continuous. The main theorem in this section concerns a family of bounded linear operators, and asserts that the family is uniformly bounded (and hence equicontinuous) if it is pointwise bounded. We begin by defining these terms precisely.

**Definition 1.1** Let $A$ be a family of linear operators from a normed space $X$ to a normed space $Y$. We say that $A$ is **pointwise bounded** if $\sup_{A \in A} \{\|Ax\|\} < \infty$ for every $x \in X$. We say $A$ is **uniformly bounded** if $\sup_{A \in A} \{\|A\|\} < \infty$.

It is possible for a single linear operator to be pointwise bounded without being bounded (Exercise), so the hypothesis in the next theorem that each individual operator is bounded is essential.

**Theorem 1.2 (The Principle of Uniform Boundedness)** Let $A \subseteq \mathcal{L}(X,Y)$ be a family of bounded linear operators from a Banach space $X$ to a normed space $Y$. Then $A$ is uniformly bounded if and only if it is pointwise bounded.

**Proof** We will assume that $A$ is pointwise bounded but not uniformly bounded, and obtain a contradiction. For each $x \in X$, define $M(x) := \sup_{A \in A} \{\|Ax\|\}$; our assumption is that $M(x) < \infty$ for every $x$. Observe that if $A$ is not uniformly bounded, then for any pair of positive numbers $\epsilon$ and $C$ there must exist some $A \in A$ with $\|A\| > C/\epsilon$, and hence some $x \in X$ with $\|x\| = \epsilon$ but $\|Ax\| > C$. We can therefore choose sequences $\{A_n\} \subseteq A$ and $\{x_n\} \subset X$ as follows. First, choose $x_1$ and $A_1$ so that $\|x_1\| = 1/2$ and $\|A_1x_1\| \geq 2$. Having chosen $x_1, \ldots, x_{n-1}$ and $A_1, \ldots, A_{n-1}$, choose $x_n$ and $A_n$ to satisfy

$$\|x_n\| \leq 2^{-n} \left(\sup_{k<n} \|A_k\|\right)^{-1}, \quad \|A_nx_n\| \geq \sum_{k=1}^{n-1} M(x_k) + 1 + n. \quad (1)$$

Now let $x = \sum_{n=1}^{\infty} x_n$. The sum converges, since $X$ is complete, and for any $n \geq 2$,

$$\|A_nx\| = \|A_nx_n + \sum_{k \neq n} A_nx_k\| \geq \|A_nx_n\| - \|\sum_{k \neq n} A_nx_k\|. \quad (2)$$
Using (1), we see that the subtracted norm is bounded by
\[ \sum_{k=1}^{n-1} M(x_k) + \sum_{k=n+1}^{\infty} \|A_n x_k\| \leq \sum_{k=1}^{n-1} M(x_k) + 1, \] (3)
so that \(\|A_n x\| \geq n\), contradicting the assumption that \(A\) is pointwise bounded. ■

The theorem has some elementary corollaries which we will state here; it also has many deeper consequences which will be explored in later sections.

Corollary 1.3 Let \(X\) be a Banach space. A set \(A \subseteq X^*\) is bounded if and only if \(\sup_{A \in A}\{|Ax|\} < \infty\) for every \(x \in X\).

Corollary 1.4 Let \(X\) be a normed space. A set \(E \subseteq X\) is bounded if and only if \(\sup_{x \in E}\{Tx\} < \infty\) for all \(T \in X^*\).

Proof Consider \(X\) as a subset of \(X^{**}\) via the canonical map \(\hat{x}(T) = Tx\). The corollary follows, since \(X^*\) is complete and \(\|\hat{x}\| = \|x\|\). ■

2 The Closed Graph Theorem

We saw in the introduction that we cannot expect the operators of quantum mechanics to be bounded. On the other hand, we do expect them to be closed: any self-adjoint operator must be closed, and even if we only required symmetric operators they would be closable and there would probably be no loss of generality in taking their closures. It is of some practical interest, therefore, and not just a theoretical question, as to when a closed operator must be bounded. The answer is given by the next theorem. The theorem remains true, with essentially the same proof, in the Banach space setting (Exercise), but we will use only the Hilbert space case.

Theorem 2.1 Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be Hilbert spaces, and let \(A : \mathcal{H}_1 \to \mathcal{H}_2\) be closed with \(D(A) = \mathcal{H}_1\). Then \(A\) is bounded.

Proof Since \(D(A)\) is dense in \(\mathcal{H}_1\), \(A^*\) is defined. For each \(y \in D(A^*)\), define a linear functional \(T_y\) on \(\mathcal{H}_1\) by \(T_y x = \langle A^* y, x \rangle\), and let \(A\) be the set \(\{L_y : y \in D(A^*), \|y\| = 1\}\). The computation
\[ |\langle A^* y, x \rangle| = |\langle y, Ax \rangle| \leq \|Ax\|, \] for all \(x \in \mathcal{H}_1\), (4)
shows that $A$ is pointwise bounded. By the Principle of Uniform Boundedness, $\sup_A\{\|T_y\|\} < \infty$, so there exists a constant $C < \infty$ such that
\[
|\langle A^*y, x \rangle| \leq C\|x\|,
\]
for all $x \in X$ and all $y \in D(A^*)$ with $\|y\| = 1$. Taking $x = A^*y$, we see that $A^*$ is bounded on $D(A^*)$. Since $A^*$ is closed, this implies that $D(A^*)$ is closed in $\mathcal{H}_2$, but it is also dense, because $A$ is closed. Thus $A^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is bounded, which implies $A = A = (A^*)^* \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is bounded.

A very useful corollary of this theorem follows almost immediately from the simple observation that if a closed operator is invertible, its inverse must be closed.

**Corollary 2.2 (The Inverse Mapping Theorem)** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ a linear bijection. If $A$ is bounded, then $A^{-1}$ is bounded.

**Proof** If $A$ is bounded, then it is closed, so $A^{-1}$ is closed. Since $A^{-1}$ is defined on all of $\mathcal{H}_2$, it is bounded by the theorem.

Since boundedness and continuity are equivalent for linear operators, the Inverse Mapping Theorem says that a continuous linear bijection $A$ has a continuous inverse, or is an open map; i.e. if $E$ is an open set in $\mathcal{H}_1$, $A(E)$ is open in $\mathcal{H}_2$. This conclusion holds even if $A$ is not injective.

**Theorem 2.3 (The Open Mapping Theorem)** Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear surjection. Then $A$ is open.

The proof is left to the reader, together with the production of simple examples to show that the surjectivity condition cannot be removed.

We now return to the quantum-mechanical question raised at the beginning of this section. Here the Closed Graph Theorem has a much less useful consequence.

**Theorem 2.4 (The Hellinger-Toeplitz Theorem)** Let $A$ be a symmetric operator on a Hilbert space $\mathcal{H}$, with $D(A) = \mathcal{H}$. Then $A$ is bounded.

**Proof** $A$ is closable, because it is symmetric, but $\mathcal{H} = D(A) \subseteq D(\overline{A})$ implies that $D(\overline{A}) = D(A)$, so $A$ is actually closed. The result now follows from the Closed Graph Theorem.
Hilbert space of quantum states of the system. It appears as though the technical difficulties associated with domains of unbounded operators are unavoidable.