Geometry of the unification of quantum mechanics and relativity of a single particle*

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The paper summarizes, generalizes and reveals the physical content of a recently proposed framework that unifies the standard formalisms of special relativity and quantum mechanics. The framework is based on Hilbert spaces H of functions of four space-time variables $\mathbf{x}, t$, furnished with an additional indefinite inner product invariant under Poincaré transformations. The indefinite metric is responsible for breaking the symmetry between space and time variables and for selecting a family of Hilbert subspaces that are preserved under Galileo transformations. Within these subspaces the usual quantum mechanics with Shrödinger evolution and t as the evolution parameter is derived. Simultaneously, the Minkowski space-time is embedded into H as a set of point-localized states, Poincaré transformations obtain unique extensions to H and the embedding commutes with Poincaré transformations. Furthermore, the framework accommodates arbitrary pseudo-Riemannian space-times furnished with the action of the diffeomorphism group.

I. PRELIMINARIES

Resolving the tension between quantum mechanics and relativity is one of the most important problems of modern physics. The tension can be traced back to an asymmetric way in which space and time variables enter into the non-relativistic quantum mechanics. In the coordinate representation the state of a system at a given moment of time is described by a function of spatial (and possibly other, "internal") variables. The principle of superposition is only applicable to states considered at the same time. States of a system considered at different times belong to different spaces of states, which precludes their meaningful superposition. Imagine now that detectors in a quantum-mechanical experiment move at a constant velocity with respect to the original ones. In this case quantum effects due to superposition of states are known to be still present. However, the usual quantum mechanics is not capable of predicting those effects. In fact, relative to the rest frame of the moving detectors the superposition of the original states of the investigated system involves summing states at different times, which, as just discussed, is not meaningful in the standard quantum mechanics Ref.[1].

A formal relativistic covariance of quantum mechanics of a single particle is well known to be achieved by replacing the Schrödinger equation with a relativistic equation, e.g., the Klein-Gordon or the Dirac equation. However, as known after Newton and Wigner Ref.[2], the usual interpretation of a state function as a probability density to find particle at a point is lost. Moreover, Hegerfeldt showed that under free evolution governed by these equations any

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form of localization of the initial state function leads to superluminal spreading of the wave packet at $t > 0$ Ref.[3].

Quantum field theory achieves covariance under Poincaré transformations by employing space and time variables symmetrically as parameters and considering operator-valued functions of these parameters as dynamical variables. In the case of electrodynamics the resulting theory is well known to be one of the most precise theories in all of physics. It is therefore often considered to be the ultimate quantum theory that encompasses the single particle quantum mechanics as a particular case. However, the theory is powerful primarily in description of scattering processes. Simple quantum effects, such as interference in the double-slit experiment are difficult to describe. The treatment of those experiments rely on Klein-Gordon or Dirac equations and therefore inherits the already mentioned difficulties in interpretation of the state function satisfying these equations.

An alternative way of achieving relativistic covariance was put forward in the theory of Stueckelberg, and Horwitz and Piron (Refs.[4],[5]). This theory introduces Hilbert spaces of square-integrable functions of four space-time variables and considers space and time variables symmetrically (with time playing the role similar to position in the standard quantum mechanics). The role of time as an evolution parameter is played by a new invariant (observer independent) parameter. Evolution of a system in this parameter is described by a relativistic equation of motion (Stueckelberg-Schrödinger equation). The theory can be used for computation of interference effects, scattering processes and bound state structures alike. It preserves sharp locality in space and time and resolves some of the mentioned difficulties discovered in the works of Newton and Wigner, and Hegerfeldt. Calculation of bound states for a pair of particles gives a spectrum consistent with the usual spectrum of Scrödinger mechanics, up to relativistic corrections. In general however, the issue of experimental verification of the theory and its agreement in a limit with Scrödinger mechanics and quantum field theory is still open.

In this paper a new relativistic covariant geometric framework that is fully compatible with quantum mechanics and relativity is presented. The framework employs the basic correspondence between points in the classical space $\mathbb{R}^3$ and states of a particle found at a point. Being properly generalized, this correspondence yields a physically meaningful embedding of space-time into a Hilbert space of states of the particle. This in turn gives an unprecedented opportunity to include both relativity theory and quantum mechanics on an equal footing as parts of the same Hilbert space framework. As the theory of Stueckelberg, Horwitz and Piron, the new theory deals with Hilbert spaces of functions of space and time variables, involves a new evolution parameter to describe motions in those spaces, it is compatible with the sharp locality in space and time and predicts the effect of interference in time. In addition, the theory explains the observed asymmetry in space and time variables in quantum experiments, clarifies the concept of relativistic covariance and provides a reason for linearity of the quantum theory. Furthermore, the theory can be generalized to curved space-times furnished with the action of a diffeomorphism group as well.

It is particularly important that the new geometric framework contains the formalisms of the standard quantum mechanics and the classical relativity theory as certain limits. This ensures agreement of predictions of the theory with current experiments, at least in the limit. At the same time the employed geometric construction is rigid. In particular, the space-time is uniquely determined by the considered Hilbert space of states. This rigidity indicates that the framework is not just a loose union of quantum mechanics and relativity. On the other hand, any theory that contains quantum mechanics must deal with a Hilbert space of states. In this sense the proposed framework based on embedding of space-time into a Hilbert space of states is the most economical framework possible for the unification of these theories.

In Sec.II the results of Refs.[6],[7] on a geometric approach to quantum mechanics based on the embedding of the classical space $\mathbb{R}^3$ into a space of states are reviewed. In Sec.III this approach is generalized to the embedding of Minkowski space-time and the theory of relativity is extended to a Hilbert space. In Sec.IV the concept of relativistic covariance in quantum theory is investigated and the reason for asymmetry between space and time is revealed. Sec.III and IV are based on Ref.[8]. Sec.V is dedicated to two new results on extension of the obtained formalism to curved space-times.

II. ENCODING SPACE $\mathbb{R}^3$ INTO A HILBERT SPACE OF STATES

Consider a system consisting of a single free microscopic particle. The classical Euclidean space $\mathbb{R}^3$, as a set, can be identified with the set of all points $\mathbf{a}$ at which the particle can be found as a result of an observation (measurement). The state of the particle found at $\mathbf{a}$ is the Dirac delta $\delta^3_{\mathbf{a}}(\mathbf{x}) = \delta^3(\mathbf{x}-\mathbf{a})$. The one-to-one correspondence between points $\mathbf{a} \in \mathbb{R}^3$ and states $\delta^3_{\mathbf{a}}(\mathbf{x})$ can be used to identify classical space with a subset in the Hilbert space of states of the particle. The plan is to take this identification seriously, introduce it in a mathematically rigorous way and explore its benefits for resolving the problem of unification of quantum mechanics and relativity. We will see that such an identification of the classical space and eventually of space-time with a subset of a space of states allows one to extend classical physics from space-time to the space of states. The extension will be shown to go beyond the framework of representations of groups. Simultaneously, the embedding puts processes on the space of states in a clear relationship with the classical space submanifold.

The idea of identification of points with delta states runs into an immediate difficulty: delta functions are not in the common space $L_2(\mathbb{R}^3)$ of Lebesgue square-integrable functions on $\mathbb{R}^3$. The usual way out of this difficulty is to approximate this singular generalized function by a square-integrable function, e.g., a Gaussian function. However, the new function is not then an eigenstate of the position operator. A common rigorous way of working with delta states is to use the rigged Hilbert space construction of Gelfand Ref.[9]. In this case one "sandwiches" the space $L_2(\mathbb{R}^3)$ between a topological subspace $\Phi \subset L_2(\mathbb{R}^3)$ and its dual Φ^*, where Φ^* contains delta functions. However, the delta functions in the resulting rigged Hilbert space $\Phi \subset L_2(\mathbb{R}^3) \subset \Phi^*$ have no norm and cannot be treated on equal footing with the square-integrable states. An alternative way to be followed here is to modify the Hilbert space of state functions so that it contains delta functions and is "approximately equal" to the space $L_2(\mathbb{R}^3)$. For this, note that the inner product of functions φ, ψ in the space $L_2(\mathbb{R}^3)$ can be formally written as follows:

$$(\varphi, \psi)_{L_2} = \int \delta^3(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})\overline{\psi}(\mathbf{y})d^3\mathbf{x}d^3\mathbf{y}. \tag{1}$$

In particular, the fact that delta functions are not in $L_2(\mathbb{R}^3)$ is related to the singularity of the kernel $\delta^3(\mathbf{x}-\mathbf{y})$ of the metric. Let's approximate this kernel by the Gaussian function $\left(\frac{L}{\sqrt{2\pi}}\right)^3 e^{-\frac{L^2}{2}(\mathbf{x}-\mathbf{y})^2}$ for some positive constant L and introduce the inner product

$$(\varphi, \psi)_{\mathbf{H}} = \left(\frac{L}{\sqrt{2\pi}}\right)^3 \int e^{-\frac{L^2}{2}(\mathbf{x}-\mathbf{y})^2}\varphi(\mathbf{x})\overline{\psi(\mathbf{y})}d^3\mathbf{x}d^3\mathbf{y}. \tag{2}$$

The norm of the delta function $\delta^3_{\mathbf{a}}$ is then a constant independent of $\mathbf{a}$. The facts that Eq.(2) is indeed an inner product, that the corresponding norm of functions in $L_2(\mathbb{R}^3)$ is finite, and that for any $\varphi \in L_2(\mathbb{R}^3)$ and a sufficiently large L this norm can be made arbitrarily close to the usual $L_2(\mathbb{R}^3)$-norm were verified in Refs.[7],[8]. In this sense the Hilbert space $\mathbf{H}$ obtained by completing the space $L_2(\mathbb{R}^3)$ in the above norm approximates $L_2(\mathbb{R}^3)$ (symbolically, $\mathbf{H} \approx L_2(\mathbb{R}^3)$). Accordingly, if $\mathbf{H}$ is used in place of $L_2(\mathbb{R}^3)$ in the usual quantum mechanics, the expected values, probabilities of transition and other measured quantities remain practically the same, ensuring consistency with experiment.

Note that by an appropriate choice of units of length one can always ensure that the constant L in Eq.(2) is equal to one. Furthermore, the coefficient in front of the integral in Eq.(2) can be dropped without changing relative probabilities of events. The resulting inner product is not only consistent with the formalism of the standard quantum mechanics, but also yields physically meaningful results for systems in a position eigenstate. Thus, the expected value of the position operator for such a system gives

$$(\delta^3_{\mathbf{a}}, \widehat{\mathbf{x}}\delta^3_{\mathbf{a}})_{\mathbf{H}} = \int e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2}\delta^3(\mathbf{x}-\mathbf{a})\overline{\widehat{\mathbf{y}}\delta^3(\mathbf{y}-\mathbf{a})}d^3\mathbf{x}d^3\mathbf{y} = \mathbf{a}. \tag{3}$$

Following the plan, consider now the set M_3 of all delta functions in $\mathbf{H}$. As shown in Refs.[7],[8], the set M_3 forms a 3-dimensional submanifold of $\mathbf{H}$ and the Hilbert metric in $\mathbf{H}$ being restricted to M_3 yields the ordinary Euclidean metric. So as a manifold with metric, M_3 is identical to the Euclidean space $\mathbb{R}^3$. At the same time M_3 is not a vector subspace of $\mathbf{H}$: adding two delta functions does not yield a delta function in general. Rather, because the norm of any delta function $\delta^3_{\mathbf{a}}$ in $\mathbf{H}$ is the same, M_3 is a submanifold of the sphere in $\mathbf{H}$. Nevertheless, a vector structure on M_3 exists. Namely, the operations of addition $\oplus$ and multiplication by a scalar $\lambda\odot$ can be defined via $\omega(\mathbf{a}) \oplus \omega(\mathbf{b}) = \omega(\mathbf{a}+\mathbf{b})$ and $\lambda \odot \omega(\mathbf{a}) = \omega(\lambda\mathbf{a})$, where the map $\omega : \mathbb{R}^3 \longrightarrow \mathbf{H}$ is given by $\omega : \mathbf{a} \longrightarrow \delta^3_{\mathbf{a}}$. The resulting operations are continuous in the topology of $M_3 \subset \mathbf{H}$.

These results signify that the space $\mathbb{R}^3$ is mathematically identical to the manifold M_3. But what about their physical equivalence? In the context of the single particle mechanics the question is whether the motion of a point particle in the classical mechanics can be identified with a motion of state in M_3. Note first of all that a classical particle has a definite position $\mathbf{a}(t)$ at any moment of time t so that its quantum state is given by the delta function $\delta^3_{\mathbf{a}(t)}$. The motion of the particle is then represented by a path on M_3. Of course, the full physical equivalence of the classical motion of a particle with a motion of state in M_3 requires a dynamical law of motion. Presumably, such a law must be consistent with the Schrödinger evolution of microscopic particles and the classical Newtonian motion of material points. Although beginning with the Ehrenfest theorem there are many indications that Schrödinger dynamics is indeed consistent with the Newtonian limit, a complete understanding of this consistency is yet to be found.

III. EXTENDING SPECIAL RELATIVITY TO A HILBERT SPACE

The next step is to extend the embedding results to Minkowski space-time and to build a framework that incorporates relativity on Minkowski space-time and quantum mechanics of a single particle. To obtain a relativistic framework one needs to work with spaces of functions of four variables $\mathbf{x}, t$. It is straightforward to generalize the described embedding of $\mathbb{R}^3$ to the case of the Euclidean space $\mathbb{R}^4$. For this it suffices to begin with the Hilbert space $L_2(\mathbb{R}^4)$ of square-integrable functions of four variables $x = (\mathbf{x}, t)$ and complete it in the metric given by the kernel $e^{-\frac{1}{2}(x-y)^2}$. As before, the space $\widetilde{H}$ contains delta functions δ_a^4 and the set M_4 of all delta functions forms a submanifold of $\widetilde{H}$ with the induced Euclidean metric.

To encode the Minkowski space N consider the Hermitian form $(f,g)_{H_\eta}$ given by

$$(f,g)_{H_\eta} = \int e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2+\frac{1}{2}(t-s)^2} f(\mathbf{x},t)\overline{g(\mathbf{y},s)} d^3\mathbf{x} dt d^3\mathbf{y} ds \tag{4}$$

and let $(f,f)_{H_\eta} \equiv \|f\|^2_{H_\eta}$ be the corresponding quadratic form, or the squared H_η-norm. Because of the positive term in the exponent of the kernel in Eq.(4), not every function in the Hilbert space $\widetilde{H}$ has a finite H_η-norm. In addition, the quadratic form $(f,f)_{H_\eta}$ is not positive-definite. More precisely, if $f \neq 0$ and $f(\mathbf{x},t)$ is even in t, then $(f,f)_{H_\eta} > 0$. Likewise, if $f \neq 0$ and $f(\mathbf{x},t)$ is odd in t, then $(f,f)_{H_\eta} < 0$. So, let H be the set of functions in $\widetilde{H}$ whose even and odd in t components have a finite H_η-norm. As shown in Ref.[8], H is exactly the set of all functions $f(\mathbf{x},t) = e^{-t^2}\varphi(\mathbf{x},t)$ with $\varphi \in \widetilde{H}$. Moreover, H furnished with the inner product $(f,g)_{H_+} = (\varphi,\psi)_{\widetilde{H}}$, where $f(\mathbf{x},t) = e^{-t^2}\varphi(\mathbf{x},t)$, $g(\mathbf{x},t) = e^{-t^2}\psi(\mathbf{x},t)$, is a Hilbert space. The Hermitian form Eq.(4) defines an indefinite, non-degenerate inner product on H. Finally, H contains the delta functions $\delta_a^4(x) = \delta^4(x-a)$ and their derivatives for all $a \in N$.

Consider now the set M_4 of all delta functions in H. One reason why the space H is useful for the issues of unification is because the Minkowski space-time N can be identified with M_4. Namely, similarly to the case of the Euclidean space $\mathbb{R}^3$, the set M_4 is a submanifold in H that is diffeomorphic to N. Moreover, the indefinite metric on H yields the Minkowski metric on M_4, while the $\widetilde{H}$-metric yields the ordinary Euclidean metric on M_4. So, similarly to the space $\mathbb{R}^3$, the Minkowski space N is "encoded" into the space H. As before, the map $\omega : N \longrightarrow H$, defined by $\omega(a) = \delta_a^4$ is not linear, but its image M_4 can be furnished with a linear structure induced from N.

A motion of a macroscopic particle in relativistic mechanics is now realized as a motion in $M_4 \subset H$. This appears to be only a relatively minor achievement as it is always possible to embed a manifold into another one of a sufficiently high dimension. However, the spaces M_4 and H are related in a very unusual, rigid way. It will be shown that because of that, physics on N can be "lifted" in a unique way to the entire space H. Moreover, the space H will be shown to be directly related to the space $\mathbf{H}$ of the previous section. This allows for the standard quantum mechanics to be interpreted as a theory on H.

Let's first "lift" the relativity theory from Minkowski space-time N onto the Hilbert space H. Because N is identified with the subset M_4 of H, the Poincaré group P acts on M_4. Namely, if Π is a Poincaré transformation on N and $\omega : N \longrightarrow H$ is defined as before by $\omega(a) = \delta^4_a$, then $\widetilde{\delta_\Pi} = \omega \Pi \omega^{-1}$ acts on M_4 mapping δ^4_a to $\delta^4_{\Pi a}$. The kernel of the transformation $\widetilde{\delta_\Pi}$ is $\delta^4(\Pi^{-1}x' - x)$, so that $\int \delta^4(\Pi^{-1}x' - x)\delta^4(x - a)d^4x = \delta^4(\Pi^{-1}x' - a) = \delta^4(x' - \Pi a)$. Now a very special way in which M_4 is embedded into H comes into play. Roughly speaking, the set M_4 plays the role of a basis in H. More precisely, one can check (Ref.[7]) that delta functions in any finite subset of M_4 are linearly independent and that M_4 is a complete set in H (i.e., there is no function in H that is orthogonal to all delta functions in M_4). Because of that there is a unique linear extension of the transformation $\widetilde{\delta_\Pi}$ from M_4 onto H. The resulting transformation δ_Π is defined by the same kernel via $(\delta_\Pi f)(x') = \int \delta^4(\Pi^{-1}x' - x)f(x)d^4x = f(\Pi^{-1}x')$. So, despite the fact that M_4 is 4-dimensional and H is infinite-dimensional, there is only one linear extension of the action of the Poincaré group from M_4 onto H!

At first sight the map $T_\Pi : \Pi \longrightarrow \delta_\Pi$ seems to be a representation of the Poincaré group P on H. So it seems that the discussed extension of relativity to H is just a representation of P on H. A closer look reveals the following more complicated picture. First of all, the Hilbert metric on H is not invariant under general Poincaré transformations and the operator δ_Π is not bounded as a map into H. Accordingly, the map T_Π is not a representation of the Poincaré group P. Second, even if one wants to interpret T_Π as a representation of some sort of P, the fact that the original Minkowski space on which P acts is itself embedded into the space H of the "representation" is a new addition to the theory that is responsible for the new results. Uniqueness of the constructed extension is one such result. Existence and uniqueness of a similar extension for the diffeomorphism group of a curved space-time (see Sec.V) is another one.

Note that the set of all functions $f \circ \Pi^{-1}$ with a fixed Π and all $f \in H$ form a Hilbert space H' with the inner product defined by $(\delta_\Pi f, \delta_\Pi g)_{H'_+} = (f, g)_{H_+}$. The map $\delta_\Pi : H \longrightarrow H'$ is then an isomorphism of Hilbert spaces. Hilbert spaces obtained in such a way can be thought of as different realizations of one and the same abstract Hilbert space $\mathbf{S}$. A particular isomorphism $\Gamma : \mathbf{S} \longrightarrow H$ can be thought of as a coordinate chart on $\mathbf{S}$ (see Ref.[7]).

These results can be summarized by the following commutative diagram

$$\begin{array}{ccccc} \mathbf{S} & \xrightarrow{\Gamma} & H & \xrightarrow{\delta_\Pi} & H' \\ \uparrow{\scriptstyle \Omega} & & \uparrow{\scriptstyle \omega} & & \uparrow{\scriptstyle \omega} \\ N & \xrightarrow{\gamma} & \mathbb{R}^{1,3} & \xrightarrow{\Pi} & \mathbb{R}^{1,3} \end{array} \tag{5}$$

In the diagram, $\Gamma : \mathbf{S} \longrightarrow H$ is an isomorphism of the abstract Hilbert space $\mathbf{S}$ with an additional indefinite metric onto the space H defined above; $\gamma : N \longrightarrow \mathbb{R}^{1,3}$ is a global coordinate chart from the Minkowski space-time onto the coordinate space of the observer in an inertial reference frame K; Π is a Poincaré transformation that relates coordinates associated with frames K and K'. Given these maps the remaining two maps, the metric preserving embedding Ω and the isomorphism δ_Π are uniquely determined by the diagram. Here δ_Π preserves both the Hilbert and the indefinite metrics.

What is a physical significance of the diagram (5)? According to the diagram a particular choice of coordinates on Minkowski space-time N determines a particular functional realization of the abstract Hilbert space $\mathbf{S}$. So an inertial observer picks up not just coordinates on N, but also a specific realization of $\mathbf{S}$. The diagram demonstrates that within the considered assumptions the embedding ω preserves the structure of special relativity and extends it in a unique way to the abstract Hilbert space $\mathbf{S}$.

Note that δ_Π maps delta functions to delta functions, so that in accord with the diagram (5) the image of the manifold M_4 in H is the submanifold M_4 in H'. This together with the fact that ω is a metric preserving embedding is what allows for the usual theory of relativity on Minkowski space-time to be a part of the new framework. At the same time completeness of the set M_4 together with linear independence of its elements makes the entire construction rigid, ensuring uniqueness of the extension. It will be shown in the following section that the standard non-relativistic quantum mechanics is a theory on the Hilbert space H. These results together with their generalization to curved space-times in Sec.V mean that diagram (5) may serve a basis for the unification of relativity and quantum mechanics.

The proposed method of extension of the Poincaré group action from N onto $\mathbf{S}$ can be applied to *any* group of continuous transformations of N. The transformations become linear when extended to $\mathbf{S}$. As in the case of the Poincaré group, this is due to the fact that moving across N corresponds to going "across dimensions" of H so that a linear extension of the transformation becomes possible. Completeness of the set M_4 in H ensures then that such an extension is unique. Note that considered transformations and their extensions are "passive", i.e. they describe changes in coordinate realizations of the fixed, invariant spaces N and $\mathbf{S}$. The corresponding "active" version of the diagram is also possible (see Ref.[10]).

IV. THE SPACE H AND QUANTUM MECHANICS OF A SINGLE PARTICLE

Having constructed an extension of special relativity to the Hilbert space $\mathbf{S}$ let's now turn to the embedding of quantum mechanics into the framework. The first thing to do is to relate the Hilbert space H of functions of four variables $\mathbf{x}, t$ to the usual Hilbert spaces of functions of three variables $\mathbf{x}$ with t as a parameter of evolution. For this consider the family of subspaces H_τ of H each consisting of all functionals $\varphi_\tau(\mathbf{x}, t) = \psi(\mathbf{x}, t)\delta(t-\tau)$ for some fixed $\tau \in \mathbb{R}$. The inner product of any two functionals in H_τ in either $\widetilde{H}$ or H_η metrics of the previous section is simply the inner product in the space $\mathbf{H}$ introduced in Sec.II. This is the case because the factor $\delta(t-\tau)$ eliminates integration in t and makes the term $(t-s)^2$ in the exponent of the kernel of the metric vanish. In the following H_τ will be understood as a Hilbert space with this inner product. The map $I : H_\tau \longrightarrow \mathbf{H}$ defined by $I(\varphi_\tau)(\mathbf{x}) = \psi(\mathbf{x}, \tau)$ is then an isomorphism of Hilbert spaces.

Because $\mathbf{H} \approx L_2(\mathbb{R}^3)$, the map I basically identifies each subspace H_τ with the usual space $L_2(\mathbb{R}^3)$ of state functions on $\mathbb{R}^3$ considered at time τ. The reason why these particular subspaces are physically meaningful becomes clear from the following result that relates the dynamics on the family of subspaces H_τ and the usual space $L_2(\mathbb{R}^3)$ of states of a spinless non-relativistic particle. Let $\widehat{h} = -\Delta + V(\mathbf{x}, t)$ be the usual Hamiltonian of non-relativistic quantum mechanics of a single particle. Then the function $\psi(\mathbf{x}, t)$ satisfies the Schrödinger equation $\frac{\partial \psi(\mathbf{x},t)}{\partial t} = -i\widehat{h}\psi(\mathbf{x}, t)$ if and only if the path $\varphi_\tau(\mathbf{x}, t) = \psi(\mathbf{x}, t)\delta(t-\tau)$ in H satisfies the equation $\frac{d\varphi_\tau}{d\tau} = \left(-\frac{\partial}{\partial t} - i\widehat{h}\right)\varphi_\tau$. In particular, the usual Schrödinger dynamics of a single particle can be derived from a dynamics on the Hilbert space H of functions of four variables. Namely, the ordinary Schrödinger evolution is recovered from the evolution φ_τ in the space H by projecting the path φ_τ onto the "co-moving" subspace H_τ identified via I with $\mathbf{H} \approx L_2(\mathbb{R}^3)$. The component $-i\widehat{h}\varphi_\tau$ of the velocity describes the motion within the subspace H_τ, while the orthogonal ("vertical") component $-\frac{\partial \varphi_\tau}{\partial t}$ of the velocity is due to the motion of the subspace H_τ itself (see Ref.[8]).

Under integration over time on functions in H_τ the time variable gets replaced with the parameter τ. In other words, for motions within the family of subspaces H_τ the evolution parameter τ that describes motions in the space H of functions of four variables becomes identified with the usual time variable that appears in Schrödinger equation. Furthermore, the delta factor $\delta(t-\tau)$ in functions in H_τ removes integration over time and therefore eliminates the effect of interference in time that is present for more general elements of H. In fact, the norm of superposition $\psi_1(\mathbf{x}, t)\delta(t-\tau) + \psi_2(\mathbf{x}, t)\delta(t-\tau)$ of functions in H_τ gives $\|\psi_1(\mathbf{x}, \tau) + \psi_2(\mathbf{x}, \tau)\|_{\mathbf{H}}$, which approximates the standard expression due to the relationship $\mathbf{H} \approx L_2(\mathbb{R}^3)$.

Recall that the space H was introduced to identify Minkowski space with the submanifold $M_4 \subset H$ and to extend relativity to H. The resulting embedding also gives a simple reason for selection of subspaces H_τ in H, needed to obtain the standard quantum mechanical framework. In fact, elements of the space H have the form $e^{-t^2}\varphi(\mathbf{x}, t)$, where φ is in the space $\widetilde{H} \approx L_2(\mathbb{R}^4)$ (and the meaning of approximation is the same as in Sec.II). Define the space H_T by application of the isomorphism $(\delta_{\Pi_\tau} f)(\mathbf{x}, t) = f(\mathbf{x}, t-\tau)$ to the space H. The space H_T (the space of the "time co-moving" representation) consists of the functions $e^{-(t-\tau)^2}\varphi(\mathbf{x}, t)$, with $\varphi \in \widetilde{H} \approx L_2(\mathbb{R}^4)$. The variables $\mathbf{x}, t$ enter symmetrically in the definition of $L_2(\mathbb{R}^4)$, while the factor $e^{-(t-\tau)^2}$ breaks the symmetry between $\mathbf{x}$ and t by making a typical element of H_T well localized in the time variable. In a sufficiently small scale the factor $e^{-(t-\tau)^2}$ as a function of $t-\tau$ quickly falls off to almost zero and can be replaced with the delta function $\delta(t-\tau)$. This yields the family of subspaces H_τ and allows for the usual formalism of quantum mechanics. In particular, the fact that interference in time is not observed in typical quantum mechanical experiments can be traced in the theory back to the indefinite metric Eq.(4) on the Hilbert space H.

Subspaces H_τ are not preserved under the general Poincaré maps δ_Π. In fact, δ_Π mixes space and time coordinates and therefore does not preserve the form $\varphi(\mathbf{x}, t)\delta(t-\tau)$ of elements of H_τ in general. This is not surprising because standard quantum mechanics is non-relativistic. However, to provide a valid foundation of the non-relativistic quantum mechanics these subspaces must be preserved under Galileo transformations. A Galileo transformation G yields the map $\delta_G : H \longrightarrow H'$ defined by $\delta_G f = f \circ G^{-1}$ for all $f \in H$. This map transforms the state $\varphi(\mathbf{x}, t)\delta(t-\tau)$ into the state $\varphi(A\mathbf{x}+\mathbf{v}t+\mathbf{b},t+c)\delta(t+c-\tau)$, where A is an orthogonal transformation, $\mathbf{v} and \mathbf{b}$ are 3-vectors and c is a real number. Recall now that φ is an element of the Hilbert space $\mathbf{H}$ with metric given by the kernel $e^{-\frac{1}{2}(\mathbf{x}-\mathbf{y})^2}$. This kernel is obviously invariant under Galileo transformations so that the function $\varphi(A\mathbf{x}+\mathbf{v}t+\mathbf{b},t+c)$ is still an element of $\mathbf{H}$. One concludes that Galileo transformations yield isomorphisms between subspaces H_τ (and that the map $G \longrightarrow \delta_G$, where δ_G is considered as acting on $\mathbf{H}$ is a unitary representation of the Galileo group).

Note that the equation $\frac{d\varphi_\tau}{d\tau} = \left(-\frac{\partial}{\partial t} - i\widehat{h}\right)\varphi_\tau$ with usual Hamiltonian is a well known non-relativistic limit of the Stueckelberg-Schrödinger equation in the theory of Stueckelberg (Ref.[4]) and Horwitz & Piron (Ref.[5]). This theory treats space and time symmetrically and predicts interference in time (Refs.[11],[12]). The non-relativistic limit of Stueckelberg theory was investigated by Horwitz and Rotbart in Ref.[13]. The approximate equality of the time variable t with the evolution parameter τ obtained in Ref.[13] is consistent with the definition of H_τ.

Newton and Wigner (Ref.[2]) argue that delta functions δ^4_a cannot represent spatially localized states in a relativistic theory. However, their derivation is based on the condition of orthogonality of a localized state and its spatial displacement, which is not valid in the proposed framework. Newton and Wigner mention other difficulties with delta states: (1) they are not square integrable, (2) if a sequence of square integrable functions converges to δ^4_a in one frame, it will not do so in general in a Lorentz transformed frame. Both statements also lose their validity in the new setting. The operator having δ^4_a as an eigenfunction with the eigenvalue a^μ is the operator $\widehat{x^\mu}$ of multiplication by the variable x^μ, $\mu = 0,1,2,3$.

Note that the delta function locality is present in the relativistic Stueckelberg theory, which is off-shell. If the Stueckelberg expectation value of the dynamical variable $\widehat{x^\mu}$ (the four-point position operator of Horwitz and Rotbart Ref.[13]) is decomposed into a direct integral over mass, then for each definite mass in the integral, the Newton-Wigner operator (having Newton-Wigner localized states as eigenstates) emerges. Locality is restored in the result of the integral (Refs.[5],[13]).

The covariant property of the states δ^4_a and the operator $\widehat{x^\mu}$ does not mean by itself that the found objects are physical. There are well known difficulties: (1) the wave packet $\delta^3_{\mathbf{a}}$ contains negative energy components; (2) if such a packet is allowed to evolve by the usual relativistic equations it will evolve out of the light cone (Ref.[3]). Although these difficulties are typical for relativistic on-shell wave equations and were partially understood within the Stueckelberg approach (Ref.[5]), they must be reexamined in the new setting.

V. GENERALIZING THE FRAMEWORK TO CURVED SPACE-TIME MANIFOLDS

So far the discussion involved only the classical 3-dimensional Euclidean space and the Minkowski space-time. If the approach is taken seriously, it becomes essential to check its validity for more general space-times N. It is also important to see whether the Hilbert space into which N is embedded can be defined without specifying the manifold first. For example, is it possible to identify an *arbitrary* curved space-time N with the set of all delta functions in a Hilbert space of functions on $\mathbb{R}^4$ (rather than on N)?

A global result proved in Ref.[10] says that for an arbitrary manifold N there exists a Hilbert space $H_{\mathbb{R}^n}$ of continuous functions on $\mathbb{R}^n$, such that the set M_n of all delta functions in the dual space $H^*_{\mathbb{R}^n}$ is an embedded submanifold of $H^*_{\mathbb{R}^n}$ diffeomorphic to N. In essence, the theorem claims that an arbitrary n-dimensional manifold can be "encoded" into an appropriate Hilbert space of functions on $\mathbb{R}^n$.

To get an idea of how to find the Hilbert space $H_{\mathbb{R}^n}$, especially when the topology of the manifold is not trivial, consider the case of a circle S^1. In this case the space $H_{\mathbb{R}}$ must be a Hilbert space of continuous functions on $\mathbb{R}$. To ensure that the image M_1 of the map $\omega : \mathbb{R} \longrightarrow H^*_{\mathbb{R}}$, $\omega(a) = \delta_a$ is a circle, one needs $\delta_a = \delta_{a+2\pi}$ for all $a \in \mathbb{R}$, which means that functions in $H_{\mathbb{R}}$ must be 2π-periodic. To satisfy these conditions, consider the Sobolev space of continuous 2π-periodic functions on $\mathbb{R}$ with the inner product $(f,g) = \int_{-\pi}^{\pi} \left(f(x)\overline{g}(x) + f'(x)\overline{g'}(x) \right) dx$. It is easy to check that the set of all delta functions in the dual space $H^*_{\mathbb{R}}$ with the induced topology is a circle S^1.

Note that a particular manifold in the theorem is encoded by fixing the *functional content* of the Hilbert space rather than fixing the domain of the functions. To put it differently, the manifold M_n is "made of" functions and not points in the domain of the functions. In other words, the manifold structure is not presupposed, but is *derived* from a purely functional framework.

The above global result says nothing about the metric induced on the set M_4. Is it possible to choose the space H so that the restriction of its metric to M_4 is a given pseudo-Riemannian metric on the space-time N? In mathematical terms, the question is whether the described embedding of N into H is isometric for some H. It turns out that the answer to this question is positive, at least locally. Namely, for an arbitrary pseudo-Riemannian manifold N and any point $x \in N$ there is a neighborhood W of x in N and a Hilbert space H with an additional indefinite metric that contains delta functions $\delta^{(n)}_a$ for all a in an open set U in $\mathbb{R}^n$ such that the set M_n of all these delta functions is a submanifold of H isometric to W (see Ref.[10]). In simple words it means that by changing a Hilbert space of functions on $\mathbb{R}^n$ one can locally make its set of delta functions to be a neighborhood in an arbitrary curved space-time.

These statements allow one to extend the results of Sec.III to neighborhoods in arbitrary pseudo-Riemannian space-times. In this case the Poincaré group acting on Minkowski space-time is replaced by the group of diffeomorphisms of a particular neighborhood. For instance, this yields the following analogue of diagram Eq.(5):

$$\begin{array}{ccccc} \mathbf{S} & \xrightarrow{\Gamma} & H & \xrightarrow{\delta_D} & H' \\ \uparrow{\scriptstyle \Omega} & & \uparrow{\scriptstyle \omega} & & \uparrow{\scriptstyle \omega} \\ W & \xrightarrow{\gamma} & U & \xrightarrow{D} & U \end{array} \tag{6}$$

Here W is a neighborhood in curved space-time, γ is a chart on W and U is the corresponding set in $\mathbb{R}^4$, D is an arbitrary diffeomorphism of U and δ_D is its extension to the space H. The existence and uniqueness of the isomorphism δ_D and the space H' can be proved as in the flat case.

Recall that the set M_4 is invariant under transformations δ_Π, making it possible to "separate" special relativity from the Hilbert space framework. In Sec.IV it was verified that the "Galileo maps" δ_G map subspaces H_τ onto themselves. This explains why the non-relativistic quantum mechanics of a spinless particle could also be developed within a single Hilbert space of functions of three variables. Diagram (5) provides us with a "covariant" extension of special relativity. Likewise, diagram (6) together with its active version yield a local geometric extension of general relativity. Those extensions are based on isomorphisms of separable Hilbert spaces. If such a scheme is adopted in physics, that would mean that specific functional realizations of the abstract Hilbert space **S**, at least within the considered class of realizations, are not physical but rather are similar to various choices of coordinates on space-time. One may disregard this point by saying that considered isomorphisms of Hilbert spaces of functions are direct analogues of well known changes in representation in quantum theory. However, unlike changes of representation that are simply passive changes in the description of physical reality, the transformations considered here can be realized actively. Active transformations create a new physical reality and make the construction physically meaningful.

If the discussed embedding of the classical space $\mathbb{R}^3$ into **H** as well as the embeddings of Minkowski space-time and local embeddings of arbitrary curved space-times into the corresponding Hilbert spaces are taken seriously, then the linearity of quantum theory appears in a completely new light. In fact, the geometry of the abstract Hilbert space **S** and its realizations, like H, is linear. It is the non-linearity of submanifolds M_3 and M_4 that seems to be responsible for the non-linear way in which classical world appears to us. By replacing the restricted, "space-time based" view of the world with its extension to the space **S** one can perhaps obtain a tool for reconciliation of quantum theory and relativity.

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