INVARIANT DIFFERENTIAL OPERATORS AND FCR FACTORS OF ENVELOPING ALGEBRAS

IAN M. MUSSON AND JEB F. WILLENBRING

Abstract. If $\mathfrak{g}$ is a semisimple Lie algebra, we describe the prime factors of $\mathcal{U}(\mathfrak{g})$ that have enough finite dimensional modules. The proof depends on some combinatorial facts about the Weyl group which may be of independent interest. We also determine, which finite dimensional $\mathcal{U}(\mathfrak{g})$-modules are modules over a given prime factor. As an application we study finite dimensional modules over some rings of invariant differential operators arising from Howe duality.

1. Introduction

1.1. Let F denote an algebraically closed field of characteristic zero, and let $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be the enveloping algebra of a reductive Lie algebra $\mathfrak{g}$ over F. Starting with the pioneering work of Soergel ([Soe90]), there has been some effort to understand the prime spectrum $X = Spec(\mathcal{U})$ (see [BJ01], [P05], [P]). The key open problem is to determine the order relations between prime ideals.

Intersections of prime ideals are important in relating the algebraic and topological properties of X. For example, if $P \in X$, then P is primitive if and only if P is locally closed in X (see [Dix96], 8.5.7). We can formulate this result as follows, for $P \in X$, set

$$X(P) = \{Q \in X | P \subset Q\}.$$

(Throughout this paper we use the symbol $\subset$ to denote proper inclusion.) Then exactly one of the following holds:

(1) P is primitive,
(2) $P = \bigcap_{Q \in X(P)} Q$

Now set $Y(P) = \{Q \in X(P) | \dim \mathcal{U}/Q < \infty\}$. In view of the above it is natural to ask when P is equal to $\bigcap_{Q \in Y(P)} Q$. This is closely related to the property that $\mathcal{U}/P$ is an FCR-algebra, (we recall the definition shortly). In fact, it is easy to see that if $\mathfrak{g}$ is semisimple then $\mathcal{U}/P$ is FCR if and only if one of the following mutually exclusive conditions holds:

(3) $\dim \mathcal{U}/P < \infty$,
(4) $P = \bigcap_{Q \in Y(P)} Q$.

1.2. Given an F-algebra A, we say that A *has enough finite dimensional representations* if the intersection of the annihilators of the finite dimensional A-modules is zero. If in addition all finite dimensional representations are completely reducible A is an *FCR* algebra (see [KS94]).

Let $\mathfrak{g}$ be a semisimple Lie algebra with enveloping algebra $\mathcal{U}(\mathfrak{g})$. In this paper, we describe the prime factors of $\mathcal{U}(\mathfrak{g})$ that are FCR algebras, in terms of Soergel's theory. If A is a prime factor of $\mathcal{U}(\mathfrak{g})$, then any finite dimensional A-module is a $\mathfrak{g}$-module, and hence completely reducible. So the issue is whether or not A has enough finite dimensional modules. This last condition has the following topological interpretation. For any algebra A, let

$$\hat{A} = \varprojlim \{A/I \mid I \text{ is an ideal with } \dim A/I < \infty\},$$

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the profinite completion of A. Then the natural ring homomorphism $A \to \hat{A}$ is injective if and only if A has enough finite dimensional modules.

1.3. We begin with some basic notation used to state the main theorem. Suppose that $\mathfrak{g}$ is a reductive Lie algebra. Let $\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}]$, be the semisimple part of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and set $\mathfrak{h}_{ss} = \mathfrak{h} \cap \mathfrak{g}_{ss}$. Then, $\mathfrak{h} = \mathfrak{h}_{ss} \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$.

Let R denote the root system corresponding to the pair $(\mathfrak{g}, \mathfrak{h})$ with Weyl group W. Choose a set of positive roots, R^+, such that $R = R^+ \cup -R^+$, and let $B = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ denote the simple roots in R^+.

Set $\rho := \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, and define the "dot action" of W on $\mathfrak{h}^*$ as $w.\xi := w(\xi + \rho) - \rho$ for $\xi \in \mathfrak{h}^*$ and $w \in W$. The dot action of W on $\mathfrak{h}^*$ induces an action on $\mathcal{S} = \mathcal{S}(\mathfrak{h})$, the symmetric algebra on $\mathfrak{h}$. Let $\kappa(\ ,\ )$ denote the Killing form on $\mathfrak{g}_{ss}$. For $\mu \in \mathfrak{h}^*$, let h_μ be the element of $\mathfrak{h}_{ss}$ determined by $\kappa(h_\mu, h) = \mu(h)$ for all $h \in \mathfrak{h}_{ss}$. The isomorphism $\mathfrak{h}_{ss}^* \simeq \mathfrak{h}_{ss}$ sending μ to h_μ induces a bilinear form $(\ ,\ )$ on $\mathfrak{h}_{ss}^*$. Recall that this form is positive definite on the real span of the roots. For $\alpha \in R$ let $H_\alpha = \frac{2h_\alpha}{(\alpha,\alpha)}$ denote the coroot to α, and set $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. Let $\overline{\omega}_i \in \mathfrak{h}_{ss}^*$ be the fundamental weights, so that $(\overline{\omega}_i, \alpha_j^\vee) = \delta_{i,j}$. We write H_i (resp. h_i) in place of H_{α_i} (resp. h_{α_i}). Thus $\mu(H_i) = (\mu, \alpha_i^\vee)$.

Each $\lambda \in \mathfrak{h}^*$ determines a subroot system, $R_\lambda = \{\alpha \in R | (\lambda, \alpha^\vee) \in \mathbb{Z}\}$. Let W_λ denote the Weyl group of R_λ. Let B_λ be the unique basis of simple roots in $R_\lambda^+ := R^+ \cap R_\lambda$, and set $W^\lambda := \{w \in W | w(B_\lambda) \subseteq R^+\}$. The elements of W^λ are left coset representatives for W_λ in W. Thus, $W = W^\lambda W_\lambda$.

1.4. To any prime ideal Ω of $\mathcal{S}$ we can associate a prime ideal I_Ω of $\mathcal{U}$, and a "tautological highest weight" $\lambda := \lambda_\Omega$ (see[Soe90]). We recall the details in Section 3. Any prime ideal of $\mathcal{U}$ has the form I_Ω for a suitable prime ideal Ω in $\mathcal{S}$. Set $\mathcal{U}_\Omega := \mathcal{U}/I_\Omega$.

For $\mu \in \mathfrak{h}^*$, let $L(\mu)$ be the simple module with highest weight μ. Let

$$P^+ = \{\mu \in \mathfrak{h}^* | \dim L(\mu) < \infty\}.$$

We say that μ is *dominant* if $\mu \in P^+$. We say that a prime ideal Ω is *strongly dominant* if $\mathcal{V}(\Omega) \cap P^+$ is Zariski dense in $\mathcal{V}(\Omega)$, where $\mathcal{V}(\Omega)$ denotes the zero set of Ω in $\mathfrak{h}^*$. If Ω is strongly dominant then Ω is dominant as defined in [P], see Lemma 3.2.

Main Theorem. *Assume $\mathfrak{g}$ is semisimple and $\mathcal{U} = \mathcal{U}(\mathfrak{g})$.*

1. *If $\mathcal{U}_\Omega$ is FCR then for some $w \in W$, $w.\Omega$ is strongly dominant.*
2. *Let Ω be a strongly dominant ideal in $\mathcal{S}$, and $w \in W$.*
   The following conditions are equivalent:
   - (a) *$\mathcal{U}_{w.\Omega}$ is FCR*
   - (b) $I_{w.\Omega} = I_\Omega$
   - (c) *$w \in W^\lambda$ where $\lambda = \lambda_\Omega$.*

If $\mathfrak{g}$ is reductive we can still give sufficient conditions for $\mathcal{U}$ to have enough finite dimensional representations, see Proposition 4.4 and Lemma 4.5.

This paper is organized as follows. In Section 2 we prove some results about root systems and the Weyl group which may be of independent interest. Then in Section 3 we collect some results about prime and primitive ideals in enveloping algebras. The main theorem is proved in Section 4. In Sections 5, 6, and 7, we apply our results to some examples of rings of invariant differential operators related to Howe duality.

2. Weyl group combinatorics

We introduce some notation. Let B_1 be a subset of B, R_1 be the subroot system of R with simple roots B_1 and let R_1^+ be the corresponding set of positive roots. Let W_1 be the Weyl group of R_1 and set

$$\rho' = \frac{1}{2} \sum_{\alpha \in R_1^+} \alpha.$$

For $w \in W$ define:

$$Q(w) = \left\{ \alpha \in R^+ | w\alpha \in -R^+ \right\},$$

and set $\ell(w) = |Q(w)|$. We have ([GW98] Lemma 7.3.6),

$$\rho - w^{-1}\rho = \sum_{\alpha \in Q(w)} \alpha. \tag{1}$$

Lemma 2.1. *Suppose $w \in W$ is such that $w(R_1) = R_1$ and set*

$$T(w) = \{\alpha \in R_1^+ | w\alpha \in -R_1^+\}.$$

Then,

$$\rho' - w^{-1}\rho' = \sum_{\alpha \in T(w)} \alpha.$$

Proof. Note that R_1^+ is a disjoint union $R_1^+ = w(R_1^+ \backslash T(w)) \cup (-wT(w))$. The rest of the proof is similar to the proof of (1). □

Proposition 2.2. *Suppose that $w \in W$ and $w(R_1) = R_1$. Then*

$$(\rho', \sum_{\alpha \in Q(w)} \alpha) \geq 0,$$

with equality if and only if $w(B_1) = B_1$.

We introduce some notation needed for the proof of this result. If Q is a finite subset of $\mathfrak{h}^*$ we set $\langle Q \rangle = \sum_{\alpha \in Q} \alpha$. Let

$$W_0 = \{w \in W | w(R_1) = R_1\}.$$

Lemma 2.3. *For $w \in W_0$ the following are equivalent,*

(1) *$w(B_1) = B_1$,*
(2) *$w(B_1) \subseteq R_1^+$,*
(3) *$\ell(ws_\alpha) > \ell(w)$ for all $\alpha \in B_1$.*

Proof. Clearly (1) implies (2). For the reverse implication, suppose $\beta \in R_1^+$ and write $\beta = \sum_{\alpha \in B_1} c_\alpha \alpha$. Define $ht(\beta) = \sum_\alpha c_\alpha$. From (2) it follows that $ht(w(\beta)) \geq ht(\beta)$ and $w(R_1^+) = R_1^+$. This implies that $w^{-1}\gamma \in B_1$ for $\gamma \in B_1$. Hence (1) holds. The equivalence of (2) and (3) follows from [Hu90] Lemma 1.6 a), b). □

Corollary 2.4. *Set $T = \{w \in W | w(B_1) = B_1\}$. Then,*

(1) *$W_0 = W_1 T$ is the semidirect product of the normal subgroup W_1 by T.*
(2) *For $w \in W_1$ and $t \in T$ we have $\ell(wt) = \ell(w) + \ell(t)$.*

Proof. (1) Since W_1 is generated by reflections, s_α with $\alpha \in B_1$ and $ts_\alpha t^{-1} = s_{t(\alpha)}$ for $t \in T$, it follows that T normalizes W_1. From the implication (1) $\Longrightarrow$ (3) in Lemma 2.3 and [Hu90] Proposition 1.10 (c) we see that T is a transversal to W_1 in W_0. It follows easily that $W_0 = W_1 T$ is the semidirect product of W_1 by T.
(2) This holds by [Hu90] Proposition 1.10 (c). □

Recall that if $w \in W$ and $\alpha \in B$ such that $\ell(ws_\alpha) = \ell(w) + 1$ we have $Q(ws_\alpha) = s_\alpha Q(w) \cup \{\alpha\}$ ([GW98] Corollary 7.3.4). If $w_1, w_2 \in W$ and $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, it follows by induction on $\ell(w_2)$ that $Q(w_1 w_2) = w_2^{-1} Q(w_1) \cup Q(w_2)$, a disjoint union. Now for $t \in T$ and $w \in W_1$ we have $\ell(wt) = \ell(w) + \ell(t)$ by Corollary 2.4. Since $t\rho' = \rho'$, it follows that

$$(\rho', \langle Q(wt) \rangle) = (\rho', \langle Q(w) \rangle) + (\rho', \langle Q(t) \rangle).$$

Hence Proposition 2.2 follows from the following result.

Let v be the longest element of W_1. Then, $v(R_1^+) = -R_1^+$ and since $\ell(v) = |R_1^+|$, we have $v(R^+ \backslash R_1^+) = R^+ \backslash R_1^+$. Note that $v^2 = 1$, and if $t \in T$, then since $t^{-1}vt \in W_1$ and $t^{-1}vt(R_1^+) = -R_1^+$ we have $tv = vt$. Suppose $t \in T$, and set

$$Q^+ = \{\alpha \in Q(t) | (\rho', \alpha) > 0\},$$

$$Q^- = \{\alpha \in Q(t) | (\rho', \alpha) < 0\}.$$

Lemma 2.5.

(1) *If $w \in W_1$ then $Q(w) = T(w)$. Hence, if $w \neq 1$, then $(\rho', \langle Q(w) \rangle) > 0$.*
(2) *The map $\kappa : \alpha \mapsto v\alpha$ is a permutation of $Q(t)$, which interchanges Q^+ and Q^-.*
(3) *If $t \in T$, then $(\rho', \langle Q(t) \rangle) = 0$.*

Proof. (1) Obviously $T(w) \subseteq Q(w)$, and it follows from [Hu90] Corollary 1.7 and Proposition 1.10 (b) that $\ell(w) = |T(w)|$, giving equality. The second statement follows from ([GW98] Lemma 2.5.12).
(2) First observe that if $\alpha \in Q(t)$ then $\alpha \notin R_1^+$, since $t(R_1^+) = R_1^+$; hence $t\alpha = -\beta$ with $\beta \in R^+ \backslash R_1^+$. We must show that $v\alpha$ is in R^+ and $tv\alpha \in -R^+$. The first statement follows since $v(R^+ \backslash R_1^+) = R^+ \backslash R_1^+$, and the second follows from $tv\alpha = vt\alpha = -v\beta \in -R^+$. We have shown that κ is a permutation of $Q(t)$. Finally, for $\alpha \in Q(t)$,

$$(\rho', v\alpha) = (v\rho', \alpha) = -(\rho', \alpha).$$

The result follows easily from this fact.
(3) follows immediately from (2). □

Next we relate the dot action of W on $\mathcal{S}$ to the usual action.

Lemma 2.6. *If $h \in \mathfrak{h}$ and $w \in W$ then,*

$$w.h = wh - \sum_{\alpha \in Q(w)} h(\alpha).$$

Proof. For $\lambda \in \mathfrak{h}^*$ we have,

$$\begin{aligned} (w.h)(\lambda) &= h(w^{-1}.\lambda) \\ &= h(w^{-1}(\lambda + \rho) - \rho) \\ &= (wh)(\lambda) + h(w^{-1}\rho - \rho). \end{aligned}$$

We now apply equation (1). □

3. FCR factors of Enveloping algebras.

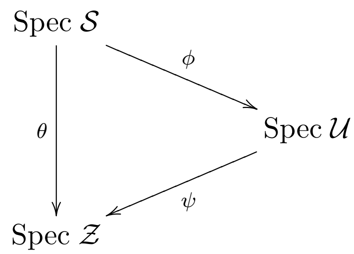
We recall Soergel's work on prime ideals in the enveloping algebra $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ where $\mathfrak{g}$ is a reductive Lie algebra. Let $\mathcal{Z}$ denote the center of $\mathcal{U}$.

If R is a ring we let $\mathrm{Spec}(R)$ denote the prime spectrum of R. For $\Omega \in \mathrm{Spec}\, \mathcal{S}$, define an ideal I_Ω in $\mathrm{Spec}\, \mathcal{U}$ as follows: Let $\mathbb{F} := Quot(\mathcal{S}/\Omega)$ (the quotient field of the commutative domain defined by Ω). We will let $\mathfrak{g}_{\mathbb{F}} = \mathbb{F} \otimes_{\mathrm{F}} \mathfrak{g}$ denote the Lie algebra obtained from $\mathfrak{g}$ by extension of scalars. We write λ_Ω for the $\mathbb{F}$-linear map from $\mathfrak{h}_{\mathbb{F}}$ to $\mathbb{F}$ whose restriction to $\mathfrak{h}$ is:

$$\lambda_\Omega : \mathfrak{h} \hookrightarrow \mathcal{S} \twoheadrightarrow \mathcal{S}/\Omega \hookrightarrow \mathbb{F}.$$

Let $L(\lambda_\Omega)$ be the simple highest weight module for $\mathfrak{g}_{\mathbb{F}}$ with highest weight λ_Ω. Let I_Ω be the annihilator of $L(\lambda_\Omega)$ in $\mathcal{U}(\mathfrak{g})$ and set $\phi(\Omega) = I_\Omega$. The maximal ideals in $\mathcal{S}$ have the form, $M_\mu := \{f \in \mathcal{S} | f(\mu) = 0\}$ for $\mu \in \mathfrak{h}^*$, and we write I_μ instead of I_{M_μ}, so that I_μ is the annihilator of the simple module $L(\mu)$ with highest weight μ. We write $\mathfrak{h}^*_{\mathbb{F}}$ for $\mathfrak{h}^* \otimes \mathbb{F} = (\mathfrak{h}_{\mathbb{F}})^*$.

For $P \in \operatorname{Spec} \mathcal{U}$, let $\psi(P) = P \cap Z$ ($\in \operatorname{Spec} \mathcal{Z}$). Lastly, θ denotes the map induced by the Harish-Chandra isomorphism $\mathcal{Z} \xrightarrow{\sim} \mathcal{S}^W \subseteq \mathcal{S}$. Then (by [Soe90], Section 2.1) we have the following commutative diagram:



By [Soe90] Section 2.2, these maps are continuous in the Jacobson topology on prime ideals. In particular these maps preserve inclusions.

Remark 3.1. When $\lambda = \lambda_\Omega$ we have:

$$R_\lambda = \{\alpha \in R | H_\alpha - n \in \Omega \text{ for some } n \in \mathbb{Z}\}.$$

In this situation we define $R^+_\lambda = \{\alpha \in R_\lambda | H_\alpha - n \text{ for some } n \in \mathbb{N}\}$. In [P], the ideal Ω is called *dominant* if $R^+_\lambda \subseteq R^+$.

Lemma 3.2. *If Ω is strongly dominant then Ω is dominant.*

Proof. If Ω is not dominant, then

$$H_\alpha + (\rho, \alpha^\vee) - m \in \Omega,$$

for some $\alpha \notin R^+$ and $m \in \mathbb{N}$. If $\beta = -\alpha$ and $n = -m$, then for $\mu \in \mathcal{V}(\Omega)$ we have

$$(\mu, \beta^\vee) = n - (\rho, \beta^\vee) < 0,$$

so $\mathcal{V}(\Omega) \cap P^+ = \emptyset$, and Ω is not strongly dominant. □

Note that the sets R_λ, W_λ, B_λ depend only on the coset $\Lambda := \lambda + P(R) \in \mathfrak{h}^*_{\mathbb{F}}/P(R)$. For a coset Λ, let $B_\Lambda = B_\lambda$ for any $\lambda \in \Lambda$.

Set

$$\Lambda^+ = \left\{\lambda \in \Lambda | (\lambda + \rho, \alpha^\vee) \geq 0, \ \forall \alpha \in B_\Lambda \right\}, \text{and}$$
$$\Lambda^{++} = \left\{\lambda \in \Lambda | (\lambda + \rho, \alpha^\vee) > 0, \ \forall \alpha \in B_\Lambda \right\}.$$

Lemma 3.3 ([Jan83] Satz 5.16)**.** *If $w \in W^\lambda$ then for all $\mu \in \lambda + P(R)$ we have $I_\mu = I_{w.\mu}$*

We recall the definition of the τ–invariant for primitive ideals. For $w \in W$, set

$$\tau_\Lambda(w) = \left\{\alpha \in B_\Lambda | w.\alpha \in -R^+ \right\}.$$

Next define $X_\lambda = \{I_{w.\lambda} | w \in W\}$, and write 2^{B_Λ} for the poset of subsets of B_Λ ordered by inclusion. We have:

Theorem 3.4 ([Jan83] Satz 5.7)**.** *Let $\Lambda \in \mathfrak{h}^*/P(R)$ and $\lambda \in \Lambda^{++}$. Then there is a well defined order reversing map:*

$$\tau_\Lambda : X_\lambda \to 2^{B_\Lambda}$$

such that $\tau_\Lambda(I_{w.\lambda}) = \tau_\Lambda(w)$.

Lemma 3.5. *Let D be a dense subset of an affine algebraic set X. Let:*

$$X = X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_r$$

be the decomposition of X into irreducible components. Then $D \cap X_i$ is dense in X_i for each i.

Proof. Consider i such that $1 \leq i \leq r$. There exists a function $p_i \in \mathcal{O}(X)$ which vanishes on X_j for all $j \neq i$ and p_i is not identically zero on X_i. Let $f \in \mathcal{O}(X_i)$ be a function that vanishes on $D \cap X_i$. We will show that f is identically zero on X_i. We may extend f to a function $g \in \mathcal{O}(X)$. The function gp_i vanishes on D and so is identically zero on X. On X_i, $gp_i = fp_i$. This means that f vanishes on the set $D' = \{x \in X_i | p_i(x) \neq 0\}$, which is dense in X_i. Therefore, f vanishes on X_i. □

4. Proof of the main result.

Until further notice $\mathfrak{g}$ will be a reductive Lie algebra over F. In general, it is unknown when a prime ideal I_Ω is contained in a given primitive ideal I_μ. However for $\mu \in P^+$ the answer is easy.

We recall the definition of Joseph's characteristic variety [Jo77]. Let $p : \mathcal{U} \to \mathcal{U}(\mathfrak{h})$ be the projection defined by the decomposition

$$\mathcal{U} = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^- \mathcal{U} + \mathcal{U}\mathfrak{n}^+).$$

For I an ideal of $\mathcal{U}$, set $\mathbb{V}(I) = \mathcal{V}(p(I))$.

Lemma 4.1. *Suppose I, J are ideals of $\mathcal{U}$ and Ω is a prime ideal of $\mathcal{S}$.*

1. *If $I \subseteq J$ then $\mathbb{V}(J) \subseteq \mathbb{V}(I)$.*
2. *If $\nu \in \mathbb{V}(I_\Omega)$ then $I_\Omega \subseteq I_\nu$.*
3. *$\mathcal{V}(\Omega) \subseteq \mathbb{V}(I_\Omega)$. In particular, $\mu \in \mathbb{V}(I_\mu)$.*
4. *$\mathbb{V}(I_\Omega) \subseteq W.\mathcal{V}(\Omega)$.*

Proof. The proofs are easily adapted from [Jo77] (Lemma on page 103). □

Theorem 4.2. *For $\mu \in P^+$, $I_\Omega \subseteq I_\mu$ if and only if $W.\mu \cap \mathcal{V}(\Omega) \neq \emptyset$.*

Proof. Using the Lemma one can justify the steps below.

(⇒): If $I_\Omega \subseteq I_\mu$, then $\mu \in \mathbb{V}(I_\mu) \subseteq \mathbb{V}(I_\Omega) \subseteq W.\mathcal{V}(\Omega)$. Therefore, $W.\mu \cap \mathcal{V}(\Omega) \neq \emptyset$.

(⇐): If $w.\mu \in \mathcal{V}(\Omega)$ for some $w \in W$, then $w.\mu \in \mathbb{V}(I_\Omega)$ since $\mathcal{V}(\Omega) \subseteq \mathbb{V}(I_\Omega)$. Therefore, since $\mu \in P^+$, $I_\Omega \subseteq I_{w.\mu} \subseteq I_\mu$.

□

Definition 4.3. Set $\Lambda(\Omega) := \{\mu \in P^+ | I_\Omega \subseteq I_\mu\}$, and for each $w \in W$, set

$$\Lambda(\Omega, w) := \{\mu \in P^+ | w.\mu \in \mathcal{V}(\Omega)\}.$$

Note that, $I_\Omega \subseteq \bigcap_{\mu \in \Lambda(\Omega)} I_\mu$ and equality holds if and only if U_Ω is FCR. Also, by Theorem 4.2 we have:

$$\Lambda(\Omega) = \bigcup_{w \in W} \Lambda(\Omega, w).$$

Proposition 4.4. *If Ω is strongly dominant then U_Ω has enough finite dimensional modules.*

Proof. If $P^+ \cap \mathcal{V}(\Omega)$ is dense in $\mathcal{V}(\Omega)$ then $I_\Omega = \bigcap_{\mu \in P^+ \cap \mathcal{V}(\Omega)} I_\mu$ by [Soe90] Proposition 1. Thus by the remark after Definition 4.3, it suffices to show that $P^+ \cap \mathcal{V}(\Omega) \subseteq \Lambda(\Omega)$ because this will imply $\bigcap_{\mu \in \Lambda(\Omega)} I_\mu \subseteq \bigcap_{\mu \in P^+ \cap \mathcal{V}(\Omega)} I_\mu$. The former inclusion follows since if $\mu \in P^+ \cap \mathcal{V}(\Omega)$ then $\Omega \subseteq M_\mu$, so $I_\Omega \subseteq I_\mu$, because ϕ preserves inclusions. □

Lemma 4.5. *If Ω is a strongly dominant ideal in $\mathcal{S}$, and $w \in W^\lambda$ where $\lambda = \lambda_\Omega$ then $I_{w.\Omega} = I_\Omega$.*

Proof. If $w \in W^\lambda$, then $I_\lambda = I_{w.\lambda}$ by Lemma 3.3. Thus, $I_\Omega = I_{w.\Omega}$ by [Soe90] Theorem 1 part (ii). □

For the remainder of this section we assume that $\mathfrak{g}$ is semisimple.

Proposition 4.6. *If Ω is strongly dominant and $\lambda = \lambda_\Omega$, then $B_\lambda \subset B$.*

Proof. Let $B = \{\alpha_1, \cdots, \alpha_n\}$. It suffices to show that if $\alpha \in R_\lambda \cap R^+$ and $\alpha = \sum_{i=1}^n c_i\alpha_i$ ($c_i \in \mathbb{Z}, c_i \geq 0$) then $c_i > 0$ implies $\alpha_i \in R_\lambda$. Now $\alpha \in R_\lambda$ means $H_\alpha - m' \in \Omega$ for some $m' \in \mathbb{Z}$. If $\mu \in \mathcal{V}(\Omega) \cap P^+$ we have $\mu_j = \mu(H_j) \in \mathbb{N}$ for $1 \leq j \leq n$. Also, $h_\alpha = \sum c_j h_j$. We can write $H_\alpha = \sum \frac{r_j}{s} H_j$ where $s, r_j \in \mathbb{N}$, $s > 0$ are such that $\frac{r_j}{s} = c_j \frac{(\alpha_j,\alpha_j)}{(\alpha,\alpha)}$. Then, $m' = \mu(H_\alpha) = \sum \frac{r_j}{s}\mu_j$. Hence, if $c_i > 0$, then $r_i > 0$ and $0 \leq \mu_i \leq \frac{m's}{r_i} = m$.

Suppose $c_i > 0$, and for $0 \leq p \leq m$, let $\Pi_{i,p} := \mathcal{V}(H_i - p)$. The above establishes that $\mathcal{V}(\Omega) \cap P^+ \subseteq \cup_{p=0}^m \Pi_{i,p}$. Since Ω is strongly dominant, $\mathcal{V}(\Omega) \subseteq \bigcup_{p=0}^m \Pi_{i,p}$, but $\mathcal{V}(\Omega)$ is irreducible so $\mathcal{V}(\Omega) \subseteq \Pi_{i,p}$ for some p. This means $H_i - p \in \Omega$, so $\alpha_i \in R_\lambda$. □

To prove an important special case of the main theorem. Before proceeding however, observe that for any ideal $\Omega \subseteq \mathcal{S}$ and $w \in W$ we have:

$$\mathcal{V}(w\Omega) = w\mathcal{V}(\Omega), \qquad \text{and} \qquad \mathcal{V}(w.\Omega) = w.\mathcal{V}(\Omega).$$

Furthermore, if Ω' is an ideal such that $\mathcal{V}(\Omega') = \mathcal{V}(\Omega) + \rho$, then for any $w \in W$, $\mathcal{V}(w\Omega') = w.\mathcal{V}(\Omega) + \rho$. In addition $H_\alpha - n \in \Omega$ implies $H_\alpha - n - (\rho, \alpha^\vee) \in \Omega'$.

Theorem 4.7. *Suppose Ω and $w.\Omega$ are prime ideals of $\mathcal{S}$ such that $P^+ \cap \mathcal{V}(\Omega)$ and $P^+ \cap \mathcal{V}(w.\Omega)$ are nonempty. Then $w \in W^\lambda$, where $\lambda := \lambda_\Omega$ is the tautological highest weight corresponding to Ω.*

Proof. Suppose that $\mu \in P^+ \cap \mathcal{V}(\Omega)$ and $\nu \in P^+ \cap \mathcal{V}(w.\Omega)$. Assume to the contrary that $w \in W$ is such that there exists $\alpha \in B_\Lambda$ such that $w\alpha = -\beta$ for $\beta \in R^+$. Since $\alpha \in B_\lambda$ we have $H_\alpha - n \in \Omega$ for some $n \in \mathbb{Z}$. Because $\mu \in P^+ \cap \mathcal{V}(\Omega)$, it follows that $\mu(H_\alpha - n) = 0$, and $\mu(H_\alpha) \in \mathbb{N}$. This implies that $n \in \mathbb{N}$. Similarly $H_\beta - m \in w.\Omega$ for some $m \in \mathbb{N}$.

Let Ω' be the ideal in $\mathcal{S}$ defined by the closed algebraic set $\mathcal{V}(\Omega) + \rho$. Then $H_\alpha - n - (\rho, \alpha^\vee) \in \Omega'$, and so $-w(H_\alpha - n - (\rho, \alpha^\vee)) = H_\beta + n + (\rho, \alpha^\vee) \in w\Omega'$. Since $\nu \in \mathcal{V}(w.\Omega) = w.\mathcal{V}(\Omega) = \mathcal{V}(w\Omega') - \rho$, we have, $\nu + \rho \in \mathcal{V}(w\Omega')$. Therefore,

$$(\nu + \rho)(H_\beta + n + (\rho, \alpha^\vee)) = 0,$$

since an element of $w\Omega'$ evaluates as 0 on $\mathcal{V}(w\Omega')$. This means that:

$$m + (\rho, \beta^\vee) + n + (\rho, \alpha^\vee) = 0.$$

But this is a contradiction because $(\rho, \gamma^\vee) > 0$ for all $\gamma \in R^+$. □

In the next two lemmas we assume that Ω is strongly dominant and $\lambda = \lambda_\Omega$. By Proposition 4.6, $B_\lambda \subseteq B$. Thus, we may apply the results of Section 2 with B_1 replaced by B_λ etc. Let I the subset of $\{1, 2, \cdots, \ell\}$ such that $B_\lambda = \{\alpha_i | i \in I\}$.

Lemma 4.8. *Assume that Ω is strongly dominant and $\lambda = \lambda_\Omega$. Then if $w \in W$, $w(R_\lambda) = R_\lambda$ and $w(B_\lambda) \nsubseteq R^+_\lambda$ then*

$$w.h_{\rho'} = wh_{\rho'} - c.$$

for some $c > 0$.

Proof. By Lemma 2.6,

$$w.h_{\rho'} = wh_{\rho'} - h_{\rho'}(\sum_{\alpha\in Q(w)} \alpha).$$

Since Ω is strongly dominant $B_\lambda \subseteq B$ by Proposition 4.6. Therefore by Proposition 2.2, $c := h_{\rho'}(\sum_{\alpha\in Q(w)} \alpha) > 0$. □

Lemma 4.9. *If $w \in W$ and $w.\Omega = \Omega$ then $w(B_\lambda) = B_\lambda$.*

Proof. Suppose that $w.\Omega = \Omega$. We proceed in two steps.

Step 1. $w(R_\lambda) = R_\lambda$: Indeed, if $\alpha \in R_\lambda$, then $H_\alpha - m \in \Omega$ for some $m \in \mathbb{Z}$. If $w\alpha = \beta$ then by Lemma 2.6, Ω contains

$$w.(H_\alpha - m) = H_\beta - \sum_{\gamma\in Q(w)} (\gamma, \alpha^\vee) - m,$$

and this implies that $\beta \in R_\lambda$.

Step 2: Since the α_i $(i \in I)$ belong to R_λ there are non-negative rational numbers a_i $(i \in I)$ such that:

$$h_i - a_i \in \Omega;$$

similarly, $h_{\rho'} - a \in \Omega$ for some non-negative $a \in \mathbb{Q}$. By Lemma 2.1 we have

$$w\rho' - \rho' = \sum_{\alpha\in T(w)} w\alpha = -\sum_{i\in I} b_i\alpha_i$$

for non-negative integers b_i. Hence,

$$wh_{\rho'} = h_{\rho'} - \sum_{i\in I} b_ih_i. \tag{2}$$

Assume for a contradiction that $w(B_\lambda) \not\subseteq B_\lambda$. Then by Lemma 2.3 $w(B_\lambda) \not\subseteq R_\lambda^+$, so by Lemma 4.8,

$$w.h_{\rho'} = wh_{\rho'} - c \tag{3}$$

for some $c > 0$. Since $w.\Omega = \Omega$, it follows from equations (2) and (3) that Ω contains,

$$\begin{aligned} w.(h_{\rho'} - a) &= wh_{\rho'} - a - c \\ &= h_{\rho'} - \sum_{i\in I} b_ih_i - a - c. \end{aligned}$$

However, $h_i - a_i \in \Omega$ for $i \in I$, so

$$h_{\rho'} - \sum_{i\in I} b_ia_i - a - c \in \Omega.$$

Since $h_{\rho'} - a \in \Omega$ we deduce that

$$\sum_{i\in I} b_ia_i + c \in \Omega.$$

This is a contradiction since $c > 0$ and $b_i, a_i \geq 0$ for all $i \in I$. □

Theorem 4.10. *If $U_{\Omega'}$ is FCR there exists a strongly dominant ideal Ω in $\mathcal{S}$ such that $\Omega' = w.\Omega$ for some $w \in W^\lambda$ (where $\lambda := \lambda_\Omega$ is the tautological highest weight of Ω).*

Proof. Let

$$X := \overline{\Lambda(\Omega')} = X_1 \cup \cdots \cup X_r.$$

define the decomposition of X into irreducible components. Set $Y_i := \Lambda(\Omega') \cap X_i$. Since $U_{\Omega'}$ is FCR, and $\Lambda(\Omega') = \bigcup Y_i$,

$$I_{\Omega'} = \bigcap_{\mu \in \Lambda(\Omega')} I_\mu = \bigcap_{i=1}^{r} \bigcap_{\mu \in Y_i} I_\mu.$$

Because $I_{\Omega'}$ is prime, we have that $I_{\Omega'} = \bigcap_{\mu \in Y_i} I_\mu$, for some i . Let Ω denote the ideal of elements vanishing on X_i. Note that Y_i is dense in X_i for each i since $\Lambda(\Omega')$ is dense in X (see Lemma 3.5). Therefore $I_{\Omega'} = I_\Omega$ by [Soe90] Proposition 1. This implies that $\theta(I_{\Omega'}) = \theta(I_\Omega)$, and therefore by [Soe90] Theorem 1 part (i) we have $\Omega' = w.\Omega$ for some $w \in W$.

Because $\Lambda(\Omega) \cap X_i$ is dense in X_i, and $\Lambda(\Omega) \subseteq P^+$, it follows that $P^+ \cap \Lambda(\Omega)$ is also dense in X_i. Therefore Ω is strongly dominant. Since $I_\Omega = I_{w.\Omega}$, the proof of [Soe90] Theorem 1 part (ii) shows that there exists $w_1 \in W$ such that $w \cdot \Omega = w_1.\Omega$ and $I_\lambda = I_{w_1 \cdot \lambda}$. It follows from Lemma 4.9 that $w_1^{-1} w(B_\lambda) = B_\lambda$. Hence to prove the result it is enough to show that $w_1 \in W^\lambda$.

Write $w_1 = uv$ with $u \in W^\lambda$, $v \in W_\lambda$. By Lemma 3.3 we have $I_{w_1.\lambda} = I_{v.\lambda}$. Hence $I_\lambda = I_{v.\lambda}$. By Theorem 3.4, $\tau_\Lambda(v) = \tau_\Lambda(1)$. This implies that $v = 1$, since $\tau_\Lambda(1) = \emptyset$ and if $v \neq 1$ and $v = us_\alpha$ is a reduced decomposition in W_λ, then $\alpha \in \tau_\Lambda(v)$. Therefore, $w_1 = u \in W^\lambda$. □

Lemma 4.11. *Suppose $\Omega, \Omega_1 \in \operatorname{Spec} \mathcal{S}$ and let λ, λ_1 respectively be the tautological highest weights.*

1. *If $\Omega_1 = v.\Omega$ with $v \in W^\lambda$ then $B_{\lambda_1} = vB_\lambda$.*
2. *If in addition, $u \in W^{\lambda_1}$ then $uv \in W^\lambda$.*

Proof. (1) From remark 3.1 we see that

$$v(R_\lambda) = \{\alpha \in R | v^{-1} H_\alpha - m \in \Omega \text{ for some } m \in \mathbb{Z}\}$$

and one can check that $v(R_\lambda) = R_{\lambda_1}$. Since $v \in W^\lambda$,

$$vB_\lambda \subseteq R^+ \cap R_{\lambda_1} = R^+_{\lambda_1}.$$

This fact implies that $vR^+_\lambda \subseteq R^+_{\lambda_1}$, and therefore $R_{\lambda_1} \cap \mathbb{N}\, vB_\lambda = R^+_{\lambda_1}$. Since there is a unique choice of simple roots in $R^+_{\lambda_1}$, the result follows.

(2) If $\alpha \in B_\lambda$, then $v\alpha \in B_{\lambda_1}$ by part (1). Hence $uv\alpha \in R^+$ for all such α, so $uv \in W^\lambda$. □

Proof of the Main Theorem. (1) this follows from Theorem 4.10.
For (2) assume Ω is strongly dominant, and $w \in W$.
(b) $\implies$ (a): If $I_{w.\Omega} = I_\Omega$, then $U_{w.\Omega}$ is FCR by Proposition 4.4.
(c) $\implies$ (b): This follows from Lemma 4.5.
(a) $\implies$ (c): Suppose that $\mathcal{U}_{w.\Omega}$ is FCR. By Theorem 4.10, there is a strongly dominant prime Ω_1, and $u \in W^{\lambda_1}$ such that $w.\Omega = u.\Omega_1$, where λ_1 is the tautological highest weight corresponding to Ω_1. If $v = u^{-1}w$, then Ω and $v.\Omega$ are strongly dominant, so by Theorem 4.7, $v \in W^\lambda$. By Lemma 4.11 we have $w = uv \in W^\lambda$. □

5. Invariant differential operators

An interesting class of examples arises from dual pairs ([GW98] Section 4.5, [How89], [LS89]). The application of our method is fairly routine, so we give only the most interesting examples rather than conduct an exhaustive study.

Let $M_{p,k}$ denote the set of $p \times k$ matrices over F, and consider cases A,B,C as follows.

Case	K	V	Action of K on V	$\mathfrak{g}$
A	$GL_k(F)$	$M_{p,k} \times M_{k,q}$	$(g,(a,b)) \mapsto (ag^{-1}, gb)$	$g\ell_{p+q}$
B	$O_k(F)$	$M_{k,n}$	$(g,a) \mapsto ga$	sp_{2n}
C	$Sp_{2k}(F)$	$M_{k,n}$	$(g,a) \mapsto ga$	so_{2n}

In case A we set $n = p + q$. If $p = 0$ (resp. $q = 0$) we set $V = M_{k,n}$ (resp. $V = M_{n,k}$). In all cases the algebra of invariant differential operators $\mathcal{D}(V)^K$ is generated (as an associative algebra) by a Lie algebra isomorphic to $\mathfrak{g}$. Hence there is a surjective homomorphism

$$\phi : U(\mathfrak{g}) \longrightarrow \mathcal{D}(V)^K. \tag{4}$$

The image of $\mathfrak{g}$ under ϕ is described explicitly on pages 69–70 of [LS89].

Let T be a maximal torus of K. In [MR], it is shown that if zero is not a T-weight of V then $\mathcal{D}(V)^K$ has enough finite dimensional representations. Thus if $\mathfrak{g}$ is semisimple, then $\mathcal{D}(V)^K$ is FCR since it is an image of $\mathcal{U}(\mathfrak{g})$. We remark that zero is not a weight of V in cases A and C of the above table and in case B for even k.

There is a multiplicity free decomposition of $\mathcal{O}(V)$ as a $F[K] \otimes \mathcal{U}(\mathfrak{g})$-module ([GW98] Theorem 4.5.14). Furthermore, as a $\mathcal{U}(\mathfrak{g})$–module, $\mathcal{O}(V)$ is a direct sum of simple highest weight modules. Let

$$\Lambda = \{\lambda \in \mathfrak{h}^* | \mathcal{O}(V) \text{ has a } \mathfrak{g}\text{-submodule isomorphic to } L(\lambda)\}.$$

If Ω is a radical ideal of $\mathcal{S}$, and $\Omega_1, \cdots, \Omega_t$ are the prime ideals of $\mathcal{S}$ that are minimal over Ω, we set $I_\Omega = \cap_{i=1}^t I_{\Omega_i}$.

Lemma 5.1. *If Ω is the radical ideal of $\mathcal{S}$ such that $\mathcal{V}(\Omega) = \overline{\Lambda}$, the Zariski closure of Λ, then*

$$\ker \phi = I_\Omega. \tag{5}$$

Proof. Suppose that $\overline{\Lambda} = X_1 \cup \cdots \cup X_t$ is the decomposition of $\overline{\Lambda}$ into irreducible components, and for $1 \le i \le t$, let Ω_i be the prime ideals of $\mathcal{S}$ such that $\mathcal{V}(\Omega_i) = X_i$. By Lemma 3.5, $\Lambda_i = \Lambda \cap X_i$ is dense in X_i, and so by ([Soe90] Proposition 1) we have:

$$I_{\Omega_i} = \bigcap_{\lambda \in \Lambda_i} ann_{\mathcal{U}} L(\lambda)$$

for $1 \le i \le t$. Therefore, since $\mathcal{O}(V)$ is a faithful $\mathcal{D}(V)^K$-module and $\Lambda = \bigcup_{i=1}^t \Lambda_i$ we have:

$$\begin{aligned} \ker \phi &= \{u \in \mathcal{U} | \phi(u)\mathcal{O}(V) = 0\} \\ &= \bigcap_{\lambda \in \Lambda} ann_{\mathcal{U}} L(\lambda) \\ &= \bigcap_{i=1}^t \bigcap_{\lambda \in \Lambda_i} ann_{\mathcal{U}} L(\lambda) \\ &= I_\Omega. \end{aligned}$$

□

Note that I_Ω is a completely prime ideal. We identify a situation where it is zero.

Theorem 5.2. *Assume that rank $\mathfrak{g} \le$ rank K then $\mathcal{U} \cong \mathcal{D}(V)^K$.*

Proof. Let $\{\mathcal{U}_N\}$ be the usual filtration on $\mathcal{U}$ and $\{D_N\}$ the Bernstein filtration on $\mathcal{D}(V)$. Then K preserves each D_N, so acts on $gr\mathcal{D}(V)$ and $gr\left(\mathcal{D}(V)^K\right) = (gr\mathcal{D}(V))^K$. Now $\mathfrak{g}$ maps onto D_2^K and this induces a surjection $\mathcal{U}_N \to D_{2N}^K$ with kernel $I_\Omega \cap \mathcal{U}_N$. Passing to the associated graded rings we obtain a surjection

$$\mathcal{S}(\mathfrak{g}) = gr\mathcal{U} \to gr\mathcal{D}(V)^K = \mathcal{S}(V \oplus V^*)^K$$

with kernel grI_Ω. Then if rank $K \geq$ rank $\mathfrak{g}$ then we can apply the Second Fundamental Theorem of Invariant Theory in the free case ([GW98] Corollary 4.2.5) to $V \oplus V^*$. We conclude that $grI_\Omega = (0)$ so $I_\Omega = (0)$. □

Our next aim is to describe the irreducible constituents of $\mathcal{O}(V)$ as a K-module and as a $\mathcal{U}(\mathfrak{g})$-module. With this goal in mind we set up some standard notation.

By a partition we will mean a finite sequence of weakly decreasing non-negative integers. We will denote partitions by lower case Greek letters. The *length* of a partition is denoted $\ell(\alpha) = \max\{i | \alpha_i > 0\}$. The *conjugate* of a partition λ is a partition λ' whose i^{th} part is given by $|\{j | \lambda_j \geq i\}|$. Note that $\ell(\lambda) = \lambda'_1$.

The highest weights of representations of $\mathfrak{g}$ (resp. K) are given by m-tuples where m is the rank of $\mathfrak{g}$ (resp. K). The m-tuple $(a_1, \cdots, a_m)$ corresponds to the weight $\sum_{i=1}^m a_i \epsilon_i$ where the ϵ_i are defined in Section 2.3.1 of [GW98].

At this point we consider the cases separately.

6. Case A

As a basis for $\mathfrak{h}$ we take $\{E_1, \cdots, E_n\}$ where E_i is the $n \times n$ matrix with a 1 in the row i, column i entry and zeros elsewhere.

Given non-negative integer partitions α and β with $\ell(\alpha) \leq p$, $\ell(\beta) \leq q$, and $\ell(\alpha) + \ell(\beta) \leq k$, let $V_{(\alpha,\beta)}$ denote the irreducible $\mathfrak{gl}_n(F)$ module with highest weight:

$$(\alpha, \beta)_{\mathfrak{g}} \quad := \quad \underbrace{(-k - \alpha_p, -k - \alpha_{p-1}, \cdots, -k - \alpha_1, \beta_1, \beta_2, \cdots, \beta_q)}_{n}.$$

Let $V^{(\alpha,\beta)}$ denote the irreducible representation of GL_k with highest weight:

$$(\alpha, \beta)_K \quad := \quad \underbrace{(\alpha_1, \alpha_2, \cdots, \alpha_p, 0, \cdots, 0, -\beta_q, \cdots, -\beta_2, -\beta_1)}_{k}.$$

Lemma 6.1. *There is a multiplicity free decomposition under the joint action of K and $\mathfrak{g}$, and we have the multiplicity free decomposition:*

$$\mathcal{O}(V) = \bigoplus V^{(\alpha,\beta)} \otimes V_{(\alpha,\beta)},$$

where the direct sum is over the set

$$W_k(p,q) := \left\{ (\alpha,\beta) \;\middle|\; \begin{array}{l} \alpha \text{ and } \beta \text{ partitions such that} \\ \ell(\alpha) \leq p,\, \ell(\beta) \leq q, \quad \text{and} \\ \ell(\alpha) + \ell(\beta) \leq k \end{array} \right\}$$

Proof. See [EW04] p. 356. □

Define $\Lambda_k(p,q) = \{(\alpha,\beta)_{\mathfrak{g}} | (\alpha,\beta) \in W_k(p,q)\}$, and let $Z_k(p,q) := \overline{\Lambda_k(p,q)}$ denote the Zariski closure of these weights.

If $k < n$ define:

$$\Omega_m := \begin{cases} (E_{k+1}, E_{k+2}, \cdots, E_n), & \text{if } m = 0; \\ (E_1 + k, \cdots, E_m + k, E_{m+k+1}, \cdots, E_n), & \text{if } 0 < m < n-k; \\ (E_1 + k, \cdots, E_{n-k} + k), & \text{if } m = n-k. \end{cases}$$

Lemma 6.2. *The decomposition of $Z_k(p,q)$ into irreducible components is as follows:*

(1) $Z_k(p,q) = \mathfrak{h}^*$, *if $n \leq k$.*

(2) *If $k < n \leq 2k$, then*

$$Z_k(p,q) = \bigcup_{m \in \Phi_p} \mathcal{V}(\Omega_m)$$

where:

$$\Phi_p = \begin{cases} \{0, \cdots, p\}, & \text{if } p \leq n-k; \\ \{0, \cdots, n-k\}, & \text{if } n-k \leq p \leq k; \\ \{p-k, \cdots, n-k\}, & \text{if } k \leq p \leq n. \end{cases}$$

(3) *If $2k \leq n$ then*

$$Z_k(p,q) = \bigcup_{m \in \Phi_p} \mathcal{V}(\Omega_m)$$

where:

$$\Phi_p = \begin{cases} \{0, \cdots, p\}, & \text{if } p \leq k; \\ \{p-k, \cdots, p\}, & \text{if } k \leq p \leq n-k; \\ \{p-k, \cdots, n-k\}, & \text{if } p \geq n-k. \end{cases}$$

Proof. Statement (1) follows because if $n \leq k$, then $\mathcal{O}(V)$ contains the irreducible $\mathfrak{gl}_n(F)$ module with highest weight $(\alpha,\beta)_{\mathfrak{g}}$ for any partitions α, β with $\ell(\alpha) \leq p$, and $\ell(\beta) \leq q$. For statements (2) and (3) we define, for $0 \leq i \leq p$

$$W_k^i(p,q) = \{(\alpha,\beta) \in W_k(p,q) | \ell(\alpha) = p-i\},$$

and

$$\Lambda_k^i(p,q) = \{(\alpha,\beta)_{\mathfrak{g}} | (\alpha,\beta) \in W_k(p,q)\}.$$

Suppose $(\alpha,\beta) \in W_k^i(p,q)$. Then, $0 \leq i \leq p$ since $0 \leq \ell(\alpha) \leq p$. Also, $\ell(\alpha) \leq k$ implies that $p-k \leq i$. In addition, the set of weights $(\alpha,\beta)_{\mathfrak{g}}$ such that $\ell(\alpha) = p-i$ and $\ell(\beta) < k-\ell(\alpha) = k+i-p$ is contained in the Zariski closure of the set of weights $(\alpha,\beta)_{\mathfrak{g}}$ with $\ell(\beta) = k+i-p$. Since we require that $\ell(\beta) \leq q$, we can restrict our attention to the highest weights $(\alpha,\beta)_{\mathfrak{g}}$ such that $\ell(\alpha) = p-i$ and $k+i-p \leq q$. That is, $i \leq n-k$. It follows that $Z_k(p,q) = \bigcup \overline{\Lambda_k^i(p,q)}$ where the union is over all i such that $\max(0, p-k) \leq i \leq \min(p, n-k)$. It is easy to check that for these values of i, $\overline{\Lambda_k^i(p,q)} = \mathcal{V}(\Omega_i)$. The result follows by considering individual cases. □

Proposition 6.3. *If n and k are fixed, then for $0 \leq i < j \leq n-k$ we have $I_{\Omega_i} = I_{\Omega_j}$. Moreover, Ω_0 and Ω_{n-k} are strongly dominant.*

Proof. To see that Ω_0 is strongly dominant note that

$$\sum_{i=1}^{k} \mathbb{N}\overline{\omega}_i \subseteq \mathcal{V}(\Omega_0) = \sum_{i=1}^{k} \mathrm{F}\overline{\omega}_i.$$

Similarly Ω_{n-k} is strongly dominant. Now set $\Omega = \Omega_{n-k}$, so that:

$$\Omega = (E_1 + k, E_2 + k, \cdots, E_{n-k} + k),$$

and let $\lambda = \lambda_\Omega$. For $0 \leq i \leq n-k$ we define $w_i \in W = S_n$ by:

$$w_i(j) = \begin{cases} j, & \text{for } 1 \leq j \leq i; \\ j+k, & \text{for } i+1 \leq j \leq n-k; \\ j-n+k+i, & \text{for } n-k+1 \leq j \leq n. \end{cases}$$

Then, it is easy to see that $w_i.\Omega = \Omega_i$, and that $w_i \in W^\lambda$. The result follows from the Lemma 4.5. □

Remark 6.4. Explicit generators for I_{Ω_0} are given in [Pro04] using Capelli identities.

7. Cases B and C

We first turn our attention to Case B.

Lemma 7.1. *There is a multiplicity free decomposition under the joint action of K and $\mathfrak{g}$ given by*

$$\mathcal{O}(V) = \bigoplus_{\mu \in M} V^{\mu} \otimes V_{\mu},$$

where we index the summands by the partitions μ in the set:

$$M := \left\{\mu \text{ is a partition such that } \mu_1' + \mu_2' \leq k \text{ and } \ell := \mu_1' \leq n\right\}.$$

The highest weight of the sp_{2n}-module, V_μ is given by:

$$\underbrace{(-\frac{k}{2}, -\frac{k}{2}, \cdots, -\frac{k}{2} - \mu_\ell, \cdots, -\frac{k}{2} - \mu_1)}_{n}$$

Proof. See [EW04] p. 353. □

We describe the Zariski closure, Z in $\mathfrak{h}^*$ of the set of weights of $\mathcal{O}(V)$ as a $\mathfrak{g}$-module. If $k \geq 2n$ then $Z = \mathfrak{h}^*$, so we consider the case where $k < 2n$. For k odd, we can see using Theorem 4.2 that $\mathcal{D}(V)^K$ has no finite dimensional modules, so we consider the k even case.

Suppose that $k = 2p$. Fix i with $1 \leq i \leq \min(p, n-p)$ and set $r = n-p-i$, $s = n-p+i$. Then define

$$\Omega_i = (H_1, H_2, \cdots, H_{r-1}, H_r + 1, H_{r+1}, \cdots, H_{s-1}, H_s + H_{s+1} + \cdots H_n + p + 1).$$

We also set

$$\Omega_0 = (H_1, \cdots, H_{n-p-1}, H_{n-p} + H_{n-p+1} + \cdots + H_n + p).$$

Lemma 7.2. *The decomposition of Z into irreducible components is given by*

$$Z = \bigcup_{i=0}^{\min(p,n-p)} \mathcal{V}(\Omega_i).$$

Proof. For $0 \leq i \leq \min(p, n-p)$, set

$$M_i = \{\mu \text{ is a partition such that } \mu_1' + \mu_2' \leq k, \text{ and } \mu_1' = p + i\},$$

and let $\Lambda_i \subseteq \mathfrak{h}^*$ be the set of highest weights of the modules V_μ with $\mu \in M_i$. Note that if $\mu_1' + \mu_2' \leq k = 2p$ and $\mu_1' < p$ then the highest weight of V_μ is in $\overline{\Lambda}_0$.

If $0 \leq i < j \leq \min(p, n-p)$ neither of the ideals Ω_i, Ω_j is contained in the other. Therefore, it suffices to show that the Zariski closure of Λ_i is $\mathcal{V}(\Omega_i)$. Suppose that $\mu \in M_i$ and let $\alpha = \sum_{i=1}^{n} a_i \epsilon_i$ be the highest weight of V_μ. Then $a_i = -p$ for $1 \leq i \leq r = n-p-i$, and $a_i = -p-1$ for $r+1 \leq i \leq s = n-p+i$. The result follows easily. □

Note that GK-dim $\mathcal{S}/\Omega_i = p - i$, where GK-dim is the Gelfand-Kirillov dimension. Let $I_i = I_{\Omega_i}$ for $0 \leq i \leq p$.

Recall the homomorphism ϕ from equation (4).

Lemma 7.3. *We have* $\ker \phi = I_0$.

Proof. By Lemma 5.1, $\ker \phi = \cap_{i=0}^{p} I_i$. Since $\ker \phi$ is prime, it follows that $I_i \subseteq \ker \phi \subseteq I_0$ for some i. Hence $I_i \cap \mathcal{Z} \subseteq I_0 \cap \mathcal{Z}$, that is $\Omega_i \cap \mathcal{Z} \subseteq \Omega_0 \cap \mathcal{Z}$. If $i \geq 1$ then since $\mathcal{S}$ is a finitely generated $\mathcal{Z}$-module we would have:

$$\begin{aligned} p - i &= \text{GK-dim}(S/\Omega_i) = \text{GK-dim}(Z/Z \cap \Omega_i) \\ &\geq \text{GK-dim}(Z/Z \cap \Omega_0) = p \end{aligned}$$

a contradiction. □

We show next that $\mathcal{U}/I_0$ is FCR. To do this define

$$\Omega = (H_{p+1}, \cdots, H_n),$$

and define $w \in W$ so that if $a = (a_1, \cdots, a_n) \in \mathfrak{h}^*$, we have $w(a) = b$, where:

$$b_i = \begin{cases} a_{i+p}, & \text{for } 1 \leq i \leq n-p; \\ a_{i+p-n}, & \text{for } n-p+1 \leq i \leq n. \end{cases}$$

It is easy to check that Ω is strongly dominant, $w.\Omega = \Omega_0$ and $w \in W^\lambda$ where $\lambda = \lambda_\Omega$. Thus by the main theorem, $I_0 = I_\Omega$ and $\mathcal{U}_\Omega$ is FCR. We expect that

$$I_0 \subset I_1 \subset \cdots \subset I_p$$

with all inclusions strict. If $p = 1$ this follows from Lemmas 5.1 and 7.3.

Case C is relatively uninteresting, the Zariski closure $\mathcal{V}(\Omega)$ of the set of highest weight occurring in $\mathcal{O}(V)$ as a $\mathfrak{g}$-module is irreducible. Also $\mathcal{U}_\Omega \cong D(V)^K$ has enough finite dimensional modules from [MR] and thus is FCR. Furthermore, $\mathcal{V}(\Omega) = \mathfrak{h}^*$ if and only if $n \leq k$.

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Ian M. Musson, University of Wisconsin - Milwaukee, Department of Mathematical Sciences, P.O. Box 0413, Milwaukee WI 53201-0413

E-mail address: musson@uwm.edu

Jeb F. Willenbring, University of Wisconsin - Milwaukee, Department of Mathematical Sciences, P.O. Box 0413, Milwaukee WI 53201-0413

E-mail address: jw@uwm.edu