

Stable Hilbert series as related to the measurement of quantum entanglement

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Abstract

We compute a stable formula for the Hilbert series of the invariant algebra of polynomial functions on $\bigotimes_{i=1}^r \mathbb{C}^{n_i}$ under the action of $U(n_1) \times \cdots \times U(n_r)$, when viewed as real vector space. This situation has a physical interpretation as it is the quantum analog of an r -particle classical system in which the i^{th} particle has n_i classical states. The stable formula involves only elementary combinatorics, while its derivation involves the representation theory of the symmetric group. In particular, the Kronecker coefficients play an important role.

Key words: Hilbert series, Kronecker coefficients, quantum entanglement, Schur-Weyl duality

1 Introduction

A common approach to problems in quantum computation exploits the physical effect known as *entanglement*. The motivation for a systematic study of this notion appeared already in the first half of the twentieth century in [2]. Later in [3] these effects were realized to be relevant to computation. An axiomatic treatment of entanglement would include, in part, the following fact: Two vectors in a Hilbert space, \mathcal{H} , have the same entanglement if they are in the same orbit of a certain group of unitary operators acting on \mathcal{H} . More precisely, we consider a situation where the group of unitary operators is of the form $U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_r)$, which acts on $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_r$. From our viewpoint, this setting captures the notion of entanglement of r quantum particles, each with n_i classical states, $i = 1, 2, \dots, r$.

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More generally, let K be a compact Lie group acting \mathbb{C} -linearly on a finite dimensional complex vector space V . It is a difficult problem in representation theory to provide a description of the K -orbits in V . One approach set out in [8] and [10] is to use the invariant theory of K to separate orbits. More precisely: Set $\mathcal{P}_{\mathbb{R}}(V)$ to be the algebra of complex valued polynomial functions on the vector space V when viewed as a real vector space. The group K acts in the standard way on $\mathcal{P}_{\mathbb{R}}(V)$ by $g \cdot f(v) = f(g^{-1}v)$ for $g \in K$, $f \in \mathcal{P}_{\mathbb{R}}(V)$ and $v \in V$. Let the algebra of K -invariants in $\mathcal{P}_{\mathbb{R}}(V)$ be denoted as $\mathcal{P}_{\mathbb{R}}(V)^K$. We have:

Theorem (c.f. Thm. 3.1 of [8]) If $v, w \in V$ then $f(v) = f(w)$ for all $f \in \mathcal{P}_{\mathbb{R}}(V)^K$ if and only if v and w are in the same K -orbit.

Fix a sequence of positive integers $\mathbf{n} = (n_1, \dots, n_r)$. Let

$$V(\mathbf{n}) = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \dots \otimes \mathbb{C}^{n_r}$$

be the representation of $K(\mathbf{n}) = \prod_{i=1}^r U(n_i)$ under the standard action on each tensor² factor. (Here $U(n)$ denotes the unitary group.)

Well known results of Hilbert establish that the $K(\mathbf{n})$ -invariant subalgebra of $\mathcal{P}_{\mathbb{R}}(V(\mathbf{n}))$ is finitely generated. In spite of this result, our situation lacks a complete description of such generators, except for certain small values of the parameter space, $\mathbf{n} = (n_1, n_2, \dots, n_r)$.

Moreover, the $K(\mathbf{n})$ -invariant subalgebra inherits a gradation from $\mathcal{P}_{\mathbb{R}}(V(\mathbf{n}))$. Thus, let $\mathcal{P}_{\mathbb{R}}^d(V(\mathbf{n}))$ denote the subspace of degree d homogeneous polynomial function of $\mathcal{P}_{\mathbb{R}}(V(\mathbf{n}))$. We set $\mathcal{P}_{\mathbb{R}}^d(V(\mathbf{n}))^{K(\mathbf{n})} = \mathcal{P}_{\mathbb{R}}^d(V(\mathbf{n})) \cap \mathcal{P}_{\mathbb{R}}(V(\mathbf{n}))^{K(\mathbf{n})}$. One will see easily from Section 2 that $\mathcal{P}_{\mathbb{R}}^d(V(\mathbf{n}))^{K(\mathbf{n})} = (0)$ for d odd. However, the dimension of $\mathcal{P}_{\mathbb{R}}^d(V(\mathbf{n}))^{K(\mathbf{n})}$ for even d is more subtle.

We define the Hilbert series of $\mathcal{P}_{\mathbb{R}}(V(\mathbf{n}))^{K(\mathbf{n})}$ as:

$$H_t(\mathbf{n}) = \sum_{m=0}^{\infty} h_m t^{2m} \quad \text{where: } h_m = h_m(\mathbf{n}) = \dim \mathcal{P}_{\mathbb{R}}^{2m}(V(\mathbf{n}))^{K(\mathbf{n})}.$$

In Section 4, we show that for fixed m and r the value of $h_m(\mathbf{n})$ stabilizes as the components of \mathbf{n} grow large. Consequently, we can define

$$\tilde{h}_{m,r} = \lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \dots \lim_{n_r \rightarrow \infty} h_m(n_1, \dots, n_r).$$

Several papers in the recent literature investigate Hilbert series related to measurements of quantum entanglement. See, for example, [1, 7, 8, 10]. Despite the fact that the value of $h_m(\mathbf{n})$ is not known in general, the value of $\tilde{h}_{m,r}$ has a surprisingly simple description, which we present next.

We first set up the standard notation for partitions, which we define as weakly decreasing finite sequences of positive integers. We will always use lower case Greek letters to denote

² Here we tensor over \mathbb{C} .

partitions. If $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell > 0)$ is a partition we will write $\ell(\lambda) = \ell$ for the length (or depth) of λ and $|\lambda|$ for the size of λ . We will write $\lambda \vdash m$ to indicate that λ is a partition of size m . Lastly, if λ has a_1 ones, a_2 twos, a_3 threes etc., let

$$z_\lambda = 1^{a_1} 2^{a_2} 3^{a_3} \dots a_1! a_2! a_3! \dots$$

The result of this paper is

Theorem 1 *For any integers $m \geq 0$ and $r \geq 1$,*

$$\tilde{h}_{m,r} = \sum_{\lambda \vdash m} z_\lambda^{r-2}$$

Apart from the physical motivation, the stable behavior of the dimension of the $K(\mathbf{n})$ -invariant polynomials of degree m may be interpreted as a statement about the representation theory of “ $U(\infty)$ ”.

We prove this result in Section 4. The proof involves a reduction of the problem to the representation theory of the complex general linear group which is set up in Section 2 and an exposition of the Kronecker coefficients obtained from Schur-Weyl duality in Section 3. In the last section we provide some small tables giving an indication of the growth rate for $h_m(\mathbf{n})$ and $\tilde{h}_{m,r}$.

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2 Preliminaries

We will now complexify the picture so that we may work in the situation where all groups are complex reductive, and all representations involve rational linear actions on complex vector spaces. First we introduce some notation. Let $\mathcal{P}(V)$ denote the algebra of complex valued polynomial function on a complex vector space V . We denote the standard gradation on $\mathcal{P}(V)$ as $\mathcal{P}(V) = \bigoplus_{d=0}^{\infty} \mathcal{P}^d(V)$, where $\mathcal{P}^d(V)$ denotes the homogeneous polynomials of degree d . As in the last section, $\mathcal{P}_{\mathbb{R}}(V)$ denotes the graded vector space of complex valued polynomial functions on V when V is regarded as a real vector space.

Suppose that a compact Lie group, K , acts \mathbb{C} -linearly on V . The K -action on V gives rise to an action on $\mathcal{P}(V)$ by $k \cdot f(v) = f(k^{-1}v)$ for $k \in K$, $f \in \mathcal{P}(V)$ and $v \in V$. As a graded representation, $\mathcal{P}_{\mathbb{R}}(V) \cong \mathcal{P}(V \oplus \bar{V})$, where \bar{V} denotes the complex vector space with the opposite complex structure (see [8]). Note that as a representation of K , \bar{V} is equivalent to the dual representation, which we denote by V^* .

Let G denote the complexification of K . By analytic continuation a complex representation of K extends to G . Let \widehat{G} denote the equivalence classes of irreducible rational representations of G .

Proposition 2 *Let G be a complex reductive group and V a finite dimensional complex rational representation. Assume that an irreducible representation of G occurs in $\mathcal{P}(V)$ at most in one degree. Then $\mathcal{P}^{2m+1}(V \oplus V^*)^G = (0)$ and*

$$\dim P^{2m}(V \oplus V^*)^G = \sum_{\xi \in \widehat{G}} \text{mult}(m, \xi)^2.$$

where $\text{mult}(m, \xi)$ denotes the multiplicity of the representation ξ in $\mathcal{P}^m(V)$.

Proof: As finite dimensional representations, $\mathcal{P}^d(V \oplus V^*) = \bigoplus_{d_1+d_2=d} \mathcal{P}^{d_1}(V) \otimes \mathcal{P}^{d_2}(V^*)$, and $\mathcal{P}^{d_2}(V^*) \cong \mathcal{P}^{d_2}(V)^*$. We may identify $[\mathcal{P}^{d_1}(V) \otimes \mathcal{P}^{d_2}(V)^*]^G$ with the equivariant homomorphisms, $\text{Hom}_G(\mathcal{P}^{d_2}(V), \mathcal{P}^{d_1}(V))$. By Schur's Lemma one obtains an invariant polynomial in $\mathcal{P}^d(V \oplus V^*)$ every time an irreducible representation in $\mathcal{P}^{d_1}(V)$ is paired with an irreducible representation in $\mathcal{P}^{d_2}(V)$. For $\xi \in \widehat{G}$, we obtain $\dim P^d(V \oplus V^*)^G = \sum \text{mult}(d_1, \xi) \text{mult}(d_2, \xi)$. If $d = 2m$ for some m then since any irreducible representation of G occurs in at most one degree we must have $d_1 = d_2 = m$. Otherwise only the zero function is invariant. **QED**

We now specialize to our situation. The complexification of $K(\mathbf{n})$ is $G(\mathbf{n}) = \prod_{i=1}^r GL_{n_i}$ (Here $GL_n = GL_n(\mathbb{C})$.) For $\xi \in \widehat{G(\mathbf{n})}$, let $\mathcal{M}(m, \xi)$ denote the multiplicity of ξ in $\mathcal{P}^m(V(\mathbf{n}))$. It is easy to see that an element of the center of the group $G(\mathbf{n})$ acts on $\mathcal{P}^m(V(\mathbf{n}))$ by a scalar depending on m . Hence, an irreducible representation of $G(\mathbf{n})$ occurs in at most one degree. By Proposition 2 we see that

$$h_m(\mathbf{n}) = \sum_{\xi \in \widehat{G}} \mathcal{M}(m, \xi)^2. \quad (1)$$

So we turn now to the question of computing the multiplicities, $\mathcal{M}(m, \xi)$. As in [5, 6], we parameterize the equivalence classes of finite dimensional rational irreducible representations of GL_n using partitions.

Let $n \in \mathbb{Z}^+$. For a partition λ with $\ell(\lambda) \leq n$, let $F_{(n)}^\lambda$ denote the irreducible rational representation of the general linear group with highest weight indexed by λ as in Chapter 9 of [5]. Since the parts, λ_j , of λ are non-negative, the matrix coefficients of $F_{(n)}^\lambda$ are polynomial functions of the matrix entries of GL_n .

If V is a representation of a group G then the vector space $V \otimes V$ is also a representation of G , under the diagonal action. However, the group $G \times G$ also acts on $V \otimes V$ by $(g_1, g_2) \cdot v_1 \otimes v_2 = (g_1 v_1) \otimes (g_2 v_2)$. We denote this latter action by $V \widehat{\otimes} V$ to distinguish it from the diagonal action.

Lastly, note that if G_1 and G_2 denote reductive linear algebraic groups over \mathbb{C} then each rational irreducible representation of $G_1 \times G_2$ is equivalent to $E^\alpha \widehat{\otimes} E^\beta$ for some rational irreducible representations E^α and E^β of G_1 and G_2 respectively. Consequently, the irreducible

finite dimensional representations of $G(\mathbf{n})$ with polynomial matrix coefficients are of the form $F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} F_{(n_2)}^{\mu^{(2)}} \widehat{\otimes} \cdots \widehat{\otimes} F_{(n_r)}^{\mu^{(r)}}$ where $\ell(\mu^{(j)}) \leq n_j$ for each j .

3 Schur-Weyl duality

The symmetric group acts on the space $\otimes^m \mathbb{C}^n$ by permuting the tensor factors. That is, if $\sigma \in S_m$ then

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)},$$

for $v_1, \dots, v_m \in \mathbb{C}^n$. This action then extends by linearity to all of $\otimes^m \mathbb{C}^n$. At the same time, GL_n acts on the same space by the diagonal action. That is, for $x \in GL_n$

$$x \cdot (v_1 \otimes \cdots \otimes v_m) = (xv_1) \otimes \cdots \otimes (xv_m).$$

It is the case that the subalgebra of $\text{End}(\otimes^m \mathbb{C}^n)$ generated by the image of each of these actions is the commutant of the other. A consequence of this fact is that we have a multiplicity free decomposition

$$\otimes^m \mathbb{C}^n \cong \bigoplus_{\lambda} F_{(n)}^{\lambda} \widehat{\otimes} W_{(m)}^{\lambda},$$

where $W_{(m)}^{\lambda}$ is the irreducible complex representation of S_m indexed by the partition λ as in [5, 6]. Note that the direct sum is over all partitions, λ , of size m with at most n parts. Note that if $n \geq m$ then all irreducible representations of S_m occur in the decomposition.

Using this decomposition, one can interpret statements about the representation theory of S_m in terms of the representation theory of GL_n , and visa-versa. An example of this phenomena is useful in our situation, which we will describe next.

Definition 1: Let $\mu^{(1)}, \dots, \mu^{(r)}$ and λ be partitions of m . The tensor product $W_{(m)}^{\mu^{(1)}} \otimes \cdots \otimes W_{(m)}^{\mu^{(r)}}$ is a representation of S_m under the diagonal action. We let $g_{\mu^{(1)}, \dots, \mu^{(r)}, \lambda}$ denote the multiplicity of the irreducible S_m -representation, $W_{(m)}^{\lambda}$, in the decomposition of this tensor product. That is,

$$W_{(m)}^{\mu^{(1)}} \otimes \cdots \otimes W_{(m)}^{\mu^{(r)}} \cong \bigoplus_{\lambda} g_{\mu^{(1)} \dots \mu^{(r)} \lambda} W_{(m)}^{\lambda}$$

where the sum is over all partitions of m .

We now turn to the group GL_n when $n = n_1 \cdots n_r$. Observe first that we can map $G(\mathbf{n}) = GL_{n_1} \times \cdots \times GL_{n_r}$ into GL_n by the Kronecker product of matrices. That is to say, if $x_i \in GL_{n_i}$ for each $1 \leq i \leq r$, then $x_1 \otimes \cdots \otimes x_r$ defines a linear transformation of $\otimes_{i=1}^r \mathbb{C}^{n_i}$. If we identify $\otimes_{i=1}^r \mathbb{C}^{n_i}$ with \mathbb{C}^n then we have obtained a mapping, $G(\mathbf{n}) \rightarrow GL_n$ that we will call the *Kronecker map*.

Definition 2: Let $\mu^{(1)}, \dots, \mu^{(r)}$ and λ be partitions with $\ell(\mu^{(i)}) \leq n_i$ for all i and $\ell(\lambda) \leq n$ (with $n = n_1 \cdots n_r$). The irreducible GL_n -representation, $F_{(n)}^{\lambda}$, decomposes as a $G(\mathbf{n})$ -

representation when pulled back by the Kronecker map,

$$G(\mathbf{n}) \longrightarrow GL_n \longrightarrow GL(F_{(n)}^\lambda).$$

Let $k_{\mu^{(1)}\dots\mu^{(r)}\lambda}$ denote the multiplicity of $F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} \dots \widehat{\otimes} F_{(n_r)}^{\mu^{(r)}}$ in $F_{(n)}^\lambda$ after this pull back. That is, we have the decomposition

$$F_{(n)}^\lambda \cong \bigoplus_{\mu^{(1)}, \dots, \mu^{(r)}} k_{\mu^{(1)}\mu^{(2)}\dots\mu^{(r)}\lambda} F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} \dots \widehat{\otimes} F_{(n_r)}^{\mu^{(r)}}.$$

Remark: By considering the action of the center of $G(\mathbf{n})$ on $F_{(n_1\dots n_r)}^\lambda$ (under the pull back from the Kronecker map), one can easily see that if $k_{\mu^{(1)}\dots\mu^{(r)}\lambda} > 0$ then $|\mu^{(i)}| = |\lambda|$ for all i .

Proposition 3 *Let λ and $\mu^{(1)}, \dots, \mu^{(r)}$ be partitions of m with $\ell(\mu^{(i)}) \leq n_i$ for all i and $\ell(\lambda) \leq n_1 \dots n_r$. We have*

$$k_{\mu^{(1)}\mu^{(2)}\dots\mu^{(r)}\lambda} = g_{\mu^{(1)}\mu^{(2)}\dots\mu^{(r)}\lambda}.$$

Proof: The idea is to apply the Kronecker map in the context of Schur-Weyl duality. If we make the identification $\mathbb{C}^{n_1 n_2 \dots n_r} \cong \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_r}$ then we obtain:

$$\begin{aligned} \bigotimes_{i=1}^m \left(\bigotimes_{i=1}^r \mathbb{C}^{n_i} \right) &\cong \left(\bigotimes_{i=1}^m \mathbb{C}^{n_1} \right) \otimes \dots \otimes \left(\bigotimes_{i=1}^m \mathbb{C}^{n_r} \right) \\ &\cong \left(\bigoplus_{\mu^{(1)}} F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} W_{(m)}^{\mu^{(1)}} \right) \otimes \dots \otimes \left(\bigoplus_{\mu^{(r)}} F_{(n_r)}^{\mu^{(r)}} \widehat{\otimes} W_{(m)}^{\mu^{(r)}} \right) \end{aligned}$$

by applying the Schur-Weyl decomposition on each of the r tensor factors. Re-ordering the terms in the direct sum we see that as a representation of $G(\mathbf{n}) \times S_m$ we have

$$\begin{aligned} \bigotimes_{i=1}^m \left(\bigotimes_{i=1}^r \mathbb{C}^{n_i} \right) &\cong \bigoplus_{\mu^{(1)}, \dots, \mu^{(r)}} \left(F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} \dots \widehat{\otimes} F_{(n_r)}^{\mu^{(r)}} \right) \widehat{\otimes} \left(W_{(m)}^{\mu^{(1)}} \otimes \dots \otimes W_{(m)}^{\mu^{(r)}} \right) \\ &\cong \bigoplus_{\mu^{(1)}, \dots, \mu^{(r)}} \left(F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} \dots \widehat{\otimes} F_{(n_r)}^{\mu^{(r)}} \right) \widehat{\otimes} \left(\bigoplus_{\lambda} g_{\mu^{(1)}\mu^{(2)}\dots\mu^{(r)}\lambda} W_{(m)}^\lambda \right) \\ &\cong \bigoplus_{\lambda} \left(\bigoplus_{\mu^{(1)}, \dots, \mu^{(r)}} g_{\mu^{(1)}\dots\mu^{(r)}\lambda} F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} \dots \widehat{\otimes} F_{(n_r)}^{\mu^{(r)}} \right) \widehat{\otimes} W_{(m)}^\lambda \end{aligned}$$

On the other hand, from the definition of $k_{\mu^{(1)}\dots\mu^{(r)}\lambda}$ we also have

$$\begin{aligned} \bigotimes^m \left(\bigotimes_{i=1}^r \mathbb{C}^{n_i} \right) &\cong \bigotimes^m \mathbb{C}^{n_1 \dots n_r} \cong \bigoplus_{\lambda} F_{(n_1 n_2 \dots n_r)}^{\lambda} \widehat{\otimes} W_{(m)}^{\lambda} \\ &\cong \bigoplus_{\lambda} \left(\bigoplus_{\mu^{(1)}, \dots, \mu^{(r)}} k_{\mu^{(1)}\mu^{(2)}\dots\mu^{(r)}\lambda} F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} \dots \widehat{\otimes} F_{(n_r)}^{\mu^{(r)}} \right) \widehat{\otimes} W_{(m)}^{\lambda} \end{aligned}$$

as a representation of $G(\mathbf{n}) \times S_m$. The result follows since $\bigotimes^m (\bigotimes_{i=1}^r \mathbb{C}^{n_i})$ has a unique decomposition into irreducible representations. **QED**

Both $g_{\mu^{(1)}\mu^{(2)}\dots\mu^{(r)}\lambda}$ and $k_{\mu^{(1)}\mu^{(2)}\dots\mu^{(r)}\lambda}$ are called the *Kronecker coefficients*. Of particular interest is the special case when $\lambda = (m)$ for a non-negative integer m . In this situation, $F_{(n)}^{\lambda} = F_{(n)}^{(m)} \cong \mathcal{P}^m((\mathbb{C}^n)^*)$. Thus, the decomposition of $\mathcal{P}^m(V(\mathbf{n})^*)$ under the action of $G(\mathbf{n})$ is equivalent to the computation of $k_{\mu^{(1)}\dots\mu^{(r)}\lambda}$ when $\lambda = (m)$. Note that if we specialize further to the case where $r = 2$ one can obtain an even sharper result as follows. Since $g_{\mu^{(1)}\mu^{(2)}\lambda} = k_{\mu^{(1)}\mu^{(2)}\lambda}$ and $W_{(m)}^{\lambda}$ is the trivial representation of S_m when $\lambda = (m)$, we see by duality that $k_{\mu^{(1)}\mu^{(2)}(m)} = 1$ exactly when $\mu_1 = \mu_2 \vdash m$. One then recovers the following multiplicity free decomposition:

$$\mathcal{P}^m \left((\mathbb{C}^{n_1} \widehat{\otimes} \mathbb{C}^{n_2})^* \right) \cong \bigoplus_{\mu} F_{(n_1)}^{\mu} \widehat{\otimes} F_{(n_2)}^{\mu}$$

where the sum is over all partitions of m with at most $\min(n_1, n_2)$ parts.

The above decomposition is an instance of R. Howe's theory of dual pairs (see [5, 6]). At the character theoretic level the decomposition gives rise to the Cauchy identity of Schur functions (see Section 2.1.5 (b) of [6]).

4 Proof of the main result

We return to the question of computing the Hilbert series of the $G(\mathbf{n})$ -invariant algebra in $\mathcal{P}(V(\mathbf{n}) \oplus V(\mathbf{n})^*)$. Our problem has become that of decomposing $\mathcal{P}(V(\mathbf{n}))$ and $\mathcal{P}(V(\mathbf{n})^*)$ under the action of the product group $G(\mathbf{n})$. An irreducible representation of $G(\mathbf{n})$ occurs in $\mathcal{P}(V(\mathbf{n}))$ iff its dual occurs in $\mathcal{P}(V(\mathbf{n})^*)$. Furthermore, the two relevant multiplicities are equal. We will focus on the decomposition of the graded components of $\mathcal{P}(V(\mathbf{n})^*)$. In light of the remarks at the end of the last section this amounts to decomposing $F_{(n)}^{\lambda}$ when $\lambda = (m)$ and $n = n_1 n_2 \dots n_r$ under the action obtained by pulling back from GL_n to $G(\mathbf{n})$ under the Kronecker map. Applying Proposition 3 and Definition 1 when $\lambda = (m)$ we obtain:

$$\mathcal{P}^m(V(\mathbf{n})^*) \cong \bigoplus g_{\mu^{(1)}\dots\mu^{(r)}(m)} F_{(n_1)}^{\mu^{(1)}} \widehat{\otimes} \dots \widehat{\otimes} F_{(n_r)}^{\mu^{(r)}}$$

where the sum is over all r -tuples of partitions, $\mu^{(1)} \dots \mu^{(r)} \vdash m$, with $\ell(\mu^{(j)}) \leq n_j$ for all j . That is to say $\mathcal{M}(m, \xi) = g_{\mu^{(1)}\dots\mu^{(r)}(m)}$ where the dual of ξ is indexed by $(\mu^{(1)}, \dots, \mu^{(r)})$.

The partition (m) corresponds to the trivial representation of S_m . Thus by Definition 2, $g_{\mu^{(1)}\dots\mu^{(r)}(m)} = g_{\mu^{(1)}\dots\mu^{(r)}}$. From Equation 1 we see:

$$h_m(\mathbf{n}) = \sum_{\mu^{(1)}, \dots, \mu^{(r)}} g_{\mu^{(1)}\dots\mu^{(r)}}^2 \quad (2)$$

where the sum is over all ordered r -tuples of partitions of size m and $\ell(\mu^{(j)}) \leq n_j$ for $1 \leq j \leq r$. From this expression we can easily see that as the parameters n_1, \dots, n_r go to infinity the value of $h_m(\mathbf{n})$ stabilizes to a non-negative integer depending only on m and r , which we have defined earlier as $\tilde{h}_{m,r}$. We obtain

Proposition 4 *For any non-negative integers m, r , and $\mathbf{n} = (n_1, \dots, n_r)$, $h_m(\mathbf{n}) \leq \tilde{h}_{m,r}$ with equality if m is at most $\min(n_1, \dots, n_r)$. Furthermore,*

$$\tilde{h}_{m,r} = \sum_{\mu^{(1)}, \dots, \mu^{(r)}} g_{\mu^{(1)}\dots\mu^{(r)}}^2 \quad (3)$$

where the sum is over all r -tuples of partitions of size m .

Proof. Observe that the sum in Equation 3 differs from the sum in Equation 2 in that there are no conditions bounding the number of parts of the partitions, $\mu^{(j)}$, $j = 1, \dots, r$. Thus, if m is at most the value of $\min(n_1, \dots, n_r)$ then $h_m(\mathbf{n}) = \tilde{h}_{m,r}$, otherwise we have $h_m(\mathbf{n}) \leq \tilde{h}_{m,r}$. **QED**

Equation 3 alone is not so helpful as the Kronecker coefficients are notoriously difficult to compute effectively. However, this particular combination of S_m -tensor product multiplicities can be put into a closed form, which we address next in the context of a general finite group.

Let F be a finite group. Denote by \widehat{F} the isomorphism classes of irreducible complex representations of F . For any $\lambda \in \widehat{F}$ we choose a representation V_λ in the class λ . The Kronecker-coefficients, $g_{\lambda_1 \dots \lambda_r} \in \mathbb{N}$ for $\lambda_1, \dots, \lambda_r \in \widehat{F}$ are defined as $g_{\lambda_1 \dots \lambda_r} = \dim(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_r})^F$. Equivalently,

$$V_{\lambda_1} \otimes \dots \otimes V_{\lambda_{r-1}} \cong \bigoplus_{\lambda \in \widehat{F}} g_{\lambda_1 \dots \lambda_{r-1} \lambda} V_\lambda^*.$$

Note that in the case of $F = S_m$, $V_\lambda \cong V_\lambda^*$ for all $\lambda \in \widehat{S_m}$, so the above is the same as Definition 1.

Proposition 5 *Consider the action of F on the product space $F^{r-1} = F \times \dots \times F$ by simultaneous conjugation. Then*

$$\dim \mathbb{C}[F^{r-1}]^F = \sum_{\lambda_1, \dots, \lambda_r} g_{\lambda_1 \dots \lambda_r}^2.$$

Proof: We have $\mathbb{C}[F] \cong \bigoplus_\lambda V_\lambda \otimes V_\lambda^*$. Hence,

$$\begin{aligned}
\mathbb{C}[F^{r-1}] &\cong \bigotimes^{r-1} \mathbb{C}[F] \cong \left(\bigoplus_{\lambda_1} V_{\lambda_1} \otimes V_{\lambda_1}^* \right) \otimes \cdots \otimes \left(\bigoplus_{\lambda_{r-1}} V_{\lambda_{r-1}} \otimes V_{\lambda_{r-1}}^* \right) \\
&\cong \bigoplus_{\lambda_1, \dots, \lambda_{r-1}} V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{r-1}} \otimes V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_{r-1}}^* \\
&\cong \bigoplus_{\lambda_1, \dots, \lambda_{r-1}} \left(\bigoplus_{\lambda} g_{\lambda_1 \dots \lambda_{r-1} \lambda} V_{\lambda}^* \right) \otimes \left(\bigoplus_{\mu} g_{\lambda_1 \dots \lambda_{r-1} \mu} V_{\mu} \right) \\
&\cong \bigoplus_{\lambda_1, \dots, \lambda_{r-1}, \lambda, \mu} g_{\lambda_1 \dots \lambda_{r-1} \lambda} g_{\lambda_1 \dots \lambda_{r-1} \mu} (V_{\lambda}^* \otimes V_{\mu}).
\end{aligned}$$

Taking fixed points the result follows. **QED**

On the other hand we know that for a permutation representation of F on a set X , the dimension of the invariants $\mathbb{C}[X]^F$ equals the number of orbits of F on X , and this number can be expressed in the following way

$$|X/F| = \frac{1}{|F|} \sum_{h \in F} |X^h|$$

where X^h is the set of fixed points of $h \in F$ (Burnside's formula, see [4]). In case of a diagonal action on a product $X \times Y$ we clearly have $(X \times Y)^h = X^h \times Y^h$. Since for the conjugation action of F on itself the fixed point set of h equals the centralizer $F_h \subseteq F$, we finally get

$$|F^{r-1}/F| = \frac{1}{|F|} \sum_{h \in F} |F_h|^{r-1} = \sum_{h_i} |F_{h_i}|^{r-2} \quad (4)$$

where $\{h_i\}$ is a set of representatives of the conjugacy classes in F .

We now specialize to the case where $F = S_m$, and \widehat{F} is identified with the set of partitions of m . For a permutation $h \in S_m$ with cycle type given by a partition λ we have $|F_h| = z_{\lambda}$ with $z_{\lambda} = 1^{a_1} 2^{a_2} 3^{a_3} \cdots a_1! a_2! a_3! \cdots$ where λ has a_1 ones, a_2 twos, a_3 threes etc. From Proposition 5 and Equation 4 we obtain:

For any integers $m \geq 0$ and $r \geq 1$,

$$\tilde{h}_{m,r} = \sum_{\lambda \vdash m} z_{\lambda}^{r-2}.$$

5 Data

Several special cases of the main theorem may be independently verified. A well known identity is $\sum_{\lambda \vdash m} z_{\lambda}^{-1} = 1$, which is consistent with the theorem when $r = 1$. Also, using the fact that the real group $SU(p, q)$ is Hermitian symmetric, one can independently deduce the $r = 2$ case from a result of Wilfried Schmid (see [9]). Lastly, the $m = 2$ (and r arbitrary) case is easy to see directly.

The efficient calculation of $h_m(\mathbf{n})$ and $\tilde{h}_{m,r}$ is an important and difficult problem. A theme of the present paper is that the value of $\tilde{h}_{m,n}$ is simpler to calculate than $h_m(\mathbf{n})$. Moreover, the value of the former can easily be computed via generating functions. If $B_{r,k}(x) = \sum_{n \geq 0} n!^{r-2} k^{n(r-2)} x^{nk}$ then $\prod_{k \geq 1} B_{r,k}(x) = \sum_{m \geq 0} \tilde{h}_{m,r} x^m$. Other combinatorial facts about these numbers would be of interest, such as a recursive formula.

In the following three tables we present some data related to the functions $h_m(\mathbf{n})$ and $\tilde{h}_{m,r}$.

(a) Dimension of $\mathcal{P}_{\mathbb{R}}^{2m}(\otimes^3 \mathbb{C}^n)^K$ where $K = \prod_{i=1}^3 U(n)$.

$m \setminus n$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	4	4	4	4	4
3	1	5	11	11	11	11
4	1	12	31	43	43	43
5	1	15	92	143	161	161
6	1	30	327	757	871	901

(b) Dimension of $\mathcal{P}_{\mathbb{R}}^{2m}(\otimes^4 \mathbb{C}^n)^K$ where $K = \prod_{i=1}^4 U(n)$.

$m \setminus n$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	8	8	8	8	8
3	1	20	49	49	49	49
4	1	98	536	681	681	681
5	1	293	7908	14116	14721	14721
6	1	1128	135543	450862	520596	524137

(c) Value of $\lim_{n \rightarrow \infty} \dim \mathcal{P}_{\mathbb{R}}^{2m}(\otimes^r \mathbb{C}^n) \prod_{i=1}^n U(n)$

$r \setminus m$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	5	7	11
3	1	4	11	43	161	901
4	1	8	49	681	14721	524137
5	1	16	251	14491	1730861	373486525
6	1	32	1393	336465	207388305	268749463729
7	1	64	8051	7997683	24883501301	193492277719861
8	1	128	47449	191374041	2985987361161	139314094050615817

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