THE STABILITY OF GRADED MULTIPLICITY IN THE SETTING OF THE KOSTANT-RALLIS THEOREM.

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Abstract. From a combinatorial point of view, we approach the problem of finding a graded generalization of the Kostant-Rallis theorem concerning the K-harmonic polynomials on $\mathfrak{p}$. Specifically, for each classical symmetric pair we obtain a stable range where the multiplicity of an irreducible K-representation in the degree d harmonic polynomials can be expressed in terms of Littlewood-Richardson coefficients.

We investigate a question motivated by a theorem due to Bertram Kostant and Stephen Rallis (see [KR71]), which we recall briefly: For a vector space V, let $\mathbb{C}[V]$ denote the commutative algebra of polynomial functions on V. If V carries the structure of a representation of a group K, then K acts on $\mathbb{C}[V]$ in the usual way. Observe that we have a grading by degree, and let $\mathbb{C}[V]^d$ denote the degree d homogeneous polynomial functions on V.

Let K be a symmetric subgroup of a reductive algebraic group G over the complex numbers. Denote the Lie algebra of K by $\mathfrak{k}$. We take $\mathfrak{p}$ to be the Cartan complement of $\mathfrak{k}$ in $\mathfrak{g}$. Let $\mathcal{H}_\mathfrak{p}$ denote the subspace of $\mathbb{C}[\mathfrak{p}]$ consisting of K-harmonic polynomials. Fix a maximal toral subalgebra, $\mathfrak{a}$, in $\mathfrak{p}$, and let M denote the centralizer of $\mathfrak{a}$ in K. We have:

Theorem (Kostant–Rallis)**.** *The space $\mathbb{C}[\mathfrak{p}]$ is a free module over the ring of K-invariant polynomials, with*

$$\mathbb{C}[\mathfrak{p}] \cong \mathbb{C}[\mathfrak{p}]^K \otimes \mathcal{H}_\mathfrak{p},$$

and furthermore, as a representation of K, $\mathcal{H}_\mathfrak{p}$ is equivalent to the representation algebraically induced from the trivial representation of M to K,

$$\mathcal{H}_\mathfrak{p} \cong \mathit{Ind}_M^K 1.$$

The fact that $\mathcal{H}_\mathfrak{p}$ is an induced representation implies, by Frobenius reciprocity, that the multiplicity of an irreducible K-representation, V, in $\mathcal{H}_\mathfrak{p}$ is the dimension of the subspace of M-invariant vectors in V.

The space of harmonic polynomials inherits a gradation from $\mathbb{C}[\mathfrak{p}]$. We denote $\mathcal{H}_\mathfrak{p} = \bigoplus \mathcal{H}_\mathfrak{p}^d$ with $\mathcal{H}_\mathfrak{p}^d = \mathbb{C}[\mathfrak{p}]^d \cap \mathcal{H}_\mathfrak{p}$. Central to the present work is that the Kostant-Rallis theorem does not address the distribution of the K-multiplicity among the graded components of $\mathcal{H}_\mathfrak{p}$.

One source of motivation is that as a K-representation $\mathcal{H}_\mathfrak{p}$ is isomorphic to the space of K-finite vectors in the spherical principal series for the real group corresponding to the pair (G, K). A more detailed understanding of the principal series is a central problem in representation theory. See for example [Kos69], [Joh76], [JW72], [JW77], [HT93], [Lee94], and [How00], to name just a few.

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The Kostant-Rallis theorem is a generalization of Kostant's earlier result concerning the action of G on the harmonic polynomials, $\mathcal{H}_{\mathfrak{g}}$, on $\mathfrak{g}$ (see [Kos63a] [Kos63b]). In this case, the multiplicity of an irreducible representation, V, in $\mathcal{H}_{\mathfrak{g}}$ is the dimension of the subspace of vectors in V invariant under a maximal torus. Later in [Hes80], Hesselink provided a graded multiplicity formula as well, which involved an alternation over a Weyl group of Lusztig's q-analog of the Kostant partition function. In [WW00], similar formulas were obtained for certain examples including: (GL_{2n}, Sp_{2n}), (SO_{2n+2}, SO_{2n+1}), and (E_6, F_4).

Note that such a formula does not exist in the general case. Furthermore, even when such a formula does exist, the question arises as to whether or not we may we replace it with a combinatorial statistic not involving an alternation. This more combinatorial theme is the focus of the present paper. We precisely state the main result the next section and describe the proof in a fairly uniform case-by-case manner in Section 2.

1. Preliminaries

This section is divided into four parts: In Sections 1.1 and 1.2, we set up some notation for our results which are stated precisely in 1.3. Finally, in 1.4 we explain how the main theorem may be interpreted as a description of the graded multiplicity in term of the Littlewood-Richardson coefficients.

1.1. **Statement of the Problem.**

The problem addressed in this paper is to decompose the graded components of $\mathcal{H}_K(\mathfrak{p})$ as a representation of K. There are two caveats however: First, we consider only classical groups. Secondly, we provide an "answer" only stably. That is to say, for fixed d we describe the decomposition of $\mathcal{H}_K^d(\mathfrak{p})$ only when the rank of all simple factors of K are large. The answer therefore depends on the *family* of the symmetric pair.

In order to be more precise, we set up notation to index the irreducible representations of the classical groups.

1.1.1. *Parametrization of Representations.* Let G be a classical reductive algebraic group over $\mathbb{C}$. That is the say, $G = GL_n(\mathbb{C}) = GL_n$ (the general linear group); or $G = O_n(\mathbb{C}) = O_n$, (the orthogonal group); or $G = Sp_{2n}(\mathbb{C}) = Sp_{2n}$ (the symplectic group).

We label the irreducible regular representations of G using integer partitions. If G is connected, an irreducible representation is determined by its highest weight. For the classical groups, the highest weights of representations may be parameterized by points in an integer lattice in a standard way following [GW98]. In the non-connected case (i.e., O_n), there is a minor modification that may be made to include the action of the component group in the parametrization. We recall our notation below.

A non-negative integer *partition* λ, with k parts, is an integer sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$. We may sometimes refer to λ as a *Young diagram.* We use the same notation for partitions as is done in [Mac95]. For example, we write $\ell(\lambda)$ to denote the *length* (or *depth*) of a partition. Also let $|\lambda| = \sum_i \lambda_i$ be the size of a partition λ and λ' denote the *transpose* (or *conjugate*) of λ (i.e., $(\lambda')_i = |\{\lambda_j : \lambda_j \geq i\}|$).

$\mathbf{GL_n}$ *Representations*: Given non-negative integers p, q and n such that $n \geq p + q$ and non-negative integer partitions λ^+ and λ^- with p and q parts respectively, let $F_{(n)}^{(\lambda^+,\lambda^-)}$ denote the irreducible rational representation of GL_n with highest weight given by the n-tuple:

$$(\lambda^+, \lambda^-) = \underbrace{\left(\lambda_1^+, \lambda_2^+, \cdots, \lambda_p^+, 0, \cdots, 0, -\lambda_q^-, \cdots, -\lambda_1^-\right)}_{n}$$

Note that we will also write $F_{(n)}^{\lambda^+}$ for $F_{(n)}^{(\lambda^+,(0))}$. Note that $\left(F_{(n)}^{\lambda^-}\right)^*$ is equivalent to $F_{(n)}^{((0),\lambda^-)}$. More generally, $\left(F_{(n)}^{(\lambda^+,\lambda^-)}\right)^*$ is equivalent to $F_{(n)}^{(\lambda^-,\lambda^+)}$.

$\mathbf{O_n}$ *Representations:* The complex orthogonal group O_n has two connected components. Because the group is disconnected, we cannot index irreducible representations by highest weights. There is however an analog of Schur-Weyl duality for the case of O_n in which each irreducible rational representation is indexed uniquely by a non-negative integer partition ν such that $(\nu')_1 + (\nu')_2 \leq n$. That is, the sum of the first two columns of the Young diagram of ν is at most n. We will call such a diagram O_n-admissible (see [GW98] Chapter 10 for details). Let $E_{(n)}^{\nu}$ denote the irreducible representation of O_n indexed by ν in this way.

An irreducible rational representation of SO_n may be indexed by its highest weight. In [GW98] Section 5.2.2, the irreducible representations of O_n are determined in terms of their restrictions to SO_n (which is a normal subgroup having index 2). We note that if $\ell(\nu) \neq \frac{n}{2}$, then the restriction of $E_{(n)}^{\nu}$ to SO_n is irreducible. If $\ell(\nu) = \frac{n}{2}$ (n even), then $E_{(n)}^{\nu}$ decomposes into exactly two irreducible representations of SO_n. See [GW98] Section 10.2.4 and 10.2.5 for the correspondence between this parametrization and the above parametrization by partitions.

The determinant defines an (irreducible) one-dimensional representation of O_n. This representation is indexed by the length n partition $\zeta = (1, 1, \cdots, 1)$. An irreducible representation of O_n will remain irreducible when tensored by $E_{(n)}^{\zeta}$, but the resulting representation *may* be inequivalent to the initial representation. We say that a pair of O_n-admissible partitions α and β are *associate* if $E_{(n)}^{\alpha} \otimes E_{(n)}^{\zeta} \cong E_{(n)}^{\beta}$. It turns out that α and β are associate exactly when $(\alpha')_1 + (\beta')_1 = n$ and $(\alpha')_i = (\beta')_i$ for all $i > 1$. This relation is clearly symmetric, and is related to the structure of the underlying SO_n-representations. Indeed, when restricted to SO_n, $E_{(n)}^{\alpha} \cong E_{(n)}^{\beta}$ if and only if α and β are either associate or equal.

$\mathbf{Sp_{2n}}$ *Representations:* For a non-negative integer partition ν with p parts where $p \leq n$, let $V_{(2n)}^{\nu}$ denote the irreducible rational representation of Sp_{2n} where the highest weight indexed by the partition ν is given by the n tuple:

$$\underbrace{(\nu_1, \nu_2, \cdots, \nu_p, 0, \cdots, 0)}_{n}.$$

1.1.2. *Stability.* A significant aspect of this paper is the notion of *stability.* In short, we use this word to refer to the asymptotic behavior of representations of GL_n, O_n and Sp_{2n} as $n \to \infty$. Although, in certain instances we will work with a product group such as $GL_p \times GL_q$. In this case our asymptotic statements refer to $\min(p, q) \to \infty$.

For instance, one aspect of stability is the phenomenon by which a partition (or pair of partitions) indexes a representation of GL_n (resp. O_n or Sp_{2n}) independent of n. For example, $(\lambda^+, \lambda^-) = ((1), (1))$ corresponds to the adjoint representation of GL_n for all $n \geq$

2. Thus it makes sense to take $n \to \infty$ when talking about multiplicity of the adjoint representation of GL_n.

1.2. **Background Material.**

1.2.1. *Notation for Formal Power Series.* If R is a commutative ring with 1 and q is an indeterminate, then the ring of formal power series q is denoted $R[[q]]$. For

$$f = a_0 + a_1 q^1 + a_2 q^2 + \cdots \in R[[q]]$$

we use $f|_{q^s} := a_s$ as our notation for the coefficient of q^s. In the present paper, we work with a coefficient ring $R = \mathrm{k}[x_1, x_2, x_3, \cdots]$, which is the (free) polynomial algebra over a countably infinite collection of indeterminates. We assume that the field, k, is of characteristic zero.

1.2.2. *Schur Functions.* Our notation is essentially the same as [Mac95]. We distinguish between the notions of Schur functions and Schur polynomials. We will work with the former. The latter are defined as follows: Let λ denote a partition. Given a finite collection of indeterminates $x_1, \cdots, x_n$,

$$s_\lambda(x_1, \cdots, x_n) = \frac{\sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^n x_{\sigma(i)}^{\lambda_i + n - i}}{\prod_{1 \leq i < j \leq n}(x_i - x_j)},$$

where S_n is the symmetric group on n letters and sgn denotes the usual sign function. It is a standard fact that $s_\lambda(x_1, \cdots, x_n)$ are symmetric polynomials in $x_1, \cdots, x_n$ and are a k-vector space basis for $\Lambda_n := \mathrm{k}[x_1, \cdots x_n]^{S_n}$ as λ runs over all partitions of length at most n.

By setting $x_{n+1} = 0$, we obtain a ring homomorphism from $\Lambda_{n+1} \to \Lambda_n$, which allows us to setup an inverse limit $\Lambda := \varprojlim \Lambda_n$. Since $s_\lambda(x_1, x_2, \cdots, x_n, 0) = s_\lambda(x_1, \cdots, x_n)$ we have a well-defined element $s_\lambda(\mathrm{x}) \in \Lambda$, which we define as the Schur function indexed by the partition λ.

1.2.3. *The Littlewood-Richardson Coefficients.* In [Mac95], the Littlewood-Richardson coefficients are defined as the structure constants for multiplication in the ring of symmetric polynomials at the Schur basis. That is,

$$s_\mu \, s_\nu = \sum_\lambda c_{\mu\,\nu}^\lambda \, s_\lambda$$

where s_γ denotes the Schur function indexed by the partition γ.

More generally, for a partition λ and a sequence of partitions $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$, ..., $\mu^{(k)}$, we use the standard notation

$$s_{\mu^{(1)}} s_{\mu^{(2)}} \cdots s_{\mu^{(k)}} = \sum_\lambda c_{\mu^{(1)}\mu^{(2)}\cdots\mu^{(k)}}^\lambda s_\lambda.$$

For most of [Mac95], one works in a ring having infinitely many variables, so there is no restriction on the number of parts of λ, μ and ν. If one passes to finitely many variables, the only change that needs to be made is to make sure that all partitions involved do not have more parts than the number of variables. Keeping track of this caveat, one can interpret the Schur function as the character of a representation of $GL_n(\mathbb{C})$. We obtain from this interpretation:

Proposition 1.2.3.1. *If we regard the irreducible $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ representation $F^{\mu}_{(n)} \widehat{\otimes} F^{\nu}_{(n)}$ as a representation of the diagonally embedded $GL_n(\mathbb{C})$ by restriction, then the decomposition into irreducibles is given by:*

$$F^{\mu}_{(n)} \widehat{\otimes} F^{\nu}_{(n)} \cong \bigoplus c^{\lambda}_{\mu\,\nu} F^{\lambda}_{(n)}$$

where the sum is over all partitions λ such that $|\lambda| = |\mu| + |\nu|$ and $\ell(\lambda) \leq n$.

In [GW98] and [How95] the following standard result is proved, which we will need in the proof of the main theorem.

Proposition 1.2.3.2. *If we regard the irreducible $GL_{n+m}(\mathbb{C})$-representation as a $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$-representation by restriction under the embedding:*

$$\begin{bmatrix} GL_n(\mathbb{C}) & 0 \\ 0 & GL_m(\mathbb{C}) \end{bmatrix} \subset GL_{n+m}(\mathbb{C}),$$

then we have:

$$F^{\lambda}_{(n+m)} \cong \bigoplus_{\mu,\nu} c^{\lambda}_{\mu\,\nu} F^{\mu}_{(n)} \widehat{\otimes} F^{\nu}_{(m)},$$

where the sum is over all partitions μ and ν such that $\ell(\mu) \leq n$ and $\ell(\nu) \leq m$.

1.3. **Statements of the Main Result.**

1.3.1. *Definition of the Stable Products.* To begin, we define a family of formal power series, $\{\widetilde{p}^{(I)}, \cdots, \widetilde{p}^{(X)}\}$, for the ten classical symmetric pairs by the expansions of the following formal products.

(1) Bilinear Forms.

(a) (GL_n, O_n)

$$\prod_{k=1}^{\infty} \prod_{i \leq j} \left(\frac{1}{1 - q^k x_i x_j} \right) := \sum \widetilde{p}^{(I)}_{\mu}(q) s_{\mu}(\mathrm{x}). \tag{1.1}$$

(b) (GL_{2n}, Sp_{2n})

$$\prod_{k=1}^{\infty} \prod_{i < j} \left(\frac{1}{1 - q^k x_i x_j} \right) := \sum \widetilde{p}^{(II)}_{\mu}(q) s_{\mu}(\mathrm{x}). \tag{1.2}$$

(2) Direct Sums.

(a) $(GL_{n+m}, GL_n \times GL_m)$

$$\begin{aligned} &\prod_{i,j,k=1}^{\infty} \frac{1}{(1 - q^{2k-1} x_i z_j)(1 - q^{2k-1} y_i w_j)(1 - q^{2k} x_i y_j)(1 - q^{2k} z_i w_j)} \\ &:= \sum_{\alpha^+,\alpha^-,\beta^+,\beta^-} \widetilde{p}^{(III)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)}(q) s_{\alpha^+}(\mathrm{x}) s_{\alpha^-}(\mathrm{y}) s_{\beta^+}(\mathrm{z}) s_{\beta^-}(\mathrm{w}) \end{aligned} \tag{1.3}$$

(b) $(O_{n+m}, O_n \times O_m)$

$$\prod_{k=1}^{\infty}\left(\prod_{i,j}\frac{1}{1-q^{2k-1}x_iy_j}\prod_{i\le j}\frac{1}{(1-q^{2k}x_ix_j)(1-q^{2k}y_iy_j)}\right) := \sum_{\alpha^+,\alpha^-}\widetilde{p}^{(IV)}_{(\alpha^+,\alpha^-)}(q)s_{\alpha^+}(\mathrm{x})s_{\alpha^-}(\mathrm{y}) \tag{1.4}$$

(c) $(Sp_{2(n+m)}, Sp_{2n} \times Sp_{2m})$

$$\prod_{k=1}^{\infty}\left(\prod_{i,j}\frac{1}{1-q^{2k-1}x_iy_j}\prod_{i<j}\frac{1}{(1-q^{2k}x_ix_j)(1-q^{2k}y_iy_j)}\right) := \sum_{\alpha^+,\alpha^-}\widetilde{p}^{(V)}_{(\alpha^+,\alpha^-)}(q)s_{\alpha^+}(\mathrm{x})s_{\alpha^-}(\mathrm{y}) \tag{1.5}$$

(3) Polarization.

(a) (O_{2n}, GL_n)

$$\prod_{k=1}^{\infty}\left(\prod_{i,j}\frac{1}{1-q^{2k}x_iy_j}\prod_{i\le j}\frac{1}{(1-q^{2k-1}x_ix_j)(1-q^{2k-1}y_iy_j)}\right) := \sum_{\alpha^+,\alpha^-}\widetilde{p}^{(VI)}_{(\alpha^+,\alpha^-)}(q)s_{\alpha^+}(\mathrm{x})s_{\alpha^-}(\mathrm{y}) \tag{1.6}$$

(b) (Sp_{2n}, GL_n)

$$\prod_{k=1}^{\infty}\left(\prod_{i,j}\frac{1}{1-q^{2k}x_iy_j}\prod_{i<j}\frac{1}{(1-q^{2k-1}x_ix_j)(1-q^{2k-1}y_iy_j)}\right) := \sum_{\alpha^+,\alpha^-}\widetilde{p}^{(VII)}_{(\alpha^+,\alpha^-)}(q)s_{\alpha^+}(\mathrm{x})s_{\alpha^-}(\mathrm{y}) \tag{1.7}$$

(4) Tensor Products.

(a) $(GL_n \times GL_n, GL_n)$

$$\prod_{i,j,k=1}^{\infty}\frac{1}{1-q^kx_iy_j} := \sum_{\alpha^+,\alpha^-}\widetilde{p}^{(VIII)}_{(\alpha^+,\alpha^-)}(q)s_{\alpha^+}(\mathrm{x})s_{\alpha^-}(\mathrm{y}) \tag{1.8}$$

(b) $(O_n \times O_n, O_n)$

$$\prod_{k=1}^{\infty}\left(\prod_{i<j}\frac{1}{1-q^{2k-1}x_ix_j}\prod_{i\le j}\frac{1}{1-q^{2k}x_ix_j}\right) := \sum_{\alpha}\widetilde{p}^{(IX)}_{\alpha}(q)s_{\alpha}(\mathrm{x}) \tag{1.9}$$

(c) $(Sp_{2n} \times Sp_{2n}, Sp_{2n})$

$$\prod_{k=1}^{\infty}\left(\prod_{i\le j}\frac{1}{1-q^{2k-1}x_ix_j}\prod_{i<j}\frac{1}{1-q^{2k}x_ix_j}\right) := \sum_{\alpha}\widetilde{p}^{(X)}_{\alpha}(q)s_{\alpha}(\mathrm{x}) \tag{1.10}$$

If V is an irreducible representation of a group K and W is an arbitrary K-representation, then we use the notation: $[V,W] := \dim \mathrm{Hom}_K(V,W)$ for the multiplicity of V in W.

1.3.2. *Statement.*

Theorem (Main)**.** *We must state the result in cases:*

(1) *Bilinear Forms.*

(a) (GL_n, O_n)*: For* $n \geq 2d$*,*

$$\widetilde{p}^{(I)}_{\mu}(q)\big|_{q^d} = \left[E^{\mu}_{(n)}, \mathcal{H}^d_{O_n}(\mathfrak{p})\right]. \tag{1.11}$$

(b) (GL_{2n}, Sp_{2n})*: For* $n \geq 2d$*,*

$$\widetilde{p}^{(II)}_{\mu}(q)\big|_{q^d} = \left[V^{\mu}_{(n)}, \mathcal{H}^d_{Sp_{2n}}(\mathfrak{p})\right]. \tag{1.12}$$

(2) *Direct Sums.*

(a) $(GL_{n+m}, GL_n \times GL_m)$*: For* $\min(n,m) \geq d$*,*

$$\widetilde{p}^{(III)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)}(q)\Big|_{q^d} = \left[F^{(\alpha^+,\alpha^-)}_{(n)} \widehat{\otimes} F^{(\beta^+,\beta^-)}_{(m)}, \mathcal{H}^d_{GL_n \times GL_m}(\mathfrak{p})\right]. \tag{1.13}$$

(b) $(O_{n+m}, O_n \times O_m)$*: For* $\min(n,m) \geq 2d$*,*

$$\widetilde{p}^{(IV)}_{(\alpha^+,\alpha^-)}(q)\Big|_{q^d} = \left[E^{\alpha^+}_{(n)} \widehat{\otimes} E^{\alpha^-}_{(m)}, \mathcal{H}^d_{O_n \times O_m}(\mathfrak{p})\right]. \tag{1.14}$$

(c) $(Sp_{2(n+m)}, Sp_{2n} \times Sp_{2m})$*: For* $\min(n,m) \geq d$*,*

$$\widetilde{p}^{(V)}_{(\alpha^+,\alpha^-)}(q)\Big|_{q^d} = \left[V^{\alpha^+}_{(n)} \widehat{\otimes} V^{\alpha^-}_{(m)}, \mathcal{H}^d_{Sp_{2n} \times Sp_{2m}}(\mathfrak{p})\right]. \tag{1.15}$$

(3) *Polarization.*

(a) (SO_{2n}, GL_n)*: For* $n \geq 2d$*,*

$$\widetilde{p}^{(VI)}_{(\alpha^+,\alpha^-)}(q)\Big|_{q^d} = \left[F^{\alpha^+}_{(n)} \widehat{\otimes} F^{\alpha^-}_{(n)}, \mathcal{H}^d_{GL_n}(\mathfrak{p})\right]. \tag{1.16}$$

(b) (Sp_{2n}, GL_n)*: For* $n \geq d$*,*

$$\widetilde{p}^{(VII)}_{(\alpha^+,\alpha^-)}(q)\Big|_{q^d} = \left[F^{\alpha^+}_{(n)} \widehat{\otimes} F^{\alpha^-}_{(n)}, \mathcal{H}^d_{GL_n}(\mathfrak{p})\right]. \tag{1.17}$$

(4) *Tensor Products.*

(a) $(GL_n \times GL_n, GL_n)$*: For* $n \geq 2d$*,*

$$\widetilde{p}^{(VIII)}_{(\alpha^+,\alpha^-)}(q)\Big|_{q^d} = \left[F^{\alpha^+}_{(n)} \widehat{\otimes} F^{\alpha^-}_{(n)}, \mathcal{H}^d_{GL_n}(\mathfrak{p})\right]. \tag{1.18}$$

(b) $(O_n \times O_n, O_n)$*: For* $n \geq 4d$*,*

$$\widetilde{p}^{(IX)}_{\alpha}(q)\big|_{q^d} = \left[E^{\alpha}_{(n)}, \mathcal{H}^d_{O_n}(\mathfrak{p})\right]. \tag{1.19}$$

(c) $(Sp_{2n} \times Sp_{2n}, Sp_{2n})$*: For* $n \geq d$*,*

$$\widetilde{p}^{(X)}_{\alpha}(q)\big|_{q^d} = \left[V^{\alpha}_{(n)}, \mathcal{H}^d_{Sp_{2n}}(\mathfrak{p})\right]. \tag{1.20}$$

Proof. Several of the cases already exist in the literature (see [Wil02] and [Sta84]). The remaining cases (2a, 3a, and 3b) are proved in Section 2.3 of this paper. ☐

1.4. **Graded Multiplicity in terms of the Littlewood-Richardson Coefficients.** A consequence of the main theorem is that in all classical instances of the Kostant-Rallis theorem we can describe the graded multiplicity, stably, in terms of Littlewood-Richardson coefficients. This description can be read off of the products defining the series $\widetilde{p}^{(I)}, \cdots, \widetilde{p}^{(X)}$ together with the identities:

$$\prod_{i\leq j} \frac{1}{1-tx_ix_j} = \sum s_\mu(\mathrm{x}) t^{\frac{|\mu|}{2}} \qquad \text{(summed over } \mu \text{ with even rows)}, \tag{1.21}$$

$$\prod_{i<j} \frac{1}{1-tx_ix_j} = \sum s_\mu(\mathrm{x}) t^{\frac{|\mu|}{2}} \qquad \text{(summed over } \mu \text{ with even columns)}, \tag{1.22}$$

$$\prod_{i,j} \frac{1}{1-tx_iy_j} = \sum s_\mu(\mathrm{x}) s_\mu(\mathrm{y}) t^{|\mu|} \qquad \text{(summed over all } \mu\text{)}. \tag{1.23}$$

We next illustrate this point in detail for the symmetric pair (GL_n, O_n) (corresponding to the real group $GL_n(\mathbb{R})$). The other cases are similar, although technically more cumbersome to write down.

Our goal then is to describe the coefficients of $\widetilde{p}_\mu^{(I)}(q)$ in terms of Littlewood-Richardson coefficients. This amount to expanding the product:

$$\prod_{k=1}^{\infty}\prod_{i\leq j} \frac{1}{1-q^kx_ix_j}$$

into Schur functions. For any fixed k, set $t = q^k$ in the Littlewood identity in Equation 1.21 to obtain:

$$\prod_{i\leq j} \frac{1}{1-q^kx_ix_j} = \sum_\mu s_\mu(\mathrm{x}) q^{k\frac{|\mu|}{2}} \qquad \text{(summed over } \mu \text{ with even rows)}.$$

Then we take the product over all k to obtain:

$$\prod_{k=1}^{\infty}\prod_{i\leq j} \frac{1}{1-q^kx_ix_j} = \prod_{k=1}^{\infty}\sum_{\mu^{(k)}} s_{\mu^{(k)}}(\mathrm{x}) q^{k\frac{|\mu^{(k)}|}{2}}$$

where $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$ ... are partitions with even rows. The next step is to expand the product on the right side into a product of Schur functions:

$$= \sum_{\mu^{(1)},\mu^{(2)},\cdots} \left(s_{\mu^{(1)}}(\mathrm{x}) s_{\mu^{(2)}}(\mathrm{x}) s_{\mu^{(3)}}(\mathrm{x}) \cdots \right) q^{\frac{1|\mu^{(1)}|+2|\mu^{(2)}|+3|\mu^{(3)}|+\cdots}{2}}.$$

By definition, the coefficient of $s_\mu(\mathrm{x})$ in the above sum of products is $\widetilde{p}_\mu^{(I)}(q)$. For each non-negative integer d let:

$$\Psi(d) := \left\{ (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \cdots) : \begin{array}{l} \mu^{(k)} \text{ has even rows for all } k, \textit{and} \\ 1|\mu^{(1)}| + 2|\mu^{(2)}| + 3|\mu^{(3)}| + \cdots = 2d \end{array} \right\}.$$

In terms of Littlewood-Richardson coefficients:

$$\widetilde{p}_{\mu}^{(I)}(q)\big|_{q^d} = \sum c^{\mu}_{\mu^{(1)}\mu^{(2)}\mu^{(3)}\ldots}$$

where the sum is over $(\mu^{(1)}, \mu^{(2)}, \cdots) \in \Psi(d)$. Note that each element of $\Phi(d)$ is a sequence with finite support because of the condition on the size of $\mu^{(1)}, \mu^{(2)}, \cdots$. Moreover, the same condition implies that the set $\Phi(d)$ is finite for each d.

2. Proofs of the Theorems.

This section is divided into three parts.

In Sub-sections 2.1 and 2.2, we review the required background from representation theory and symmetric function theory. In Sub-section 2.3, we prove the main theorem using the results of all previous sections.

2.1. **Background from Representation Theory.**

2.1.1. *Three Multiplicity-free Spaces.* We recall first the three most standard multiplicity free spaces. These will be used in the proof of Theorem 2.1.5.1.

Theorem 2.1.1.1. *The action $(g, X) \mapsto gXg^t$ of GL_m on the $m \times m$-matrices gives rise to a GL_m-representation on the space of degree d symmetric tensors on SM_m and AM_m respectively. Relative to this action, we have the following multiplicity free decompositions:*

$$\mathcal{S}^d(SM_m) \cong \bigoplus_{\mu} F^{2\mu}_{(m)} \qquad \textit{and} \qquad \mathcal{S}^d(AM_m) \cong \bigoplus_{\mu} F^{(2\mu)'}_{(m)} \tag{2.1}$$

where μ runs over all partitions of size d such that the length of 2μ is at most m in the former case and the length of $(2\mu)'$ is at most m in that latter case.

Furthermore, the group $GL_k \times GL_l$ acts on $M_{k\times l}$ via $((g,h), X) \mapsto gXh^t$. Under this action, the rank d symmetric tensors decompose (also multiplicity free) as:

$$\mathcal{S}^d(M_{k,l}) \cong \bigoplus_{\mu} F^{\mu}_{(k)} \widehat{\otimes} F^{\mu}_{(l)} \tag{2.2}$$

where μ runs over all partitions of size d and length at most $\min(k, l)$.

Proof. See [GW98] or [How95]. □

2.1.2. *The Cartan-Helgason Theorem.* One of the tools that we will use for the proof of the main theorem is the Cartan-Helgason theorem (see [GW98] or [How95]). This fact amounts to the assertion that if (G, K) is a symmetric pair, then the G decomposition of regular functions $\mathcal{O}(G/K)$ (on G/K) is multiplicity free. The irreducible regular G-representations that occur in $\mathcal{O}(G/K)$ are precisely those representations with a K-invariant vector. We describe here four classical instances of this theorem, the first of which we describe as a remark since it is a consequence of Schur's lemma:

Remark 2.1.2.1. For any reductive group K, we may symmetrically embedded K in $G := K \times K$ as $\{(k,k) | k \in K\}$. The irreducible regular representations of G may be taken as $V_1 \widehat{\otimes} V_2$ where (ρ_1, V_1) and (ρ_2, V_2) are irreducible representations of K. We then obtain:

$$\dim\left(V_1 \otimes V_2\right)^K = \begin{cases} 1 & \text{if } V_1 \cong V_2^*, \\ 0 & \text{otherwise,} \end{cases}$$

by noting that $\mathrm{Hom}_K(V_1, V_2) \cong (V_1^* \otimes V_2)^K$ and invoking Schur's lemma.

Symmetric Pair $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$:
An irreducible representation of $GL_n(\mathbb{C})$ may be regarded as a representation of $O_n(\mathbb{C})$ by restriction. In so doing, we have the following

Theorem 2.1.2.2. *For an integer partition* λ *with* $\ell(\lambda) \leq n$*:*

$$\dim \left(F_{(n)}^{\lambda}\right)^{O_n(\mathbb{C})} = \begin{cases} 1 & \lambda_i \text{ even for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Symmetric Pair $(GL_{2m}(\mathbb{C}), Sp_{2m}(\mathbb{C}))$:
As in the last case, an irreducible representation of $GL_{2m}(\mathbb{C})$ may be regarded as a representation of $Sp_{2m}(\mathbb{C})$ by restriction. We have the following

Theorem 2.1.2.3. *For an integer partition* λ *with* $\ell(\lambda) \leq 2m$*:*

$$\dim \left(F_{(2m)}^{\lambda}\right)^{Sp_{2m}(\mathbb{C})} = \begin{cases} 1 & (\lambda')_i \text{ even for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Symmetric Pair $(GL_{p+q}(\mathbb{C}), GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))$:
As in the last cases, an irreducible representation of $GL_{p+q}(\mathbb{C})$ may be regarded as a representation of $GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ by restriction, where we embed $GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ in $GL_{p+q}(\mathbb{C})$ as:

$$\begin{bmatrix} GL_p(\mathbb{C}) & 0 \\ 0 & GL_q(\mathbb{C}) \end{bmatrix} \subset GL_{p+q}(\mathbb{C}).$$

We have the following

Theorem 2.1.2.4. *Given* $n = p + q$*, let* μ *and* ν *be partitions with at most* p *and* q *parts respectively. Then,*

$$\dim(F_{(n)}^{(\mu,\nu)})^{GL_p \times GL_q} = \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

2.1.3. *Notation for Multiplicities.* Given completely reducible representations, V_1 and V_2 of complex algebraic groups G_1 and G_2 respectively, together with an embedding $G_1 \hookrightarrow G_2$, we let

$$[V_1, V_2] = \dim \mathrm{Hom}_{G_1}(V_1, V_2)$$

where V_2 is regarded as a representation of G_1 by restriction. If V_1 is irreducible, then $[V_1, V_2]$ is the multiplicity of V_1 in V_2. This cardinal may of course be infinite if V_1 or V_2 is infinite dimensional. For most of the paper, the restriction to G_1 will be implicit, but we mention it here to be precise.

If G is a reductive algebraic group over $\mathbb{C}$, let $(V^\lambda)_{\lambda \in \widehat{G}}$ denote representatives of irreducible regular representations of G. If V is a completely reducible representation of G, then set $m_\lambda := [V^\lambda, V]$. In this paper, we will always have $m_\lambda < \infty$, and we will indicate the decomposition of V into irreducible representations of G (with multiplicity) by the expression:

$$V \cong \bigoplus m_\lambda V^\lambda,$$

which of course is shorthand for:

$$V \;\cong\; \bigoplus_{\lambda\in\widehat{G}} \underbrace{V^\lambda \oplus V^\lambda \oplus \cdots \oplus V^\lambda}_{m_\lambda \text{ copies}}$$

2.1.4. *Three Branching Rules.* We record here the branching rules which lead directly to the proof of Proposition 2.1.5.1.

Theorem 2.1.4.1 (Stable Branching Rules).

(a) $(GL_n \times GL_n, GL_n)$**:** *For* $\ell(\mu)+\ell(\nu)\le n$,

$$\left[F_{(n)}^{(\lambda^+,\lambda^-)}, F_{(n)}^{\mu}\widehat{\otimes}\left(F_{(n)}^{\nu}\right)^*\right] = \sum_{\gamma} c_{\gamma\,\lambda^+}^{\mu} c_{\gamma\,\lambda^-}^{\nu} \tag{2.3}$$

(b) (GL_n, O_n)**:** *For* $\ell(\lambda)\le \frac{n}{2}$,

$$\left[E_{(n)}^{\mu}, F_{(n)}^{\lambda}\right] = \sum_{\nu} c_{2\nu\,\mu}^{\lambda} \tag{2.4}$$

(c) (GL_{2n}, Sp_{2n})**:** *For* $\ell(\lambda)\le n$,

$$\left[V_{(2n)}^{\mu}, F_{(2n)}^{\lambda}\right] = \sum_{\nu} c_{(2\nu)'\,\mu}^{\lambda} \tag{2.5}$$

Proof. These are known – the first appears in the work of King [Kin70], which derives from a conjecture in the Ph.D. thesis by Abramsky [Abr69]. The second two are results of D. Littlewood [Lit44a], [Lit44b]. Although the above three branching rules are the only ones required for the main result, there are many other branching rules of a similar flavor. See [HTW05] and the references within for a uniform treatment of stable formulas for branching multiplicities in the context of symmetric pairs. □

2.1.5. *Stable Graded K-multiplicity in $\mathcal{S}(\mathfrak{p})$.*

Proposition 2.1.5.1. *We proceed in cases.*

(a) Direct Sum – ($\mathbf{GL}_{n+m}$,$\mathbf{GL}_n \times \mathbf{GL}_m$): *For* $\min(n,m)\ge d$,

$$\left[F_{(n)}^{(\alpha^+,\alpha^-)}\widehat{\otimes}F_{(m)}^{(\beta^+,\beta^-)}, \mathcal{S}^d\left(M_{n,m}\oplus(M_{n,m})^*\right)\right] = \sum_{\gamma,\delta,\mu,\nu} c_{\gamma\,\alpha^+}^{\mu} c_{\gamma\,\alpha^-}^{\nu} c_{\delta\,\beta^+}^{\mu} c_{\delta\,\beta^-}^{\nu} \tag{2.6}$$

where the sum is over $|\mu|+|\nu|=d$ *and* $\ell(\mu),\ell(\nu)\le\min(n,m)$.

(b) Polarization – ($\mathbf{O}_{2n}$,$\mathbf{GL}_n$): *For* $n\ge 2d$,

$$\left[F_{(n)}^{(\lambda^+,\lambda^-)}, \mathcal{S}^d\left(AM_n\oplus(AM_n)^*\right)\right] = \sum_{\gamma,\mu,\nu} c_{\gamma\,\lambda^+}^{(2\mu)'} c_{\gamma\,\lambda^-}^{(2\nu)'} \tag{2.7}$$

where the sum is over γ, μ *and* ν *such that* $|\mu|+|\nu|=d$.

(c) Polarization – ($\mathbf{Sp}_{2n}$,$\mathbf{GL}_n$): *For* $n\ge d$,

$$\left[F_{(n)}^{(\lambda^+,\lambda^-)}, \mathcal{S}^d\left(SM_n\oplus(SM_n)^*\right)\right] = \sum_{\gamma,\mu,\nu} c_{\gamma\,\lambda^+}^{2\mu} c_{\gamma\,\lambda^-}^{2\nu} \tag{2.8}$$

where the sum is over γ, μ *and* ν *such that* $|\mu|+|\nu|=d$.

Proof. This is a straightforward combination of the classical multiplicity free spaces 2.1.1.1 with Theorem 2.1.4.1. □

2.2. **Background from Symmetric Function Theory.** We require some standard constructions and identities from the theory of symmetric functions, most of which can be found in [Mac95] and [Sta99].

2.2.1. *Three Identities for Skew Symmetric Functions.* We will use the following result in the proof of Proposition 2.2.2.1.

Proposition 2.2.1.1. *For any partitions α and β,*

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathrm{x}) s_{\lambda/\beta}(\mathrm{y}) = \frac{\sum_{\nu} s_{\beta/\nu}(\mathrm{x}) s_{\alpha/\nu}(\mathrm{y})}{\prod_{i,j}(1 - x_i y_j)}. \tag{2.9}$$

For any partition μ,

$$\sum_{\lambda} s_{2\lambda/\mu}(\mathrm{x}) = \frac{\sum_{\nu} s_{\mu/2\nu}(\mathrm{x})}{\prod_{i \le j}(1 - x_i x_j)} \tag{2.10}$$

and

$$\sum_{\lambda} s_{(2\lambda)'/\mu}(\mathrm{x}) = \frac{\sum_{\nu} s_{\mu/(2\nu)'}(\mathrm{x})}{\prod_{i<j}(1 - x_i x_j)}. \tag{2.11}$$

Proof. See [Sta99] Exercise 7.27 (p. 457) or for the second two [Mac95] Example I.5.27 (p. 93). □

2.2.2. *Three Iteration Identities.*

Definition. We define three formal expressions:

(a) Direct Sum – ($\mathbf{GL}_{n+m}$,$\mathbf{GL}_n \times \mathbf{GL}_m$):

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) := \sum_{\mu,\nu,\gamma,\delta} s_{\mu/\gamma}(\mathrm{x}) s_{\nu/\gamma}(\mathrm{y}) s_{\mu/\delta}(\mathrm{z}) s_{\nu/\delta}(\mathrm{w}) q^{|\mu|+|\nu|} \tag{2.12}$$

(b) Polarization – ($\mathbf{O}_{2n}$,$\mathbf{GL}_n$):

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) := \sum_{\mu,\nu,\gamma} s_{(2\mu)'/\gamma}(\mathrm{x}) s_{(2\nu)'/\gamma}(\mathrm{y}) q^{|\mu|+|\nu|} \tag{2.13}$$

(c) Polarization – ($\mathbf{Sp}_{2n}$,$\mathbf{GL}_n$):

$$\mathcal{M}_q^{VII}(\mathrm{x},\mathrm{y}) := \sum_{\mu,\nu,\gamma} s_{2\mu/\gamma}(\mathrm{x}) s_{2\nu/\gamma}(\mathrm{y}) q^{|\mu|+|\nu|} \tag{2.14}$$

Each of these formal sums may be expressed as an infinite product, as in Proposition 2.2.2.3, whose proof requires the next two proposotions:

Proposition 2.2.2.1. *For each $d \ge 1$ we have,*

(a)

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \frac{\mathcal{M}_q^{III}(q^d\mathrm{x}, q^d\mathrm{y}, q^d\mathrm{z}, q^d\mathrm{w})}{\prod_{k=1}^{d}\left[\prod_{i,j}(1 - q^{2k-1}x_i z_j)(1 - q^{2k-1}y_i w_j)(1 - q^{2k}x_i y_j)(1 - q^{2k}z_i w_j)\right]} \tag{2.15}$$

(b)

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \frac{\mathcal{M}_q^{VI}(q^d\mathrm{x}, q^d\mathrm{y})}{\prod_{k=1}^{d}\left[\prod_{i<j}(1 - q^{2k-1}x_i x_j)(1 - q^{2k-1}y_i y_j)\prod_{i,j}(1 - q^{2k}x_i y_j)\right]} \tag{2.16}$$

$$(c)\ \mathcal{M}_q^{VII}(\mathrm{x},\mathrm{y}) = \frac{\mathcal{M}_q^{VII}(q^d\mathrm{x}, q^d\mathrm{y})}{\prod_{k=1}^{d}\left[\prod_{i\le j}(1-q^{2k-1}x_ix_j)(1-q^{2k-1}y_iy_j)\prod_{i,j}(1-q^{2k}x_iy_j)\right]} \tag{2.17}$$

Proof. We proceed in cases:

Proof of Equation 2.15: We start from the definition of $\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w})$ and by homogeneity we obtain:

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \sum_{\mu,\nu,\gamma,\delta} s_{\mu/\gamma}(q^{\frac{1}{2}}\mathrm{x})s_{\nu/\gamma}(q^{\frac{1}{2}}\mathrm{y})s_{\mu/\delta}(q^{\frac{1}{2}}\mathrm{z})s_{\nu/\delta}(q^{\frac{1}{2}}\mathrm{w})q^{|\gamma|+|\delta|}. \tag{2.18}$$

Change the order of summation:

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \sum_{\gamma,\delta}\left[\sum_{\mu} s_{\mu/\gamma}(q^{\frac{1}{2}}\mathrm{x})s_{\mu/\delta}(q^{\frac{1}{2}}\mathrm{z})\sum_{\nu} s_{\nu/\gamma}(q^{\frac{1}{2}}\mathrm{y})s_{\nu/\delta}(q^{\frac{1}{2}}\mathrm{w})\right]q^{|\gamma|+|\delta|}. \tag{2.19}$$

We apply Equation 2.9 to obtain:

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \sum_{\gamma,\delta}\left[\frac{\sum_\alpha s_{\delta/\alpha}(q^{\frac{1}{2}}\mathrm{x})s_{\gamma/\alpha}(q^{\frac{1}{2}}\mathrm{z})}{\prod_{i,j}(1-qx_iz_j)}\frac{\sum_\beta s_{\delta/\beta}(q^{\frac{1}{2}}\mathrm{y})s_{\gamma/\beta}(q^{\frac{1}{2}}\mathrm{w})}{\prod_{i,j}(1-qy_iw_j)}\right]q^{|\gamma|+|\delta|}. \tag{2.20}$$

Replace δ by μ and γ by ν,

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \sum_{\mu,\nu}\left[\frac{\sum_\alpha s_{\mu/\alpha}(q^{\frac{1}{2}}\mathrm{x})s_{\nu/\alpha}(q^{\frac{1}{2}}\mathrm{z})}{\prod_{i,j}(1-qx_iz_j)}\frac{\sum_\beta s_{\mu/\beta}(q^{\frac{1}{2}}\mathrm{y})s_{\nu/\beta}(q^{\frac{1}{2}}\mathrm{w})}{\prod_{i,j}(1-qy_iw_j)}\right]q^{|\mu|+|\nu|}. \tag{2.21}$$

Replace α by γ and β by δ,

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \sum_{\mu,\nu}\left[\frac{\sum_\gamma s_{\mu/\gamma}(q^{\frac{1}{2}}\mathrm{x})s_{\nu/\gamma}(q^{\frac{1}{2}}\mathrm{z})}{\prod_{i,j}(1-qx_iz_j)}\frac{\sum_\delta s_{\mu/\delta}(q^{\frac{1}{2}}\mathrm{y})s_{\nu/\delta}(q^{\frac{1}{2}}\mathrm{w})}{\prod_{i,j}(1-qy_iw_j)}\right]q^{|\mu|+|\nu|}. \tag{2.22}$$

Combining the sums we obtain,

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \frac{\sum_{\mu,\nu,\gamma,\delta} s_{\mu/\gamma}(q^{\frac{1}{2}}\mathrm{x})s_{\nu/\gamma}(q^{\frac{1}{2}}\mathrm{z})s_{\mu/\delta}(q^{\frac{1}{2}}\mathrm{y})s_{\nu/\delta}(q^{\frac{1}{2}}\mathrm{w})q^{|\mu|+|\nu|}}{\prod_{i,j}(1-qx_iz_j)(1-qy_iw_j)}, \tag{2.23}$$

which we recognize as:

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \frac{\mathcal{M}_q^{III}(q^{\frac{1}{2}}\mathrm{x}, q^{\frac{1}{2}}\mathrm{z}, q^{\frac{1}{2}}\mathrm{y}, q^{\frac{1}{2}}\mathrm{w})}{\prod_{i,j}(1-qx_iz_j)(1-qy_iw_j)}. \tag{2.24}$$

Performing these last few operations again leaves us with

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \frac{\mathcal{M}_q^{III}(q\mathrm{x}, q\mathrm{y}, q\mathrm{z}, q\mathrm{w})}{\prod_{i,j}(1-qx_iz_j)(1-qy_iw_j)(1-q^2x_iy_j)(1-q^2z_iw_j)}. \tag{2.25}$$

If we iterate the above d times, we obtain:

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \frac{\mathcal{M}_q^{III}(q^d\mathrm{x}, q^d\mathrm{y}, q^d\mathrm{z}, q^d\mathrm{w})}{\prod_{k=1}^{d}\left[\prod_{i,j}(1-q^{2k-1}x_iz_j)(1-q^{2k-1}y_iw_j)(1-q^{2k}x_iy_j)(1-q^{2k}z_iw_j)\right]}. \tag{2.26}$$

Proof of Equation 2.16: Again, by homogeneity we obtain:

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \sum_{\mu,\nu,\gamma} s_{(2\mu)'/\gamma}(q^{\frac{1}{2}}\mathrm{x}) s_{(2\nu)'/\gamma}(q^{\frac{1}{2}}\mathrm{y}) q^{|\gamma|}. \tag{2.27}$$

Next, we reorganize the sum,

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \sum_{\gamma} \left[\sum_{\mu} s_{(2\mu)'/\gamma}(q^{\frac{1}{2}}\mathrm{x}) \sum_{\nu} s_{(2\nu)'/\gamma}(q^{\frac{1}{2}}\mathrm{y}) \right] q^{|\gamma|}. \tag{2.28}$$

We now use Equation 2.11 and re-indexing to obtain,

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \sum_{\gamma} \left[\frac{\sum_{\mu} s_{\gamma/(2\mu)'}(q^{\frac{1}{2}}\mathrm{x})}{\prod_{i<j}(1-qx_ix_j)} \frac{\sum_{\nu} s_{\gamma/(2\nu)'}(q^{\frac{1}{2}}\mathrm{y})}{\prod_{i<j}(1-qy_iy_j)} \right] q^{|\gamma|}. \tag{2.29}$$

Reorganizing and exploiting the homogeneity of the Schur function we obtain:

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \sum_{\mu,\nu} \left[\frac{\sum_{\gamma} s_{\gamma/(2\mu)'}(q\mathrm{x}) s_{\gamma/(2\nu)'}(q\mathrm{y})}{\prod_{i<j}(1-qx_ix_j)\prod_{i<j}(1-qy_iy_j)} \right] q^{|\mu|+|\nu|}. \tag{2.30}$$

Next, we apply Equation 2.9 as in the last case,

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \sum_{\mu,\nu} \left[\frac{\sum_{\gamma} s_{(2\nu)'/\gamma}(q\mathrm{x}) s_{(2\mu)'/\gamma}(q\mathrm{y})}{\prod_{i<j}(1-qx_ix_j)\prod_{i<j}(1-qy_iy_j)\prod_{i,j}(1-q^2x_iy_j)} \right] q^{|\mu|+|\nu|}. \tag{2.31}$$

Reorganizing, we obtain:

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \frac{\mathcal{M}_q^{VI}(q\mathrm{y},q\mathrm{x})}{\prod_{i<j}(1-qx_ix_j)(1-qy_iy_j)\prod_{i,j}(1-q^2x_iy_j)}. \tag{2.32}$$

Note that we have the symmetry, $\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \mathcal{M}_q^{VI}(\mathrm{y},\mathrm{x})$. Thus, we have deduced:

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \frac{\mathcal{M}_q^{VI}(q\mathrm{x},q\mathrm{y})}{\prod_{i<j}(1-qx_ix_j)(1-qy_iy_j)\prod_{i,j}(1-q^2x_iy_j)}. \tag{2.33}$$

Iterating the above steps d times yields the result.

Proof of Equation 2.17: This case is essentially identical to the last except that we replace Equation 2.11 with Equation 2.10.

□

Recall that if $F(q)$ is a formal series in q, we use the notation $F(q)\big|_{q^k}$ to denote the coefficient of q^k.

Proposition 2.2.2.2. *For each $k, d \in \mathbb{N}$ with $1 \leq k \leq d$, we have:*

$$\mathcal{M}_q^{III}(q^d\mathrm{x}, q^d\mathrm{y}, q^d\mathrm{z}, q^d\mathrm{w})\big|_{q^k} = \prod_{j=1}^{\infty}\left(\frac{1}{1-q^{2j}}\right)\Bigg|_{q^k} \tag{2.34}$$

$$\mathcal{M}_q^{VI}(q^d\mathrm{x}, q^d\mathrm{y})\big|_{q^k} = \prod_{j=1}^{\infty}\left(\frac{1}{1-q^{2j}}\right)\Bigg|_{q^k} \tag{2.35}$$

$$\mathcal{M}_q^{VII}(q^d\mathrm{x}, q^d\mathrm{y})\big|_{q^k} = \prod_{j=1}^{\infty}\left(\frac{1}{1-q^{2j}}\right)\Bigg|_{q^k} \tag{2.36}$$

Proof. The result follows almost directly from the definitions of $\mathcal{M}_q^{III}$, $\mathcal{M}_q^{VI}$ and $\mathcal{M}_q^{VII}$ respectively. The argument in each case is very similar so we will include only the proof of Equation 2.34.
We begin with

$$\begin{aligned}
&\mathcal{M}_q^{III}(q^d\mathrm{x}, q^d\mathrm{y}, q^d\mathrm{z}, q^d\mathrm{w})\\
&= \textstyle\sum_{\mu,\nu,\gamma,\delta} s_{\mu/\gamma}(q^d\mathrm{x})s_{\nu/\gamma}(q^d\mathrm{y})s_{\mu/\delta}(q^d\mathrm{z})s_{\nu/\delta}(q^d\mathrm{w})q^{|\mu|+|\nu|}\\
&= \textstyle\sum_{\mu,\nu,\gamma,\delta} s_{\mu/\gamma}(\mathrm{x})s_{\nu/\gamma}(\mathrm{y})s_{\mu/\delta}(\mathrm{z})s_{\nu/\delta}(\mathrm{w})q^{2d|\mu|+2d|\nu|-2d|\gamma|-2d|\delta|}q^{|\mu|+|\nu|}\\
&= \textstyle\sum_{\mu,\nu,\gamma,\delta} s_{\mu/\gamma}(\mathrm{x})s_{\nu/\gamma}(\mathrm{y})s_{\mu/\delta}(\mathrm{z})s_{\nu/\delta}(\mathrm{w})q^{2d(|\mu|-|\gamma|)+2d(|\nu|-|\delta|)}q^{|\mu|+|\nu|}.
\end{aligned}$$

Observe that $s_{\mu/\gamma} = 0$ (resp. $s_{\nu/\delta} = 0$, etc.) if $|\gamma| > |\mu|$ (resp. $|\delta| > |\nu|$, etc.). In fact, we require a containment of Young diagrams to obtain a non-zero coefficient. Therefore, if $0 \le k \le d$ then the only coefficients of q^k that occur are when $\mu = \nu = \gamma = \delta$. If we sum this part of the series we obtain:

$$\sum_{\mu,\nu} q^{|\mu|+|\nu|} = \prod_{k\ge 1}\left(\frac{1}{1-q^{2k}}\right).$$

(Recall that $s_{\alpha/\beta} = 1$ if $\alpha = \beta$.)

□

Proposition 2.2.2.3.

$$\mathcal{M}_q^{III}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w}) = \prod_{k=1}^{\infty}\frac{1}{1-q^k}\left[\prod_{i,j}\frac{1}{(1-q^{2k-1}x_iz_j)(1-q^{2k-1}y_iw_j)(1-q^{2k}x_iy_j)(1-q^{2k}z_iw_j)}\right] \tag{2.37}$$

$$\mathcal{M}_q^{VI}(\mathrm{x},\mathrm{y}) = \prod_{k=1}^{\infty}\frac{1}{1-q^k}\left[\prod_{i<j}\frac{1}{(1-q^{2k-1}x_ix_j)(1-q^{2k-1}y_iy_j)}\prod_{i,j}\frac{1}{(1-q^{2k}x_iy_j)}\right] \tag{2.38}$$

$$\mathcal{M}_q^{VII}(\mathrm{x},\mathrm{y}) = \prod_{k=1}^{\infty}\frac{1}{1-q^k}\left[\prod_{i\le j}\frac{1}{(1-q^{2k-1}x_ix_j)(1-q^{2k-1}y_iy_j)}\prod_{i,j}\frac{1}{(1-q^{2k}x_iy_j)}\right] \tag{2.39}$$

Proof. We proceed in cases.

Proof of Equation 2.37: Combine Equation 2.15 with 2.34 by comparing coefficients when expanding $\mathcal{M}_q^{III}$ in q.

Proof of Equation 2.38: Combine Equation 2.16 with 2.35 by comparing coefficients when expanding $\mathcal{M}_q^{VI}$ in q.

Proof of Equation 2.39: Combine Equation 2.17 with 2.36 by comparing coefficients when expanding $\mathcal{M}_q^{VII}$ in q.

□

2.2.3. *Three Identities Involving Littlewood-Richardson Coefficients.*

Proposition 2.2.3.1.

(a) Direct Sum – ($\mathbf{GL}_{n+m}$,$\mathbf{GL}_n \times \mathbf{GL}_m$):

$$\frac{\widetilde{p}^{(III)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)}(q)}{\prod_{i=1}^{\infty}(1-q^{2i})} = \sum_{\gamma,\delta,\mu,\nu} c^{\mu}_{\gamma\,\alpha^+} c^{\nu}_{\gamma\,\alpha^-} c^{\mu}_{\delta\,\beta^+} c^{\nu}_{\delta\,\beta^-} q^{|\mu|+|\nu|} \tag{2.40}$$

(b) Polarization – ($\mathbf{O}_{2n}$,$\mathbf{GL}_n$):

$$\frac{\widetilde{p}^{(VI)}_{(\lambda^+,\lambda^-)}(q)}{\prod_{i=1}^{\infty}(1-q^{2i})} = \sum_{\gamma,\mu,\nu} c^{(2\mu)'}_{\gamma\,\lambda^+} c^{(2\nu)'}_{\gamma\,\lambda^-} q^{|\mu|+|\nu|} \tag{2.41}$$

(c) Polarization – ($\mathbf{Sp}_{2n}$,$\mathbf{GL}_n$):

$$\frac{\widetilde{p}^{(VII)}_{(\lambda^+,\lambda^-)}(q)}{\prod_{i=1}^{\infty}(1-q^{2i})} = \sum_{\gamma,\mu,\nu} c^{2\mu}_{\gamma\,\lambda^+} c^{2\nu}_{\gamma\,\lambda^-} q^{|\mu|+|\nu|} \tag{2.42}$$

Proof. Multiply the numerator of the left side of Equations 2.40, 2.41, and 2.42 by the Schur functions

$$s_{\alpha^+}(\mathrm{x})s_{\alpha^-}(\mathrm{y})s_{\beta^+}(\mathrm{z})s_{\beta^-}(\mathrm{w}),$$
$$s_{\lambda^+}(\mathrm{x})s_{\lambda^-}(\mathrm{y}) \quad \text{and} \quad s_{\lambda^+}(\mathrm{x})s_{\lambda^-}(\mathrm{y})$$

respectively and in each case sum over the partitions. We recognize the result as the definition of the infinite products defining $\widetilde{p}^{(III)}$, $\widetilde{p}^{(VI)}$, and $\widetilde{p}^{(VII)}$ respectively.

Divide the resulting infinite products by the denominator of Equations 2.40, 2.41, and 2.42 respectively. Then, we have the infinite products of Proposition 2.2.2.3. So the left sides are precisely the formal sums $\mathcal{M}^{III}_q$, $\mathcal{M}^{VI}_q$, and $\mathcal{M}^{VII}_q$. Each of these is defined in terms of skew-Schur functions, which are in turn expressible as sums over certain Littlewood-Richardson coefficients. □

2.3. **Proof of the Main Theorem.**

2.3.1. *Direct Sum – (GL_{n+m},$GL_n \times GL_m$).* In this case, $K = GL_n \times GL_m$ and $\mathfrak{p} = M_{n,m} \oplus (M_{n,m})^*$. As a consequence of the Kostant-Rallis theorem, the q-multiplicity of an irreducible representation of K in $\mathcal{S}(\mathfrak{p})$ may be obtained from the q-multiplicity in $\mathcal{H}(\mathfrak{p})$. That is to say:

$$\frac{\sum_{d=0}^{\infty}\left[F^{(\alpha^+,\alpha^-)}_{(n)}\widehat{\otimes}F^{(\beta^+,\beta^-)}_{(m)}, \mathcal{H}^d_{GL_n\times GL_m}(\mathfrak{p})\right]q^d}{\prod_{k=1}^{\min(n,m)}(1-q^{2k})} = \sum_{d=0}^{\infty}\left[F^{(\alpha^+,\alpha^-)}_{(n)}\widehat{\otimes}F^{(\beta^+,\beta^-)}_{(m)}, \mathcal{S}^d(\mathfrak{p})\right]q^d, \tag{2.43}$$

where α^+, α^-, β^+ and β^- are non-negative integer partitions such that $\ell(\alpha^+)+\ell(\alpha^-) \le n$, and $\ell(\beta^+)+\ell(\beta^-) \le m$. Note also that we recognize the denominator on the left side as being the denominator of the Hilbert series for the $K = GL_p \times GL_q$-invariants in $\mathcal{S}(\mathfrak{p})$ using the decompositions in Theorem 2.1.1.1 and the Cartan-Helgason theorem (see Section 2.1.2). Thus the left side may be interpreted as the graded multiplicity of an irreducible K-representation in $\mathcal{S}(\mathfrak{p})^K \otimes \mathcal{H}_K(\mathfrak{p})$.

Using Proposition 2.1.5.1 Equation 2.6, we have:

$$\left[F_{(n)}^{(\alpha^+,\alpha^-)} \widehat{\otimes} F_{(m)}^{(\beta^+,\beta^-)}, \mathcal{S}^d\left(M_{n,m} \oplus (M_{n,m})^*\right)\right] = \sum_{\gamma,\delta,\mu,\nu} c^{\mu}_{\gamma\,\alpha^+} c^{\nu}_{\gamma\,\alpha^-} c^{\mu}_{\delta\,\beta^+} c^{\nu}_{\delta\,\beta^-},$$

where the sum is over $|\mu| + |\nu| = d$ and $\ell(\mu), \ell(\nu) \leq \min(n, m)$. If we multiply by q^d and sum, we recognize the above expression from Equation 2.40 of Proposition 2.2.3.1. We then obtain that the coefficients of the formal series in the left side of Equation 2.43 agree with the coefficients of

$$\frac{\widetilde{p}^{(III)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)}(q)}{\prod_{i=1}^{\infty}(1 - q^{2i})} \tag{2.44}$$

up to degree $d = \min(n, m)$. Because we are only claiming an equality of coefficients up to degree $\min(n, m)$ in q, we may replace the denominator of Expression 2.44 by $\prod_{i=1}^{\min(n,m)}(1 - q^{2i})$. This proves part 2a of the main theorem, i.e.,

$$\widetilde{p}^{(III)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)}(q)\Big|_{q^d} = \left[F_{(n)}^{(\alpha^+,\alpha^-)} \widehat{\otimes} F_{(m)}^{(\beta^+,\beta^-)}, \mathcal{H}^d_{GL_n \times GL_m}(\mathfrak{p})\right]. \tag{2.45}$$

2.3.2. *Polarization – (O_{2n},GL_n) and (Sp_{2n},GL_n).* For parts 3a and 3b of the main theorem we proceed in a similar manner.

We use the separation of variables from the Kostant-Rallis theorem to obtain an expression for the graded character that involves the Hilbert series of the K-invariants in $\mathcal{S}(\mathfrak{p})$. As before we use Theorem 2.1.1.1 and the Cartan-Helgason theorem to recognize this Hilbert series. We use Proposition 2.1.5.1 to decompose the full symmetric algebra on $\mathfrak{p}$. (Specifically, we use Equations 2.7 and 2.8 instead of 2.6.) Lastly, we replace the use of Equation 2.40 with Equation 2.41 and Equation 2.42.

References

[Abr69] J. Abramsky, Ph. D. Thesis, 1969.

[GW98] Roe Goodman and Nolan R. Wallach, *Representations and invariants of the classical groups*, Cambridge University Press, Cambridge, 1998.

[Hes80] Wim H. Hesselink, *Characters of the nullcone*, Math. Ann. **252** (1980), no. 3, 179–182.

[How00] Roger Howe, *K-type structure in the principal series of* GL_3*. I*, Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997), Adv. Stud. Pure Math., vol. 26, Math. Soc. Japan, Tokyo, 2000, pp. 77–98.

[How95] ———, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, The schur lectures (1992) (tel aviv), 1995, pp. 1–182.

[HT93] Roger E. Howe and Eng-Chye Tan, *Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations*, Bull. Amer. Math. Soc. (N.S.) **28** (1993), no. 1, 1–74.

[HTW05] Roger Howe, Eng-Chye Tan, and Jeb F. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. **357** (2005), no. 4, 1601–1626 (electronic).

[Joh76] Kenneth D. Johnson, *Composition series and intertwining operators for the spherical principal series. II*, Trans. Amer. Math. Soc. **215** (1976), 269–283.

[JW72] Kenneth Johnson and Nolan R. Wallach, *Composition series and intertwining operators for the spherical principal series*, Bull. Amer. Math. Soc. **78** (1972), 1053–1059.

[JW77] Kenneth D. Johnson and Nolan R. Wallach, *Composition series and intertwining operators for the spherical principal series. I*, Trans. Amer. Math. Soc. **229** (1977), 137–173.

[Kin70] R. C. King, *Generalized Young tableaux and the general linear group*, J. Mathematical Phys. **11** (1970), 280–293.

[Kos63a] Bertram Kostant, *Lie group representations on polynomial rings*, Bull. Amer. Math. Soc. **69** (1963), 518–526.
[Kos63b] ———, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.
[Kos69] ———, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. **75** (1969), 627–642.
[KR71] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.
[Lee94] Soo Teck Lee, *On some degenerate principal series representations of* $\mathrm{U}(n,n)$, J. Funct. Anal. **126** (1994), no. 2, 305–366.
[Lit44a] D. E. Littlewood, *Invariant theory, tensors and group characters*, Philos. Trans. Roy. Soc. London. Ser. A. **239** (1944), 305–365.
[Lit44b] ———, *On invariant theory under restricted groups*, Philos. Trans. Roy. Soc. London. Ser. A. **239** (1944), 387–417.
[Mac95] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second, The Clarendon Press Oxford University Press, New York, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
[Sta84] Richard P. Stanley, *The stable behavior of some characters of* $\mathrm{SL}(n, \mathbf{C})$, Linear and Multilinear Algebra **16** (1984), no. 1-4, 3–27.
[Sta99] ———, *Enumerative combinatorics. Vol. 2*, Cambridge University Press, Cambridge, 1999.
[WW00] Nolan R. Wallach and Jeb F. Willenbring, *On Some q-Analogs of a Theorem of Kostant-Rallis*, Canad. J. Math. **52** (2000), no. 2, 438–448.
[Wil00] Jeb F. Willenbring, *Stability properties for q-multiplicities and branching formulas for representations of the classical groups*, Ph.D. Thesis, 2000.
[Wil02] Jeb F. Willenbring, *An application of the Littlewood restriction formula to the Kostant-Rallis theorem*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4393–4419.

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