RECIPROCITY ALGEBRAS AND BRANCHING FOR CLASSICAL SYMMETRIC PAIRS

ROGER E. HOWE, ENG-CHYE TAN, AND JEB F. WILLENBRING

Abstract. We study branching laws for a classical group G and a symmetric subgroup H. Our approach is by introducing the *branching algebra*, the algebra of covariants for H in the regular functions on the natural torus bundle over the flag manifold for G. We give concrete descriptions of certain subalgebras of the branching algebra using classical invariant theory. In this context, it turns out that the ten classes of classical symmetric pairs (G, H) are associated in pairs, (G, H) and (H', G').

Our results may be regarded as a further development of classical invariant theory as described by Weyl [Wey97], and extended previously in [How89a]. They show that the framework of classical invariant theory is flexible enough to encompass a wide variety of calculations that have been carried out by other methods over a period of several decades. This framework is capable of further development, and in some ways can provide a more precise picture than has been developed in previous work.

1. Introduction

1.1. **The Classical Groups.** Hermann Weyl's book, *The Classical Groups* [Wey97], has influenced many researchers in invariant theory and related fields in the decades since it was written. Written as an updating of "classical" invariant theory, it has itself acquired the patina of a classic. The books [GW98] and [Pro07] and the references in them give an idea of the extent of the influence. The current authors freely confess to being among those on whom Weyl has had major impact.

The Classical Groups has two main themes: the invariant theory of the classical groups – the general linear groups, the orthogonal groups and the symplectic groups – acting on sums of copies of their standard representations (and, in the case of the general linear groups, copies of the dual representation also); and the description of the irreducible representations of these groups. In invariant theory, the primary results of [Wey97] are what Weyl called the First and Second Fundamental Theorems of invariant theory. The First Fundamental Theorem (FFT) describes a set of "typical basic generators" for the invariants of the selected actions, and the Second Fundamental Theorem (SFT) describes the relations between these generators. The description of the representations culminates in the Weyl Character Formula.

The two themes are not completely integrated. For the first one, Weyl uses the apparatus of classical invariant theory, including the Aronhold polarization operators and the Capelli identity, together with geometrical considerations about orbits, etc. For the second, he abandons polynomial rings and relies primarily on the Schur-Weyl duality, the remarkable connection discovered by I. Schur between representations of the general linear groups and

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the symmetric groups. This duality takes place on tensor powers of the fundamental representation of GL_n, which of course are finite dimensional. This gives Weyl's description of the irreducible representations more of a combinatorial cast. This combinatorial viewpoint, based around Ferrers-Young diagrams and Young tableaux, has been very heavily developed in the latter half of the twentieth century (see [Lit40], [Sun90], [Mac95], [Ful97], [Pro07] and the references below).

1.2. **From Invariants to Covariants.** In [How89a], it was observed that by combining the results in [Wey97] with another construction of Weyl, namely the *Weyl algebra*, aka the algebra of polynomial coefficient differential operators, it is possible to give a unified treatment of the invariants and the irreducible representations.

Introduction of the Weyl algebra brings several valuable pieces of structure into the picture. A key feature of the Weyl algebra $\mathcal{W}(V)$ associated to a vector space V is that it has a filtration such that the associated graded algebra is commutative, and is canonically isomorphic to the algebra $\mathcal{P}(V \oplus V^*)$ of polynomials on the sum of V with its dual V^*. Moreover, if one extends the natural bilinear pairing between V and V^* to a skew-symmetric bilinear (or *symplectic*) form on $V \oplus V^*$, then the symplectic group $Sp(V \oplus V^*)$ of isometries of this form acts naturally as automorphisms of $\mathcal{W}(V)$, and this action gets carried over to the natural action of $Sp(V \oplus V^*)$ on $\mathcal{P}(V \oplus V^*)$. Also, the natural action of $GL(V)$, the general linear group of V, on $\mathcal{P}(V \oplus V^*)$ embeds $GL(V)$ in $Sp(V \oplus V^*)$. The corresponding action of $GL(V)$ on $\mathcal{W}(V)$ is just the action by conjugation when both $GL(V)$ and $\mathcal{W}(V)$ are regarded as being operators on $\mathcal{P}(V)$. Finally, the Lie algebra $\mathfrak{sp}(V \oplus V^*)$ of $Sp(V \oplus V^*)$ is naturally embedded as a Lie subalgebra of $\mathcal{W}(V)$. The image of $\mathfrak{sp}(V \oplus V^*)$ in $\mathcal{P}(V \oplus V^*)$ consists of the homogeneous polynomials of degree two. The Lie bracket is then given by Poisson bracket with respect to the symplectic form [CdS01].

Given a group $G \subseteq GL(V)$, it is natural in this context to look at $\mathcal{W}(V)^G$, the algebra of polynomial coefficient differential operators invariant under the action of G, or equivalently, of operators that commute with G. One can show in a fairly general context that $\mathcal{W}(V)^G$ provides strong information about the decomposition of $\mathcal{P}(V)$ into irreducible representations for G [Goo04].

In the case of the classical actions considered by Weyl, it turns out that $\mathcal{W}(V)^G$ has an elegant structure. This structure is revealed by considering the centralizer of G inside $Sp(V \oplus V^*)$. Since $G \subseteq GL(V) \subset Sp(V \oplus V^*)$, we can consider $G' = Sp(V \oplus V^*)^G$, the centralizer of G in $Sp(V \oplus V^*)$. The Lie algebra of G' will be $\mathfrak{g}' = \mathfrak{sp}(V \oplus V^*)^G$, the Lie subalgebra of $\mathfrak{sp}(V \oplus V^*)$ consisting of elements that commute with G.

Given a group S and subgroup H, looking at H', the centralizer of H in S, is a construction that has the formal properties of a duality operation, analogous to considering the commutant of a subalgebra in an algebra. It is easy to check that $H'' = (H')'$ contains H, and that $H''' = (H')''$ is again equal to H'. Hence also $H'''' = H''$. Thus H'' constitutes a sort of "closure" of H with respect to the issue of commuting inside S, and the pair (H'', H') constitute a pair of mutually centralizing subgroups of S. In [How89a] such a pair was termed a *dual pair* of subgroups of S. Dual pairs of subgroups arise naturally in studying the structure of groups. For example, in reductive algebraic groups, the Levi component of a parabolic subgroup and its central torus constitute a dual pair. A subgroup $H \subseteq G$ belongs to a dual pair in G if and only if it is its own double centralizer.

It turns out that, if G is one of the classical groups acting on V by one of the actions specified by Weyl, and G' is its centralizer in $Sp(V \oplus V^*)$, then (G, G') constitute a dual pair in $Sp(V \oplus V^*)$. That is, $G = G''$. Moreover, by translating Weyl's FFT to the context of the Weyl algebra, one sees that $\mathfrak{g}' = \mathfrak{sp}(V \oplus V^*)^G$ generates $\mathcal{W}(V)^G$ as an associative algebra. Furthermore, the condition that $G = G''$ essentially characterizes the actions considered by Weyl, providing some insight into why the FFT has such a clean statement for these actions, but is known for hardly any other examples. The fact that $\mathfrak{g}'$ generates $\mathcal{W}(V)^G$, together with some simple structural facts about $\mathfrak{g}'$, allows a detailed description of the action of G on $\mathcal{P}(V)$. As part of this picture, one obtains a natural bijection between the irreducible representations of G appearing in $\mathcal{P}(V)$, and certain irreducible representations of $\mathfrak{g}'$.

In the case of when $G = GL_n$ and $V \simeq \mathbb{C}^n \otimes \mathbb{C}^m$, then $G' = GL_m$ acting on the factor $\mathbb{C}^m$ of V. The Lie algebra $\mathfrak{g}' \simeq \mathfrak{gl}_m$ is exactly the span of the Aronhold polarization operators. In this case, the polynomials on V decompose into jointly irreducible representations $\rho \otimes \rho'$ for $GL_n \times GL_m$, and the correspondence $\rho \leftrightarrow \rho'$ is bijective. We refer to this correspondence as (GL_n, GL_m) *duality*. The paper [How95] further studied this situation and the foundations of Weyl's Fundamental Theorems, and pointed out that these results could be understood from the point of view of multiplicity free actions. In this development, the FFT for GL_n, Schur-Weyl duality and (GL_n, GL_m) duality appear as three aspects of the same phenomenon. Each can be deduced from either of the others. In particular, the polynomial version of the theory and the combinatorial version are seen as two windows on the same landscape.

1.3. **Branching Rules via Invariant Theory.** Already when [Wey97] was being written, work was underway to extend representation theory beyond a basic enumeration of the irreducible representations to describe some aspects of their structure. Part of the motivation for doing this came from quantum mechanics, which also was the inspiration for the Weyl algebra. In [LR34], Littlewood and Richardson proposed a combinatorial description of the multiplicities of irreducible representations in the tensor product of two irreducible representations of GL_n. These multiplicities are now known as Littlewood-Richardson (LR) coefficients.

The decomposition of tensor products of representations of a group G can be regarded as a *branching problem* – the problem of decomposing the restriction of a representation of some group to a subgroup. Precisely, if ρ_1 and ρ_2 are irreducible representations of G, then the tensor product $\rho_1 \otimes \rho_2$ can be regarded as an irreducible representation of $G \times G$, and all irreducible representations of $G \times G$ are of this form [GW98, How95].

If we restrict this representation to the diagonal subgroup $\Delta G = \{(g, g) : g \in G\} \subset G \times G$, then we obtain the usual notion of $\rho_1 \otimes \rho_2$ as a representation of G. Note that ΔG is isomorphic to G. Also, the involution $(g_1, g_2) \leftrightarrow (g_2, g_1)$ of $G \times G$ has ΔG as the subgroup of fixed points. Thus, ΔG is a *symmetric subgroup* of $G \times G$ – the fixed point set of an involution (order two automorphism) of $G \times G$. We also call the pair $(G \times G, \Delta(G))$ a *symmetric pair.* In summary, computing tensor products for representations of GL_n can be viewed as studying the branching problem for the symmetric pair $(GL_n \times GL_n, \Delta(GL_n))$.

In the 1940s, Littlewood considered the branching problem from GL_n to the orthogonal group O_n. Note that O_n is the set of fixed points of the involution $g \to (g^{-1})^t$, where A^t denotes the transpose of the $n \times n$ matrix A. Thus, (GL_n, O_n) is a symmetric pair. Since all the representations involved are semisimple (thanks, e.g. to Weyl's *unitarian trick* [Wey97]), to determine them up to isomorphism, it is enough to know the *branching multiplicities* – the multiplicity with which each irreducible representation of O_n occurs in a given irreducible representation of GL_n. Under some restrictions on the representations involved, Littlewood [Lit40, Lit44] showed how to express the branching multiplicities for the pair (GL_n, O_n), in terms of LR coefficients. Over the decades since [Wey97], these results have been extended in stages, so that one now has a description of the branching multiplicities for any *classical symmetric pair* – a symmetric pair (G, K), where G is a classical group – in terms of LR coefficients [LR34], [Lit40], [Lit44], [New51], [Kin71], [Kin74a], [Kin74b], [Kin75], [BKW83], [KT87a], [KT87b], [KT87c], [KT90], [Kin90], [Sun90], [KW00a], [KW00b], [Kin01]. See also [Ful97] and [HTW05a].

The goal of this paper is to show how the invariant theory approach can be further developed to encompass much of the work on branching multiplicities cited above. The main ingredient needed for doing this is the notion of *branching algebra*, an idea used by Zhelobenko [Želobenko73], but relatively little exploited since. The general idea of branching algebra is described in §2.

The concrete description of branching algebras for the classical symmetric pairs in terms of polynomial rings is carried out in §4. More precisely, in §4, certain families of well behaved subalgebras of branching algebras are realized via polynomial rings. We refer to these subalgebras as *partial* branching algebras.

When one realizes partial branching algebras in the context of dual pairs, a lovely reciprocity phenomenon reveals itself. Suppose that a reductive group $G \subset GL(V) \subset Sp(V \oplus V^*)$ belongs to a dual pair (G, G'). Let $K \subset G$ be a symmetric subgroup. It turns out that, if $K' \subset Sp(V \oplus V^*)$ is the centralizer of K in $Sp(V \oplus V^*)$, then (K, K') also form a dual pair in $Sp(V \oplus V^*)$; and furthermore (K', G') is also a classical symmetric pair! (Note that, since $K \subset G$, we clearly will have $K' \supset G'$.) Thus, the dual pairs (G, G') and (K, K') form a *seesaw pair* in the sense of Kudla [Kud84].

Moreover, the partial branching algebra constructed for (G, K) turns out also to be a partial branching algebra for (K', G')! The coincidence implies a reciprocity phenomenon: branching multiplicities for (G, K) also describe branching multiplicities for (K', G'). For this reason, we call these partial branching algebras *reciprocity algebras*. (It should be noted that often some special infinite dimensional representations of K' may be involved in these reciprocity relationships, and sometimes infinite dimensional representations of G' are also involved.) We note that parts of this picture have appeared before. The fact that (K, K') is a dual pair, and that (K', G') is a symmetric pair is implicit in [How89b]. A numerical version of the reciprocity laws implied by reciprocity algebras was given in [How83], and the reciprocity phenomenon for the GL_n tensor product algebra was noted in [How95].

The classical symmetric pairs may be sorted into ten infinite families (see Table I in §4). It turns out that if the pair (G, K) is taken from one family, the pair (K', G') is always taken from another family that is determined by the family of (G, K). That is, the seesaw construction applied to classical symmetric pairs, pairs up the ten families into five reciprocal pairs of families. The two families in a pair have many reciprocity laws relating multiplicities of their representations.

Thus, the branching algebra approach to branching rules, when made concrete via classical invariant theory, has some highly attractive formal features. We should keep in mind that they are formal, in the sense that they don't say anything explicit about what certain branching multiplicities <u>are</u>. They just say that branching multiplicities for one pair (G, K)

can be expressed in terms of branching multiplicities for another pair (K', G'). To get more specific information, one needs some reasonably explicit description of the partial branching algebras. In this paper we give a relative description, which shows that every reciprocity algebra is related to the GL_n tensor product algebra. This is carried out in §8 for one of the symmetric pairs. The theory works best under some technical restrictions, referred to as the *stable range*. As explained in [HTW05a], these relations imply the many formulas in the literature that describe branching multiplicities for symmetric pairs in terms of LR coefficients.

To have a complete theory, one should also give concrete and explicit descriptions of the reciprocity algebras. This paper does not deal with this issue. However, it has been carried out, by the authors and others, in several papers. Using ideas of Grobner/SAGBI theory [RS90], [MS05], the paper [HTW05b] describes an explicit basis for the basic case, the GL_n tensor product algebra. Analogous bases for most of the other reciprocity algebras are given in [HL07, HL06a, HL06b]. Furthermore, the paper [HL] shows how to deduce the Littlewood-Richardson Rule from the results of [HTW05b] and representation-theoretic considerations. Also, the paper [HJL+] shows that these branching algebras have toric deformations. More precisely, it shows that they have flat deformations that are semigroup rings of lattice cones, which are explicitly described.

Thus, this paper supplemented by the work just cited shows that the invariant-theory approach using branching algebras provides a uniform framework to deal with a wide variety of issues in representation theory. These computations have been handled in the literature by largely means of combinatorics, as cited above, and more recently by quantum groups and related methods, including the path methods of Littelmann [Dri87], [Lus90], [Kas90], [Jim91], [Jos95], [Jan98], [Lit94], [Lit95a], [Lit95b], [Lit95c]. The branching algebra approach can provide another window on these phenomena.

The branching algebra approach comes with some extra structure attached. The multiplicities are seen not simply as numbers, but intrinsically as cardinalities of integral points in convex sets. (Indeed, one can see this implicitly in the Littlewood-Richardson Rule, and it was made more explicit in [BZ92], [BZ01] and [PV05]).

Moreover, these points do not simply give the correct count: individual points correspond to specific highest weight vectors in representations. Finally, the fact that all representations are bundled together inside one algebra structure implies relations between the highest weight vectors and the multiplicities of different representations. The authors suspect that this extra structure can be useful in studying certain problems, such as the actions analyzed by Kostant and Rallis [KR71], and perhaps in understanding the structure of principal series representations of semisimple groups. We hope to return to these themes in future papers.

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2. Branching Algebras

For a reductive complex linear algebraic G, let U_G be a maximal unipotent subgroup of G. The group U_G is determined up to conjugacy in G [Bor91]. Let A_G denote a maximal torus which normalizes U_G, so that $B_G = A_G \cdot U_G$ is a Borel subgroup of G. Also let $\widehat{A}_G^+$ be the set of dominant characters of A_G – the semigroup of highest weights of representations of G. It is well-known [Bor91] [How95] and may be thought of as a geometric version of the theory of the highest weight, that the space of regular functions on the coset space G/U_G, denoted by $\mathcal{R}(G/U_G)$, decomposes (under the action of G by left translations) as a direct sum of one copy of each irreducible representation V_ψ (with highest weight ψ) of G (see [Tow77]):

$$\mathcal{R}(G/U_G) \simeq \bigoplus_{\psi \in \widehat{A}_G^+} V_\psi. \tag{2.1}$$

We note that $\mathcal{R}(G/U_G)$ has the structure of an $\widehat{A}_G^+$-graded algebra, for which the V_ψ are the graded components. To be specific, we note that since A_G normalizes U_G, it acts on G/U_G by right translations, and this action commutes with the action of G by left translations.

Proposition 2.1 (see [VP72])**.** *The algebra of regular functions $\mathcal{R}(G/U_G)$ is an $\widehat{A}_G^+$-graded algebra, under the right action of A_G. More precisely, the decomposition (2.1) is the graded algebra decomposition under A_G, where V_ψ is the A_G-eigenspace corresponding to $\phi \in \widehat{A}_G^+$ with $\phi = w^*(\psi^{-1})$. Here w is the longest element of the Weyl group with respect to the root system determined by the Borel subgroup B_G.*

Now let $H \subset G$ be a reductive subgroup, and let U_H be a maximal unipotent subgroup of H. We consider the algebra $\mathcal{R}(G/U_G)^{U_H}$, of functions on G/U_G which are invariant under left translations by U_H. Let A_H be a maximal torus of H normalizing U_H, so that $B_H := A_H \cdot U_H$ is a Borel subgroup of H. Then $\mathcal{R}(G/U_G)^{U_H}$ will be invariant under the (left) action of A_H, and we may decompose $\mathcal{R}(G/U_G)^{U_H}$ into eigenspaces for A_H. Since the functions in $\mathcal{R}(G/U_G)^{U_H}$ are by definition (left) invariant under U_H, the (left) A_H-eigenfunctions will in fact be (left) B_H eigenfunctions. In other words, they are highest weight vectors for H. Hence, the characters of A_H acting on (the left of) $\mathcal{R}(G/U_G)^{U_H}$ will all be dominant with respect to B_H, and we may write $\mathcal{R}(G/U_G)^{U_H}$ as a sum of (left) A_H eigenspaces $(\mathcal{R}(G/U_G)^{U_H})^\chi$ for dominant characters χ of H:

$$\mathcal{R}(G/U_G)^{U_H} = \bigoplus_{\chi \in \widehat{A}_H^+} (\mathcal{R}(G/U_G)^{U_H})^\chi. \tag{2.3}$$

Since the spaces V_ψ of decomposition (2.1) are (left) G-invariant, they are a fortiori *left* H-invariant, so we have a decomposition of $\mathcal{R}(G/U_G)^{U_H}$ into *right* A_G-eigenspaces $(\mathcal{R}(G/U_G)^{U_H})_\psi$:

$$\mathcal{R}(G/U_G)^{U_H} = \bigoplus_{\psi \in \widehat{A}_G^+} \mathcal{R}(G/U_G)^{U_H} \cap V_\psi := \bigoplus_{\psi \in \widehat{A}_G^+} \mathcal{R}(G/U_G)_\psi^{U_H}.$$

Combining this decomposition with the decomposition (2.3), we may write

$$\mathcal{R}(G/U_G)^{U_H} = \bigoplus_{\psi \in \widehat{A}_G^+,\ \chi \in \widehat{A}_H^+} (\mathcal{R}(G/U_G)_\psi^{U_H})^\chi. \tag{2.4}$$

To emphasize the key features of this algebra, we note the resulting consequences of decomposition (2.4) in the following proposition.

Proposition 2.2. (a) *The decomposition (2.4) is an $(\widehat{A}_G^+ \times \widehat{A}_H^+)$-graded algebra decomposition of $\mathcal{R}(G/U_G)^{U_H}$.*

(b) *The subspaces* $(\mathcal{R}(G/U_G)_\psi^{U_H})^\chi$ *tell us the* χ *highest weight vectors for* B_H *in the irreducible representation* V_ψ *of* G*. Therefore, the decomposition*

$$\mathcal{R}(G/U_G)_\psi^{U_H} = \bigoplus_{\chi \in \widehat{A}_H^+} (\mathcal{R}(G/U_G)_\psi^{U_H})^\chi$$

tells us how V_ψ *decomposes as a* H*-module.*

Thus, knowledge of $\mathcal{R}(G/U_G)^{U_H}$ as a $(\widehat{A}_G^+ \times \widehat{A}_H^+)$-graded algebra tell us how representations of G decompose when restricted to H, in other words, it describes the branching rule from G to H. We will call $\mathcal{R}(G/U_G)^{U_H}$ the (G, H) *branching algebra.* When $G \simeq H \times H$, and H is embedded diagonally in G, the branching algebra describes the decomposition of tensor products of representations of H, and we then call it the *tensor product algebra* for H. More generally, we would like to understand the (G, H) branching algebras for symmetric pairs (G, H).

In the context of regular functions on G/U, such branching algebras are a little too abstract. We shall elaborate later on a more concrete construction of branching algebras, which allows substantial manipulation. We shall come back to the concrete construction at the end of §4, after some preliminaries in §3 and an example in §4.1.

3. Preliminaries and Notations

3.1. **Parametrization of Representations.** Let G be a classical reductive algebraic group over $\mathbb{C}$: $G = GL_n(\mathbb{C}) = GL_n$, the general linear group; or $G = O_n(\mathbb{C}) = O_n$, the orthogonal group; or $G = Sp_{2n}(\mathbb{C}) = Sp_{2n}$, the symplectic group. We shall explain our notations on irreducible representations of G using integer partitions. In each of these cases, we select a Borel subalgebra of the classical Lie algebra and coordinatize it, as is done in [GW98]. Consequently, all highest weights are parameterized in the standard way (see [GW98]).

A non-negative integer *partition* λ, with k parts, is an integer sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$. We may sometimes refer to λ as a *Young* or *Ferrers diagram.* We use the same notation for partitions as is done in [Mac95]. For example, we write $\ell(\lambda)$ to denote the *length* (or *depth*) of a partition, i.e., $\ell(\lambda) = k$ for the above partition. Also let $|\lambda| = \sum_i \lambda_i$ be the size of a partition and λ' denote the *transpose* (or *conjugate*) of λ (i.e., $(\lambda')_i = |\{\lambda_j : \lambda_j \geq i\}|$).

$\mathbf{GL_n}$ Representations: Given non-negative integers p, q and n such that $n \geq p + q$ and non-negative integer partitions λ^+ and λ^- with p and q parts respectively, let $F_{(n)}^{(\lambda^+,\lambda^-)}$ denote the irreducible rational representation of GL_n with highest weight given by the n-tuple:

$$(\lambda^+, \lambda^-) \;=\; \underbrace{\left(\lambda_1^+, \lambda_2^+, \cdots, \lambda_p^+, 0, \cdots, 0, -\lambda_q^-, \cdots, -\lambda_1^-\right)}_{n}$$

If $\lambda^- = (0)$ then we will write $F_{(n)}^{\lambda^+}$ for $F_{(n)}^{(\lambda^+,\lambda^-)}$. Note that if $\lambda^+ = (0)$ then $\left(F_{(n)}^{\lambda^-}\right)^*$ is equivalent to $F_{(n)}^{(\lambda^+,\lambda^-)}$. More generally, $\left(F_{(n)}^{(\lambda^+,\lambda^-)}\right)^*$ is equivalent to $F_{(n)}^{(\lambda^-,\lambda^+)}$.

$\mathbf{O_n}$ Representations: The complex orthogonal group has two connected components. Because the group is disconnected we cannot index irreducible representations by highest weights. There is however an analog of Schur-Weyl duality for the case of O_n in which each irreducible rational representation is indexed uniquely by a non-negative integer partition ν such that $(\nu')_1 + (\nu')_2 \le n$. That is, the sum of the first two columns of the Young diagram of ν is at most n. We will call such a diagram O_n-admissible (see [GW98] Chapter 10 for details). Let $E^\nu_{(n)}$ denote the irreducible representation of O_n indexed ν in this way.

An irreducible rational representation of SO_n may be indexed by its highest weight. In [GW98] Section 5.2.2, the irreducible representations of O_n are determined in terms of their restrictions to SO_n (which is a normal subgroup having index 2). We note that if $\ell(\nu) \neq \frac{n}{2}$, then the restriction of $E^\nu_{(n)}$ to SO_n is irreducible. If $\ell(\nu) = \frac{n}{2}$ (n even), then $E^\nu_{(n)}$ decomposes into exactly two irreducible representations of SO_n. See [GW98] Section 10.2.4 and 10.2.5 for the correspondence between this parametrization and the above parametrization by partitions.

The determinant defines an (irreducible) one-dimensional representation of O_n. This representation is indexed by the length n partition $\zeta = (1, 1, \cdots, 1)$. An irreducible representation of O_n will remain irreducible when tensored by $E^\zeta_{(n)}$, but the resulting representation *may* be inequivalent to the initial representation. We say that a pair of O_n-admissible partitions α and β are *associate* if $E^\alpha_{(n)} \otimes E^\zeta_{(n)} \cong E^\beta_{(n)}$. It turns out that α and β are associate exactly when $(\alpha')_1 + (\beta')_1 = n$ and $(\alpha')_i = (\beta')_i$ for all $i > 1$. This relation is clearly symmetric, and is related to the structure of the underlying SO_n-representations. Indeed, when restricted to SO_n, $E^\alpha_{(n)} \cong E^\beta_{(n)}$ if and only if α and β are either associate or equal.

3.2. **Multiplicity-Free Actions.** Let G be a complex reductive algebraic group acting on a complex vector space V. We say V is a *multiplicity-free action* if the algebra $\mathcal{P}(V)$ of polynomial functions on V is multiplicity free as a G module. The criterion of Servedio-Vinberg-Kimel′fel′d ([Ser73, VK78]) says that V is multiplicity free if and only if a Borel subgroup B of G has a Zariski open orbit in V. In other words, B (and hence G) acts prehomogeneously on V (see [SK77]). A direct consequence is that B eigenfunctions in $\mathcal{P}(V)$ have a very simple structure. Let $Q_\psi \in \mathcal{P}(V)$ be a B eigenfunction with eigencharacter ψ, normalized so that $Q_\psi(v_0) = 1$ for some fixed v_0 in a Zariski open B orbit in V. Then Q_ψ is completely determined by ψ: For $v = b^{-1}v_0$ in the Zariski open B orbit,

$$Q_\psi(v) = Q_\psi(b^{-1}v_0) = \psi(b)Q_\psi(v_0) = \psi(b), \qquad b \in B.$$

Q_ψ is then determined on all of V by continuity. Since $B = AU$, and $U = (B, B)$ is the commutator subgroup of B, we can identify a character of B with a character of A. Thus the B eigenfunctions are precisely the G highest weight vectors (with respect to B) in $\mathcal{P}(V)$. Further

$$Q_{\psi_1} Q_{\psi_2} = Q_{\psi_1\psi_2}$$

and so the set of $\widehat{A}^+(V) = \{\psi \in \widehat{A}^+ \mid Q_\psi \neq 0\}$ forms a sub-semigroup of the cone $\widehat{A}^+$ of dominant weights of A.

An element $\psi (\neq 1)$ of a semigroup is *primitive* if it is not expressible as a non-trivial product of two elements of the semigroup. The algebra $\mathcal{P}(V)^U$ has unique factorization (see [HU91]). The eigenfunctions associated to the primitive elements of $\widehat{A}^+(V)$ are prime polynomials, and $\mathcal{P}(V)^U$ is the polynomial ring on these eigenfunctions. If $\psi = \psi_1\psi_2$, then $Q_\psi = Q_{\psi_1}Q_{\psi_2}$. Thus, if ψ is not primitive, then the polynomial Q_ψ cannot be prime. An element

$$\psi = \Pi_{j=1}^k \psi_j^{c_j}$$

has c_j's uniquely determined, and hence the prime factorization

$$Q_\psi = \Pi_{j=1}^k Q_{\psi_j}^{c_j}.$$

Consider a multiplicity-free action of G on an algebra $\mathcal{W}$. In the general situation, we would like to associate this algebra $\mathcal{W}$ with a subalgebra of $\mathcal{R}(G/U)$. With this goal in mind, we introduce the following notion:

Definition 3.1. *Let $\mathcal{P} = \bigoplus_{\lambda \in \widehat{A}^+} \mathcal{P}_\lambda$ denote an algebra graded by an abelian semigroup $\widehat{A}^+$. If $\mathcal{W} \subseteq \mathcal{P}$ is a subalgebra of $\mathcal{P}$, then we say that $\mathcal{W}$ is a* **total subalgebra of** $\mathcal{P}$ *if*

$$\mathcal{W} = \bigoplus_{\lambda \in Z} \mathcal{P}_\lambda$$

where Z is a sub-semigroup of $\widehat{A}^+$, which we will denote by $\widehat{A}^+(\mathcal{W}) = Z$. Note that $\mathcal{W}$ is graded by $\widehat{A}^+(\mathcal{W})$.

In what is to follow, we will usually have $\mathcal{P} = \mathcal{P}(V)$ (polynomial functions on a vector space V) and $\widehat{A}^+$ will denote the dominant chamber of the character group of a maximal torus A of a reductive group G acting on V. In this situation, we introduce an $\widehat{A}^+$-filtration on $\mathcal{W}$ as follows:

$$\mathcal{W}^{(\psi)} = \bigoplus_{\phi \le \psi} \mathcal{W}_\phi \tag{3.1}$$

where the ordering $\le$ is the ordering on $\widehat{A}^+$ given by (see [Pop86])

$$\psi_1 \le \psi_2 \qquad \text{if } \psi_1^{-1}\psi_2 \text{ is expressible as a product of rational powers of positive roots.}$$

Note that positive roots are weights of the adjoint representation of G on its Lie algebra $\mathfrak{g}$. We refer to the abelian group structure on the integral weights multiplicatively. Also, it will turn out that we only need positive *integer* powers of the positive roots.

Next consider the more specific situation where $\mathcal{W}$ which is a G-invariant and G-multiplicity-free subalgebra of a polynomial algebra $\mathcal{P}(V)$. Suppose that $\mathcal{W}^U$ has unique factorization. Then $\mathcal{W}^U$ is a polynomial ring and $\widehat{A}^+(\mathcal{W})$ is a free sub-semigroup in $\widehat{A}^+$ generated by the highest weights corresponding to the non-zero graded components of $\mathcal{W}$. Write the G decomposition as follows:

$$\mathcal{W} = \bigoplus_{\psi \in \widehat{A}^+(\mathcal{W})} \mathcal{W}_\psi$$

noting that $\mathcal{W}_\psi$ is an irreducible G module with highest weight ψ.

If δ occurs with positive multiplicity in the tensor product decomposition

$$\mathcal{W}_\phi \otimes \mathcal{W}_\psi = \bigoplus_\delta \dim Hom_G(\mathcal{W}_\delta, \mathcal{W}_\phi \otimes \mathcal{W}_\psi)\ \mathcal{W}_\delta,$$

then $\delta \le \phi\psi$. From 3.1 we can see that if

$$\mathcal{W}_\eta \subset \mathcal{W}^{(\phi)} \quad \text{and} \quad \mathcal{W}_\gamma \subset \mathcal{W}^{(\psi)}, \quad \text{i.e., } \eta \le \phi \quad \text{and} \quad \gamma \le \psi,$$

then it follows that

$$\mathcal{W}_\eta \cdot \mathcal{W}_\gamma \hookrightarrow \mathcal{W}_\eta \otimes \mathcal{W}_\gamma \subset \mathcal{W}^{(\eta\gamma)} \subset \mathcal{W}^{(\phi\psi)}.$$

Thus

$$\mathcal{W}^{(\phi)} \cdot \mathcal{W}^{(\psi)} \subset \mathcal{W}^{(\phi\psi)}.$$

We have now an $\widehat{A}^+$-filtered algebra

$$\mathcal{W} = \bigcup_{\psi \in \widehat{A}^+(\mathcal{W})} \mathcal{W}^{(\psi)},$$

and this filtration is known as the *dominance filtration* [Pop86].

With a filtered algebra, we can form its associated algebra which is $\widehat{A}^+$ graded:

$$\mathrm{Gr}_{\widehat{A}^+}\mathcal{W} = \bigoplus_{\psi \in \widehat{A}^+(\mathcal{W})} (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\psi}$$

where

$$(\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\psi} = \mathcal{W}^{(\psi)} / \left(\bigoplus_{\phi < \psi} \mathcal{W}^{(\phi)} \right).$$

Theorem 3.2. *Consider a multiplicity-free G-module $\mathcal{W}$ with a $\widehat{A}^+$-filtered algebra structure such that $\mathcal{W}$ is a unique factorization domain. Assume that the zero degree subspace of $\mathcal{W}$ is $\mathbb{C}$. Then there is a canonical $\widehat{A}^+$-graded algebra injection:*

$$Gr_{\widehat{A}^+}\pi \ : \ Gr_{\widehat{A}^+}\mathcal{W} \hookrightarrow \mathcal{R}(G/U).$$

Note that this result immediately follows from a more general theorem (see Theorem 5 of [Pop86] and the Appendix to [Vin86]). Moreover, both the assumption on the zero degree subspace and the unique factorization can be removed. We thank the referee for providing both these observations and the references. Below is a proof cast in our present context.

Proof of Theorem 3.2. In [How95] it is shown that under the above hypothesis, $\mathcal{W}^U$ is a polynomial ring on a canonical set of generators. Now, $\mathcal{W}^U$ is a $\widehat{A}^+$-graded algebra and therefore, there exists an injective $\widehat{A}^+$-graded algebra homomorphism obtained by sending each generator of the domain to a (indeed any) highest weight vector of the same weight in the codomain:

$$\alpha \ : \ \mathcal{W}^U \hookrightarrow \mathcal{R}(G/U)^U.$$

Note that $\mathcal{W}^U = \mathrm{Gr}_{\widehat{A}^+}(\mathcal{W}^U) = (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^U$.

There exists a unique G-module homomorphism $\overline{\alpha} : \mathrm{Gr}_{\widehat{A}^+}\mathcal{W} \hookrightarrow \mathcal{R}(G/U)$ such that the following diagram commutes:

$$\begin{array}{cccc} \alpha : & \mathcal{W}^U & \hookrightarrow & \mathcal{R}(G/U)^U \\ & \cap & & \cap \\ \overline{\alpha} : & \mathrm{Gr}_{\widehat{A}^+}\mathcal{W} & \hookrightarrow & \mathcal{R}(G/U) \end{array}$$

We wish to show that $\overline{\alpha}$ is an algebra homomorphism, i.e.,

$$\begin{array}{ccccc} (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\lambda} & \times & (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\mu} & \underset{m_{\mathcal{W}}}{\longrightarrow} & (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^{\lambda+\mu} \\ \overline{\alpha}\downarrow & & \overline{\alpha}\downarrow & & \overline{\alpha}\downarrow \\ \mathcal{R}(G/U)^{\lambda} & \times & \mathcal{R}(G/U)^{\mu} & \underset{m_{\mathcal{R}(G/U)}}{\longrightarrow} & \mathcal{R}(G/U)^{\lambda+\mu} \end{array}$$

commutes.

We have two maps:

$$f_i : (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^\lambda \otimes (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^\mu \to \mathcal{R}(G/U)^{\lambda+\mu}, \qquad i = 1, 2,$$

defined by: $f_1(v \otimes w) = m_{\mathcal{R}(G/U)}(\overline{\alpha}(v) \otimes \overline{\alpha}(w))$ and $f_2(v \otimes w) = \overline{\alpha}(m_{\mathcal{W}}(v \otimes w))$.

Each of f_1 and f_2 is G-equivariant and,

$$\dim \left\{\beta \;\middle|\; \beta : (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^\lambda \otimes (\mathrm{Gr}_{\widehat{A}^+}\mathcal{W})^\mu \to \mathcal{R}(G/U)^{\lambda+\mu}\right\} = 1$$

because the Cartan product has multiplicity one in the tensor product of two irreducible G-modules V_λ and V_μ (this is a well known fact that is not difficult to prove, see for example [Pop86]).

Therefore, there exists a constant C such that $f_1 = Cf_2$. We know that $\overline{\alpha}|_{\mathcal{W}^U} = \alpha$ is an algebra homomorphism. So for highest weight vectors $v^\lambda \in \mathcal{W}^U_\lambda$ and $w^\mu \in \mathcal{W}^U_\mu$:

$$f_1(v^\lambda \otimes w^\mu) = \overline{\alpha}(v^\lambda)\overline{\alpha}(w^\mu) = \alpha(v^\lambda)\alpha(w^\mu) = \alpha(v^\lambda w^\mu) = \overline{\alpha}(v^\lambda w^\mu) = f_2(v^\lambda \otimes w^\mu).$$

(Note that $v^\lambda w^\mu$ is a highest weight vector.) Note that $C = 1$. $\square$

3.3. **Dual Pairs and Duality Correspondence.** The theory of dual pairs will feature prominently in this article. For the treatment of all the branching algebras arising from classical symmetric pairs, we will need to understand dual pairs in three different settings. However, to minimize exposition, we restrict our discussion to just one of the pairs. Details in this section including the more general cases can be found in [GW98], [How89a] or [How95].

In our context, the theory of dual pairs may be cast in a purely algebraic language. In this section, we will describe the dual pairs $(O_n, \mathfrak{sp}_{2m})$. Let O_n to be the group of invertible $n \times n$ matrices, g such that $gJg^t = J$ where J is the $n \times n$ matrix:

$$J = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdots & 1 & 0 \\ 0 & 1 & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $M_{n,m}$ be the vector space of $n \times m$ complex matrices, and consider the polynomial algebra $\mathcal{P}(M_{n,m})$ over $M_{n,m}$. The group $O_n \times GL_m$ acts on $\mathcal{P}(M_{n,m})$ by $(g,h)\cdot f(x) = f(g^t x h)$, where $g \in O_n$, $h \in GL_m$ and $x \in M_{n,m}$. The derived actions of their Lie algebras act on $\mathcal{P}(M_{n,m})$ by polynomial coefficient differential operators. Using the standard matrices entries as coordinates, we define the following differential operators:

$$\Delta_{ij} := \sum_{s=1}^n \frac{\partial^2}{\partial x_{si}\partial x_{n-s+1,\,j}}, \qquad r^2_{ij} := \sum_{s=1}^n x_{si}x_{n-s+1,\,j}, \text{ and } \qquad E_{ij} := \sum_{s=1}^n x_{si}\frac{\partial}{\partial x_{sj}}.$$

We define three spaces:

$$\begin{aligned} \mathfrak{sp}^{(1,1)}_{2m} &:= \text{Span } \left\{E_{ij} + \tfrac{n}{2}\,\delta_{i,j} \mid i,j = 1,\ldots,m\right\} \simeq \mathfrak{gl}_m, \\ \mathfrak{sp}^{(2,0)}_{2m} &:= \text{Span } \left\{r^2_{ij} \mid 1 \le i \le j \le m\right\}, \quad \text{and} \\ \mathfrak{sp}^{(0,2)}_{2m} &:= \text{Span } \left\{\Delta_{ij} \mid 1 \le i \le j \le m\right\}. \end{aligned} \tag{3.1}$$

The direct sum, $\mathfrak{g} := \mathfrak{sp}^{(2,0)}_{2m} \oplus \mathfrak{sp}^{(1,1)}_{2m} \oplus \mathfrak{sp}^{(0,2)}_{2m}$, is preserved under the usual operator bracket and is isomorphic, as a Lie algebra, to the rank m complex symplectic Lie algebra, $\mathfrak{sp}_{2m}$. This presentation defines an action of $\mathfrak{sp}_{2m}$ on $\mathcal{P}(M_{n,m})$.

Let $\mathcal{S}^2\mathbb{C}^m$ be the space of symmetric m by m matrices respectively. If V is a vector space, we denote the symmetric algebra on V by $\mathcal{S}(V)$. For a set S, we shall denote by $\mathbb{C}[S]$ by the algebra generated by elements in the set S.

Theorem 3.3. *Invariants and Harmonics of O_n*

(a) *First Fundamental Theorem of Invariant Theory and Separation of Variables***:** *The invariants*

$$\mathcal{J}_{n,m} := \mathcal{P}(M_{n,m})^{O_n} = \mathbb{C}[r_{ij}^2] \qquad \left(\cong \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)}) \cong \mathcal{P}(\mathcal{S}^2\mathbb{C}^m) \text{ if } n \geq m\right).$$

Further, if $n \geq 2m$, we have separation of variables

$$\mathcal{P}(M_{n,m}) \simeq \mathcal{H}_{n,m} \otimes \mathcal{J}_{n,m}.$$

where

$$\mathcal{H}_{n,m} = \{f \in \mathcal{P}(M_{n,m}) \mid \Delta_{ij} f = 0 \text{ for all } 1 \leq i \leq j \leq m\}$$

denotes the O_n-harmonics in $\mathcal{P}(M_{n,m})$.

(b) *Multiplicity-Free Decomposition under $O_n \times \mathfrak{sp}_{2m}$***:** *We have the decomposition*

$$\mathcal{P}(M_{n,m}) \mid_{O_n \times \mathfrak{sp}_{2m}} = \bigoplus E^\lambda_{(n)} \otimes \widetilde{E}^\lambda_{(2m)}$$

where the sum is over all partitions λ with length at most $\min(n,m)$*, and such that $(\lambda')_1 + (\lambda')_2 \leq n$. Furthermore, $\widetilde{E}^\lambda_{(2m)}$ is an irreducible (infinite dimensional) highest weight representation of $\mathfrak{sp}_{2m}$ such that as a representation of GL_m,*

$$\begin{aligned} \widetilde{E}^\lambda_{(2m)} &= \mathcal{J}_{n,m} \cdot F^\lambda_{(m)} && \text{for any } n, m \geq 0, \\ &\cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^m) \otimes F^\lambda_{(m)} && \text{provided } n \geq 2m. \end{aligned}$$

(c) *Multiplicity-Free Decomposition of Harmonics under $O_n \times \mathfrak{sp}_{2m}^{(1,1)}$***:** *The O_n-harmonics $\mathcal{H}_{n,m}$ is invariant under the action of $O_n \times \mathfrak{sp}_{2m}^{(1,1)}$. Here $\mathfrak{sp}_{2m}^{(1,1)} \simeq \mathfrak{gl}_m$, and as an $O_n \times GL_m$ representation,*

$$\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}) \cong \mathcal{H}_{n,m} = \bigoplus E^\lambda_{(n)} \otimes F^\lambda_{(m)},$$

where the sum is over all partitions λ with length at most $\min(n,m)$ *and such that $(\lambda')_1 + (\lambda')_2 \leq n$. Here $I(\mathcal{J}^+_{n,m})$ refers to the ideal generated by the positive degree O_n invariants in $\mathcal{P}(M_{n,m})$.*

4. Reciprocity Algebras

In this paper, we study branching algebras using classical invariant theory. The formulation of classical invariant theory in terms of dual pairs [How89a] allows one to realize branching algebras for classical symmetric pairs as concrete algebras of polynomials on vector spaces. Furthermore, when realized in this way, the branching algebras have a double interpretation in which they solve two related branching problems simultaneously. Classical invariant theory also provides a flexible means which allows an inductive approach to the computation of branching algebras, and makes evident natural connections between different branching algebras.

The easiest illustration of the above assertions is the realization of the tensor product algebra for GL_n presented as follows. This example also illustrates the definition of a reciprocity algebra.

4.1. **Illustration: Tensor Product Algebra for $\mathbf{GL}_n$.** This first example is in [How95], which we recall here as it is a model for the other (more involved) constructions of branching algebras as total subalgebras (see Definition 3.1) of GL_n tensor product algebras.

Consider the joint action of $GL_n \times GL_m$ on the $\mathcal{P}(M_{n,m})$ by the rule

$$(g,h)\cdot f(x) = f(g^t x h), \qquad \text{for } g \in GL_n, h \in GL_m, x \in M_{n,m}.$$

For the corresponding action on polynomials, one has the $GL_n \times GL_m$ multiplicity free decomposition (see [How95])

$$\mathcal{P}(M_{n,m}) \simeq \bigoplus_{\lambda} F^{\lambda}_{(n)} \otimes F^{\lambda}_{(m)}, \tag{4.1}$$

of the polynomials into irreducible $GL_n \times GL_m$ representations. Note that the sum is over non-negative partitions λ with length at most $\min(n,m)$.

Let $U_m = U_{GL_m}$ denote the upper triangular unipotent subgroup of GL_m. From decomposition (4.1), we can easily see that

$$\mathcal{P}(M_{n,m})^{U_m} \simeq \left(\bigoplus_{\lambda} F^{\lambda}_{(n)} \otimes F^{\lambda}_{(m)} \right)^{U_m} \simeq \bigoplus_{\lambda} F^{\lambda}_{(n)} \otimes (F^{\lambda}_{(m)})^{U_m}. \tag{4.2}$$

Since the spaces $(F^{\lambda}_{(m)})^{U_m}$ are one-dimensional, the sum in equation (4.2) consists of one copy of each $F^{\lambda}_{(n)}$. Just as in the discussion of §3.2, the algebra is graded by $\widehat{A}^{+}_{m}$, where A_m is the diagonal torus of GL_m, and one sees from (4.2) that the graded components are the $F^{\lambda}_{(n)}$.

By the arguments in §3.2, $\mathcal{P}(M_{n,m})^{U_m}$ can thus be associated to a graded subalgebra in $\mathcal{R}(GL_n/U_n)$, in particular, this is a total subalgebra as in Definition 3.1. To study tensor products of representations of GL_n, we can take the direct sum of $M_{n,m}$ and $M_{n,\ell}$. We then have an action of $GL_n \times GL_m \times GL_\ell$ on $\mathcal{P}(M_{n,m} \oplus M_{n,\ell})$. Since $\mathcal{P}(M_{n,m} \oplus M_{n,\ell}) \simeq \mathcal{P}(M_{n,m}) \otimes P(M_{n,\ell})$, we may deduce from (4.1) that

$$\begin{aligned} \mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell} &\simeq \mathcal{P}(M_{n,m})^{U_m} \otimes \mathcal{P}(M_{n,\ell})^{U_\ell} \\ &\simeq \bigoplus_{\mu,\nu} (F^{\mu}_{(n)} \otimes F^{\nu}_{(n)}) \otimes \left((F^{\mu}_{(m)})^{U_m} \otimes (F^{\nu}_{(\ell)})^{U_\ell} \right). \end{aligned} \tag{4.3}$$

Thus, this algebra is the sum of one copy of each tensor products $F^{\mu}_{(n)} \otimes F^{\nu}_{(n)}$. Hence, if we take the U_n-invariants, we will get a subalgebra of the tensor product algebra for GL_n. This results in the algebra

$$(\mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell})^{U_n} \simeq \mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell \times U_n}.$$

This shows that we can realize the tensor product algebra for GL_n, or more precisely, various total subalgebras of it, as algebras of polynomial functions on matrices, specifically as the algebras $\mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell \times U_n}$.

However, the algebra $\mathcal{P}(M_{n,m} \oplus M_{n,\ell})^{U_m \times U_\ell \times U_n}$ has a second interpretation, as a different branching algebra. We note that $M_{n,m} \oplus M_{n,\ell} \simeq M_{n,m+\ell}$. On this space we have the action of

$GL_n \times GL_{m+\ell}$, which is described by the obvious adaptation of equation (4.1). The action of $GL_n \times GL_m \times GL_\ell$ arises by restriction of the action of $GL_{m+\ell}$ to the subgroup $GL_m \times GL_\ell$ embedded block diagonally in $GL_{m+\ell}$. By (the obvious analog of) decomposition (4.2), we see that

$$\mathcal{P}(M_{n,m+\ell})^{U_n} \simeq \bigoplus_\lambda (F^\lambda_{(n)})^{U_n} \otimes F^\lambda_{(m+\ell)}.$$

This algebra embeds as a subalgebra of $\mathcal{R}(GL_{m+\ell}/U_{m+\ell})$, in particular, this is a total subalgebra as in Definition 3.1. If we then take the $U_m \times U_\ell$ invariants, we find that

$$(\mathcal{P}(M_{n,m+\ell})^{U_n})^{U_m\times U_\ell} \simeq \bigoplus_\lambda (F^\lambda_{(n)})^{U_n} \otimes (F^\lambda_{(m+\ell)})^{U_m\times U_\ell}$$

is (a total subalgebra of) the $(GL_{m+\ell}, GL_m \times GL_\ell)$ branching algebra. Thus, we have established the following result.

Theorem 4.1. (a) *The algebra $\mathcal{P}(M_{n,m+\ell})^{U_n\times U_m\times U_\ell}$ is isomorphic to a total subalgebra of the $(GL_n \times GL_n, GL_n)$ branching algebra (a.k.a. the GL_n tensor product algebra), and to a total subalgebra of the $(GL_{m+\ell}, GL_m \times GL_\ell)$ branching algebra.*

(b) *In particular, the dimension of the $\psi^\lambda \times \psi^\mu \times \psi^\nu$ homogeneous component for $A_n \times A_m \times A_\ell$ of $\mathcal{P}(M_{n,m+\ell})^{U_n\times U_m\times U_\ell}$ records simultaneously*
  (i) *the multiplicity of $F^\lambda_{(n)}$ in the tensor product $F^\mu_{(n)} \otimes F^\nu_{(n)}$, and*
  (ii) *the multiplicity of $F^\mu_{(m)} \otimes F^\nu_{(\ell)}$ in $F^\lambda_{(m+\ell)}$,*

for partitions μ, ν, λ such that $\ell(\mu) \le \min(n,m)$, $\ell(\nu) \le \min(n,\ell)$ and $\ell(\lambda) \le \min(n, m+\ell)$.

Thus, we can not only realize the GL_n tensor product algebra concretely as an algebra of polynomials, we find that it appears simultaneously in two guises, the second being as the branching algebra for the pair $(GL_{m+\ell}, GL_m \times GL_\ell)$. We emphasize two features of this situation.

First, the pair $(GL_{m+\ell}, GL_m \times GL_\ell)$, as well as the pair $(GL_n \times GL_n, GL_n)$, is a symmetric pair. Hence, both the interpretations of $\mathcal{P}(M_{n,m+\ell})^{U_n\times U_m\times U_\ell}$ are as branching algebras for symmetric pairs.

Second, the relationship between the two situations is captured by the notion of "see-saw pair" of dual pairs [Kud84]. Precisely, a context for understanding the decomposition law (4.1) is provided by observing that GL_n and GL_m (or more correctly, slight modifications of their Lie algebras) are mutual centralizers inside the Lie algebra $\mathfrak{sp}(M_{n,m})$ (of the metaplectic group) of polynomial coefficient differential operators of total degree two on $M_{n,m}$ [How89a] [How95]. We say that they define a *dual pair* inside $\mathfrak{sp}(M_{n,m})$. The decomposition (4.1) then appears as the correspondence of representations associated to this dual pair [How89a]. Further, the pairs of groups $(GL_n, GL_{m+\ell}) = (G_1, G_1')$ and $(GL_n \times GL_n, GL_m \times GL_\ell) = (G_2, G_2')$ both define dual pairs inside the Lie algebra $\mathfrak{sp}(M_{n,m+\ell})$. We evidently have the relations

$$G_1 = GL_n \subset GL_n \times GL_n = G_2, \tag{4.4}$$

and (hence)

$$G_1' = GL_{m+\ell} \supset GL_m \times GL_\ell = G_2'. \tag{4.5}$$

We refer to a pair of dual pairs related as in inclusions (4.4) and (4.5), a *see-saw pair* of dual pairs.

In these terms, we may think of the symmetric pairs (G_2, G_1) and (G'_1, G'_2) as a "*reciprocal pair*" of symmetric pairs. If we do so, we see that the algebra $\mathcal{P}(M_{n,m+\ell})^{U_n \times U_m \times U_\ell}$ is describable as $\mathcal{P}(M_{n,m+\ell})^{U_{G_1} \times U_{G'_2}}$ – it has a description in terms of the see-saw pair, and in this description the two pairs of the see-saw, or alternatively, the two reciprocal symmetric pairs, enter equivalently into the description of the algebra that describes the branching law for both symmetric pairs. For this reason, we also call this algebra, which describes the branching law for both symmetric pairs, the *reciprocity algebra* of the pair of pairs.

It turns out that any branching algebra associated to a classical symmetric pair, that is, a pair (G, H) in which G is a product of classical groups, has an interpretation as a reciprocity algebra – an algebra that describes a branching law for two reciprocal symmetric pairs simultaneously. Sometimes, however, one of the branching laws involves infinite-dimensional representations.

4.2. **Symmetric Pairs and Reciprocity Pairs.** In the context of dual pairs, we would like to understand the (G, H) branching of irreducible representations of G to H, for symmetric pairs (G, H). Table I lists the symmetric pairs which we will cover in this paper.

If G is a classical group over $\mathbb{C}$, then G can be embedded as one member of a dual pair in the symplectic group as described in [How89a]. The resulting pairs of groups are (GL_n, GL_m) or (O_n, Sp_{2m}), each inside Sp_{2nm}, and are called *irreducible* dual pairs. In general, a dual pair of reductive groups in Sp_{2r} is a product of such pairs.

Table I: Classical Symmetric Pairs

Description	**G**	**H**
Diagonal	$GL_n \times GL_n$	GL_n
Diagonal	$O_n \times O_n$	O_n
Diagonal	$Sp_{2n} \times Sp_{2n}$	Sp_{2n}
Direct Sum	GL_{n+m}	$GL_n \times GL_m$
Direct Sum	O_{n+m}	$O_n \times O_m$
Direct Sum	$Sp_{2(n+m)}$	$Sp_{2n} \times Sp_{2m}$
Polarization	O_{2n}	GL_n
Polarization	Sp_{2n}	GL_n
Bilinear Form	GL_n	O_n
Bilinear Form	GL_{2n}	Sp_{2n}

Proposition 4.2 *Let G be a classical group, or a product of two copies of a classical group. Let G belong to a dual pair (G, G') in a symplectic group Sp_{2m}. Let $H \subset G$ be a symmetric subgroup, and let H' be the centralizer of H in Sp_{2m}. Then (H, H') is also a dual pair in Sp_{2m}, and G' is a symmetric subgroup inside H'.*

Proof: This can be shown by fairly easy case-by-case checking. The basic reason that (H, H') form a dual pair is that, for any classical symmetric pair (G, H), the restriction of the standard module of G, or its dual, to H is a sum of standard modules of H, or their duals [How89a]. This is very easy to check on a case-by-case basis. The see-saw relationship of symmetric pairs organizes the 10 series of symmetric pairs as given in Table I into five pairs of series. These are shown in Table II. □

Table II: Reciprocity Pairs

Symmetric Pair $(\mathbf{G},\mathbf{H})$	$(\mathbf{H},\mathfrak{h}')$	$(\mathbf{G},\mathfrak{g}')$
$(GL_n \times GL_n, GL_n)$	$(GL_n, \mathfrak{gl}_{m+\ell})$	$(GL_n \times GL_n, \mathfrak{gl}_m \oplus \mathfrak{gl}_l)$
$(O_n \times O_n, O_n)$	$(O_n, \mathfrak{sp}_{2(m+\ell)})$	$(O_n \times O_n, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2l})$
$(Sp_{2n} \times Sp_{2n}, Sp_{2n})$	$(Sp_{2n}, \mathfrak{so}_{2(m+\ell)})$	$(Sp_{2n} \times Sp_{2n}, \mathfrak{so}_{2m} \oplus \mathfrak{so}_{2\ell})$
$(GL_{n+m}, GL_n \times GL_m)$	$(GL_n \times GL_m, \mathfrak{gl}_\ell \oplus \mathfrak{gl}_\ell)$	$(GL_{n+m}, \mathfrak{gl}_\ell)$
$(O_{n+m}, O_n \times O_m)$	$(O_n \times O_m, \mathfrak{sp}_{2\ell} \oplus \mathfrak{sp}_{2\ell})$	$(O_{n+m}, \mathfrak{sp}_{2\ell})$
$(Sp_{2(n+m)}, Sp_{2n} \times Sp_{2m})$	$(Sp_{2n} \times Sp_{2m}, \mathfrak{so}_{2\ell} \oplus \mathfrak{so}_{2\ell})$	$(Sp_{2(n+m)}, \mathfrak{so}_{2\ell})$
(O_{2n}, GL_n)	$(GL_n, \mathfrak{gl}_{2m})$	$(O_{2n}, \mathfrak{sp}_{2m})$
(Sp_{2n}, GL_n)	$(GL_n, \mathfrak{gl}_{2m})$	$(Sp_{2n}, \mathfrak{so}_{2m})$
(GL_n, O_n)	$(O_n, \mathfrak{sp}_{2m})$	$(GL_n, \mathfrak{gl}_m)$
(GL_{2n}, Sp_{2n})	$(Sp_{2n}, \mathfrak{so}_{2m})$	$(GL_{2n}, \mathfrak{gl}_m)$

Remark: Table II also amounts to another point of view on the structure on which [How89b] is based.

As alluded to in §2, we need a more concrete realization of branching algebras. With this goal in mind, we shall introduce the general definition of a reciprocity algebra through the following sequence of steps:

Step 1 Consider a symmetric pair (G,H).

Step 2 Use the theory of dual pairs to construct a multiplicity free $G \times K$ variety $\mathcal{V}$, for a group K associated to a dual pair involving G. Analogues to the Theory of Spherical Harmonics (see Theorem 3.3) allow us to consider a dual pair $(G,\mathfrak{g}')$, which has the Lie algebra of K as $\mathfrak{g}'^{(1,1)}$ (similar to the space $\mathfrak{sp}_{2m}^{(1,1)}$ in (3.1)). We note that $\mathfrak{g}'$ forms a *family* of Lie algebras, and each choice of $\mathfrak{g}'$ determines the type of G irreducible representations involved.

Step 3 Consider the coordinate ring of $\mathcal{V}$, which we denote by $\mathbb{C}[\mathcal{V}]$. Note that $\mathbb{C}[\mathcal{V}]$ is either polynomial algebra or a quotient of a polynomial algebra, depending on which dual pair we are considering. If U_K is a maximal unipotent subgroup of K, then $\mathbb{C}[\mathcal{V}]^{U_K}$ is a *partial model* of G, in other words, a collection of irreducible representations of G appearing once and only once in $\mathbb{C}[\mathcal{V}]^{U_K}$.

Step 4 Taking U_H covariants, the algebra $\mathbb{C}[\mathcal{V}]^{U_K \times U_H}$ will be our candidate. We will abuse our terminology and still call it a "branching algebra". This is because $\mathbb{C}[\mathcal{V}]^{U_K \times U_H}$ sits in $\mathcal{R}(G/U_G)^{U_H}$ as a total subalgebra (see Definition 3.1). We hasten to add, as pointed out at the end of Step 2, that we have a family of total subalgebras in $\mathcal{R}(G/U_G)^{U_H}$. Further, each total subalgebra relates two branching phenomena, and thus we call it a reciprocity algebra.

In the following table we provide the ingredients for the special cases as well as the stability range for the classical branching formula involving Littlewood-Richardson coefficients (see [HTW05a]).

Table III: Stability Range

Sym. Pair, (G,H)	K	Rep. of $G\times K$	Stability Range
(GL_k, O_k)	$GL_p\times GL_q$	$M_{k,p}\oplus M^*_{k,q}$	$k\geq 2(p+q)$
(GL_{2k}, Sp_{2k})	$GL_p\times GL_q$	$M_{2k,p}\oplus M^*_{2k,q}$	$k\geq p+q$
(O_{2k}, GL_k)	GL_{2n}	$M_{2k,2n}$	$k\geq 2n$
(Sp_{2k}, GL_k)	GL_{2n}	$M_{2k,2n}$	$k\geq 2n$
$(GL_{k+\ell}, GL_k\times GL_\ell)$	$GL_p\times GL_q$	$M_{k+\ell,p}\oplus M^*_{k+\ell,q}$	$\min(k,l)\geq p+q$
$(O_{k+\ell}, O_k\times O_\ell)$	GL_n	$M_{k+\ell,n}$	$\min(k,l)\geq 2n$
$(Sp_{2k+2\ell}, Sp_{2k}\times Sp_{2\ell})$	GL_n	$M_{2(k+\ell),n}$	$\min(k,l)\geq n$
$(GL_k\times GL_k, GL_k)$	$GL_p\times GL_q\times$ $GL_r\times GL_s$	$M_{k,p}\oplus M_{k,q}\oplus$ $M^*_{k,r}\oplus M^*_{k,s}$	$k\geq p+q+r+s$
$(O_k\times O_k, O_k)$	$GL_n\times GL_m$	$M_{k,n+m}$	$k\geq 2(n+m)$
$(Sp_{2k}\times Sp_{2k}, Sp_{2k})$	$GL_n\times GL_m$	$M_{2k,n+m}$	$k\geq (n+m)$

With the above general construction in mind, we begin with two of the reciprocity algebras in the next two sections. We have chosen the examples more to illustrate the subtleties and the general framework.

5. Branching from GL_n to O_n

Consider the problem of restricting irreducible representations of GL_n to the orthogonal group O_n. We consider the symmetric see-saw pair (GL_n, O_n) and (Sp_{2m}, GL_m). As in the discussion of §4.1, we can realize (a total subalgebra of) the coordinate ring of the flag manifold GL_n/U_n as the algebra of U_m-invariants on $\mathcal{P}(M_{n,m})$. If we then look at the U_{O_n}-invariants in this algebra, then we will have (a certain total subalgebra of) the (GL_n, O_n) branching algebra. Thus, we are interested in the algebra

$$\mathcal{P}(M_{n,m})^{U_{O_n}\times U_m}.$$

We note that, in analogy with the situation of §4.1, this is the algebra of invariants for the unipotent subgroups of the smaller member of each symmetric pair.

Let us investigate what this algebra appears to be if we first take invariants with respect to U_{O_n}. We have a decomposition of $\mathcal{P}(M_{n,m})$ as a joint $O_n\times\mathfrak{sp}_{2m}$-module (see Theorem 3.3(b)):

$$\mathcal{P}(M_{n,m})\simeq\bigoplus_{\mu} E^{\mu}_{(n)}\otimes\widetilde{E}^{\mu}_{(2m)}. \tag{5.1}$$

Recall that the sum runs through the set of all non-negative integer partitions μ such that $\ell(\mu)\leq\min(n,m)$ and $(\mu')_1+(\mu')_2\leq n$. Here $E^{\mu}_{(n)}$ denotes the irreducible O_n representation parameterized by μ. Recall from (4.1), the multiplicity free $GL_n\times GL_m$ decomposition $\mathcal{P}(M_{n,m})\simeq\bigoplus_{\mu}F^{\mu}_{(n)}\otimes F^{\mu}_{(m)}$. The module $E^{\mu}_{(n)}$ is generated by the GL_n highest weight vector in $F^{\mu}_{(n)}$. Further, $\widetilde{E}^{\mu}_{(2m)}$ is an irreducible infinite-dimensional representation of $\mathfrak{sp}_{2m}$ with lowest $\mathfrak{gl}_m$-type $F^{\mu}_{(m)}$.

Theorem 5.1. *Assume $n>2m$.*

(a) *The algebra $\mathcal{P}(M_{n,m})^{U_{O_n}\times U_m}$ is isomorphic to a total subalgebra of the (GL_n, O_n) branching algebra, and to a total subalgebra of the $(\mathfrak{sp}_{2m}, GL_m)$ branching algebra.*

(b) *In particular, the dimension of the $\phi^{\mu}\times\psi^{\lambda}$ homogeneous component for $A_{O_n}\times A_m$ of $\mathcal{P}(M_{n,m})^{U_{O_n}\times U_m}$ records simultaneously*

(i) *the multiplicity of* $E^{\mu}_{(n)}$ *in the representation* $F^{\lambda}_{(n)}$*, and*

(ii) *the multiplicity of* $F^{\lambda}_{(m)}$ *in* $\widetilde{E}^{\mu}_{(2m)}$*.*

for partitions μ*,* λ *such that* $\ell(\mu) \leq m$*, and* $\ell(\lambda) \leq m$*.*

Proof. Taking the U_{O_n}-invariants for the decomposition (5.1), we find that

$$\mathcal{P}(M_{n,m})^{U_{O_n}} \simeq \bigoplus_{\mu} (E^{\mu}_{(n)})^{U_{O_n}} \otimes \widetilde{E}^{\mu}_{(2m)}, \tag{5.2}$$

where the sum is over partitions μ such that $\ell(\mu) \leq \min(n, m)$ and $(\mu')_1 + (\mu')_2 \leq n$. Note that the stability condition $n > 2m$ guarantees the latter inequality. The space $(E^{\mu}_{(n)})^{U_{O_n}}$ is the space of highest weight vectors for $(E^{\mu}_{(n)})^{U_{O_n}}$. We would like to say that it is one-dimensional, so that $\mathcal{P}(M_{n,m})^{U_{O_n}}$ would consist of one copy of each of the irreducible representations $\widetilde{E}^{\mu}_{(2m)}$. But, owing to the disconnectedness of O_n, this is not quite true, and when it is true, the highest weight may not completely determine $E^{\mu}_{(n)}$.

However, if $n > 2m$, then $(E^{\mu}_{(n)})^{U_{O_n}}$ is one-dimensional, and does single out $E^{\mu}_{(n)}$ among the representations which appear in the sum (5.1). Hence, let us make this restriction for the present discussion. Taking the U_m invariants in the sum (5.2), we find that

$$(\mathcal{P}(M_{n,m})^{U_{O_n}})^{U_m} \simeq \bigoplus_{\mu} (E^{\mu}_{(n)})^{U_{O_n}} \otimes (\widetilde{E}^{\mu}_{(2m)})^{U_m}. \tag{5.3}$$

Note that the sum is over all partitions μ such that $\ell(\mu) \leq m$ (since $n > 2m$). The space $(\widetilde{E}^{\mu}_{(2m)})^{U_m}$ describes how the representation $\widetilde{E}^{\mu}_{(2m)}$ of $\mathfrak{sp}_{2m}$ decomposes as a $\mathfrak{gl}_m$ module, or equivalently, as a GL_m-module. In other words, $(\widetilde{E}^{\mu}_{(2m)})^{U_m}$ describes the branching rule from $\mathfrak{sp}_{2m}$ to $\mathfrak{gl}_m$ for the module $\widetilde{E}^{\mu}_{(2m)}$.

We know (thanks to our restriction to $n > 2m$) that the space $(E^{\mu}_{(n)})^{U_{O_n}}$ is one-dimensional. Let ϕ^{μ} be the A_{O_n} weight of $(E^{\mu}_{(n)})^{U_{O_n}}$. Thus, ϕ^{μ} is the restriction to the diagonal maximal torus A_{O_n} of the character ψ^{μ} of the group A_n of diagonal $n \times n$ matrices. Our assumption further implies that ϕ^{μ} determines $E^{\mu}_{(n)}$. Therefore, for a given dominant A_m weight ψ^{λ}, corresponding to the partition λ, where $\ell(\lambda) \leq m$, the ψ^{λ}-eigenspace in $(\widetilde{E}^{\mu}_{(2m)})^{U_m}$ tells us the multiplicity of $F^{\lambda}_{(m)}$ in the restriction of $\widetilde{E}^{\mu}_{(2m)}$ to $\mathfrak{gl}_m$. This is the same as the dimension of the joint $(\phi^{\mu} \times \psi^{\lambda})$-eigenspace in

$$(\mathcal{P}(M_{n,m})^{U_{O_n}})^{U_m} \simeq \mathcal{P}(M_{n,m})^{U_{O_n} \times U_m} \simeq (\mathcal{P}(M_{n,m})^{U_m})^{U_{O_n}}.$$

But we have already seen that this eigenspace describes the multiplicity of $E^{\mu}_{(n)}$ in $F^{\lambda}_{(n)}$. Thus, again the $A_{O_n} \times A_m$ homogeneous components of $\mathcal{P}(M_{n,m})^{U_{O_n} \times U_m}$ have a simultaneous interpretation, one for a branching law associated to each of the two symmetric pairs composing the symmetric see-saw pair. □

In this case, one of the branching laws involves infinite-dimensional representations. However, they are highest weight representations, which are the most tractable of infinite-dimensional representations, from an algebraic point of view.

6. Tensor Product Algebra for O_n

Using the symmetric see-saw pair $\big((O_n \times O_n, O_n), (Sp_{2(m+\ell)}, Sp_{2m} \times Sp_{2\ell})\big)$, we can construct (total subalgebras of) the tensor product algebra for O_n. To prepare for this, we should explicate the decomposition (5.1) further.

Let us recall the basic setup as in §3.3. Recall that $\mathcal{J}_{n,m} = \mathcal{P}(M_{n,m})^{O_n}$ is the algebra of O_n-invariant polynomials. Theorem 3.3(a) implies that $\mathcal{J}_{n,m}$ is a quotient of $\mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$, the symmetric algebra on $\mathfrak{sp}_{2m}^{(2,0)}$.

The natural mapping

$$\mathcal{H}_{n,m} \to \mathcal{P}(M_{n,m})/I(\mathcal{J}_{n,m}^{+})$$

is a linear $O_n \times GL_m$-module isomorphism. Further, the $O_n \times GL_m$ structure of $\mathcal{H}_{n,m}$ is as follows (see Theorem 3.3(c)):

$$\mathcal{H}_{n,m} \simeq \bigoplus_{\mu} E^{\mu}_{(n)} \otimes F^{\mu}_{(m)}.$$

Here μ ranges over the same diagrams as in (5.1).

From Theorem 3.3(b),

$$\widetilde{E}^{\mu}_{(2m)} \simeq F^{\mu}_{(m)} \cdot \mathcal{J}_{n,m} \simeq \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)}) \cdot F^{\mu}_{(m)}, \tag{6.1}$$

and it follows that

$$\widetilde{E}^{\mu}_{(2m)}/(\mathfrak{sp}_{2m}^{(2,0)} \cdot \widetilde{E}^{\mu}_{(2m)}) \simeq F^{\mu}_{(m)}.$$

In other words, we can detect the $\mathfrak{sp}_{2m}$ isomorphism class of the module $\widetilde{E}^{\mu}_{(2m)}$ by the GL_m isomorphism class of the quotient $\widetilde{E}^{\mu}_{(2m)}/(\mathfrak{sp}_{2m}^{(2,0)} \cdot \widetilde{E}^{\mu}_{(2m)})$. Also, if $W \subset \mathcal{P}(M_{n,m})$ is any $\mathfrak{sp}_{2m}$-invariant subspace, then

$$W/(\mathfrak{sp}_{2m}^{(2,0)} \cdot W) \simeq W \cap \mathcal{H}_{n,m},$$

and this subspace also reveals the $\mathfrak{sp}_{2m}$ isomorphism type of W.

We can use the above to find a total subalgebra of the tensor product algebra of O_n. One consequence of the above discussion is that

$$\big(\mathcal{P}(M_{n,m})/I(\mathcal{J}_{n,m}^{+})\big)^{U_m} \simeq \bigoplus_{\mu} E^{\mu}_{(n)} \otimes (F^{\mu}_{(m)})^{U_m}$$

consists of one copy of each irreducible representation $E^{\mu}_{(n)}$.

If we repeat the above discussion for $M_{n,\ell}$, and combine the results, we find that

$$\begin{aligned} &\big(\mathcal{P}(M_{n,m})/I(\mathcal{J}_{n,m}^{+})\big)^{U_m} \otimes \big(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}_{n,\ell}^{+})\big)^{U_\ell} \\ &\simeq \bigoplus_{\mu,\nu} \Big(E^{\mu}_{(n)} \otimes E^{\nu}_{(n)}\Big) \otimes \Big((F^{\mu}_{(m)})^{U_m} \otimes (F^{\nu}_{(\ell)})^{U_\ell}\Big) \end{aligned} \tag{6.2}$$

is a direct sum of one copy of each possible tensor product of an $E^{\mu}_{(n)}$ with an $E^{\nu}_{(n)}$. At this point, we make the assumption that $n > 2(m+\ell)$, as in this range the O_n constituents of decomposition are irreducible when restricted to the connected component of the identity in O_n. If we now take the U_{O_n}-invariants in equation (6.2), we will have (a total subalgebra of) the tensor product algebra of O_n:

$$\big((\mathcal{P}(M_{n,m})/I(\mathcal{J}_{n,m}^{+}))^{U_m} \otimes (\mathcal{P}(M_{n,\ell})/I(\mathcal{J}_{n,\ell}^{+}))^{U_\ell}\big)^{U_{O_n}}$$

$$\simeq \bigoplus_{\mu,\nu} \left(E^\mu_{(n)} \otimes E^\nu_{(n)}\right)^{U_{O_n}} \otimes \left((F^\mu_{(m)})^{U_m} \otimes (F^\nu_{(\ell)})^{U_\ell}\right).$$

We can describe this algebra in another way. Begin with the observation that $\mathcal{P}(M_{n,m}) \otimes \mathcal{P}(M_{n,\ell}) \simeq \mathcal{P}(M_{n,m+\ell})$, and

$$\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}) \simeq \mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}).$$

Thus

$$(\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}))^{U_m} \otimes (\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}))^{U_\ell} \simeq (\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_m \times U_\ell},$$

and taking U_{O_n} invariants of the above, we get

$$\begin{aligned} &\left((\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}))^{U_m} \otimes (\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}))^{U_\ell}\right)^{U_{O_n}} \\ &\quad \simeq \left((\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_m \times U_\ell}\right)^{U_{O_n}} \\ &\quad \simeq \left((\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_{O_n}}\right)^{U_m \times U_\ell}. \end{aligned}$$

Theorem 6.1. *Given positive integers n, m and ℓ with $n > 2(m+\ell)$ we have:*

(a) *The algebra*

$$\left((\mathcal{P}(M_{n,m})/I(\mathcal{J}^+_{n,m}))^{U_m} \otimes (\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^+_{n,\ell}))^{U_\ell}\right)^{U_{O_n}}$$

is isomorphic to a total subalgebra of the $(O_n \times O_n, O_n)$ branching algebra (a.k.a. the O_n tensor product algebra), and to a total subalgebra of the $(\mathfrak{sp}_{2(m+\ell)}, \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2\ell})$ branching algebra.

(b) *Specifically, the dimension of the $(\phi^\lambda \times \psi^\mu \times \psi^\nu)$-eigenspace for $A_{O_n} \times A_m \times A_\ell$ of $\left((\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_{O_n}}\right)^{U_m \times U_\ell}$ records simultaneously*

(i) *the multiplicity of $E^\lambda_{(n)}$ in $E^\mu_{(n)} \otimes E^\nu_{(n)}$, as well as*

(ii) *the multiplicity of $\widetilde{E}^\mu_{(2m)} \otimes \widetilde{E}^\nu_{(2\ell)}$ in the restriction of $\widetilde{E}^\lambda_{(2(m+\ell))}$.*

Here the partitions μ, ν, λ satisfy the following conditions:
$\ell(\mu) \leq \min(n,m)$, $\ell(\nu) \leq \min(n,\ell)$, and $\ell(\lambda) \leq \min(n, m+\ell)$.

Proof. Let us now compute the ring expressed in this way. From Theorem 3.3(b), we know that

$$\mathcal{P}(M_{n,m})^{U_{O_n}} \simeq \left(\bigoplus_\mu E^\mu_{(n)} \otimes \widetilde{E}^\mu_{(2m)}\right)^{U_{O_n}} \simeq \bigoplus_\mu (E^\mu_{(n)})^{U_{O_n}} \otimes \widetilde{E}^\mu_{(2m)}.$$

Note that within the range $n > 2(m+\ell)$ we have $\dim(E^\mu_{(n)})^{U_{O_n}} = 1$ since the O_n-representations $E^\mu_{(n)}$ remain irreducible when restricted to SO_n.

Now repeat this with m replaced by $m+\ell$:

$$\mathcal{P}(M_{n,m+\ell})^{U_{O_n}} \simeq \left(\bigoplus_\mu E^\mu_{(n)} \otimes \widetilde{E}^\mu_{(2(m+\ell))}\right)^{U_{O_n}} \simeq \bigoplus_\mu (E^\mu_{(n)})^{U_{O_n}} \otimes \widetilde{E}^\mu_{(2(m+\ell))}.$$

Hence

$$(\mathcal{P}(M_{n,m+\ell})/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell}))^{U_{O_n}} \simeq \left(\left(\bigoplus_\lambda E^\lambda_{(n)} \otimes \widetilde{E}^\lambda_{(2(m+\ell))}\right)/I(\mathcal{J}^+_{n,m} \oplus \mathcal{J}^+_{n,\ell})\right)^{U_{O_n}}$$

$$\simeq \bigoplus_{\lambda} (E^{\lambda}_{(n)})^{U_{O_n}} \otimes \left(\widetilde{E}^{\lambda}_{(2(m+\ell))} / (\mathfrak{sp}^{(2,0)}_{2m} \oplus \mathfrak{sp}^{(2,0)}_{2\ell}) \cdot \widetilde{E}^{\lambda}_{(2(m+\ell))} \right).$$

From this we finally get

$$\left((\mathcal{P}(M_{n,m+\ell}) / I(\mathcal{J}^{+}_{n,m} \oplus \mathcal{J}^{+}_{n,\ell}))^{U_{O_n}} \right)^{U_m \times U_\ell}$$

$$\simeq \bigoplus_{\lambda} (E^{\lambda}_{(n)})^{U_{O_n}} \otimes \left(\widetilde{E}^{\lambda}_{(2(m+\ell))} / (\mathfrak{sp}^{(2,0)}_{2m} \oplus \mathfrak{sp}^{(2,0)}_{2\ell}) \cdot \widetilde{E}^{\lambda}_{(2(m+\ell))} \right)^{U_m \times U_\ell}.$$

From the discussion following equation (6.1), we see that the factor

$$\left(\widetilde{E}^{\lambda}_{(2(m+\ell))} / (\mathfrak{sp}^{(2,0)}_{2m} \oplus \mathfrak{sp}^{(2,0)}_{2\ell}) \cdot \widetilde{E}^{\lambda}_{(2(m+\ell))} \right)^{U_m \times U_\ell}$$

tells us the $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2\ell}$ decomposition of $\widetilde{E}^{\lambda}_{(2(m+\ell))}$. □

Hence, again the algebra has a double interpretation, one in terms of decomposing tensor products of O_n representations, and one in terms of branching from $\mathfrak{sp}_{2(m+\ell)}$ to $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2\ell}$ (although the second branching law involves infinite-dimensional representations).

7. The Stable Range and Relations Between Reciprocity Algebras

Let us summarize our discussions this far. Given any classical symmetric pair, we can embed it in a (family of) see-saw symmetric pair(s). Doing this, we find that (a total subalgebra of) the branching algebra for the pair can equally well be interpreted as the branching algebra for a dual family of representations of the dual symmetric pair. The representations of the dual symmetric pair will frequently be infinite dimensional, but they are always highest weight modules.

An immediate consequence of this isomorphism of algebras is the isomorphisms of intertwining spaces and hence equality of multiplicities, which we have collectively described as *reciprocity laws*. These reciprocity laws are of the same nature as Frobenius Reciprocity for induced representations of groups.

From §4.2, we see that the see-saw symmetric pairs actually come in two parameter families. If one of the pairs involves many more variables than the other, then certain features of the discussions above become simpler.

Take the results of Theorem 4.1 as an illustration: Let n, m and ℓ denote positive integers. Now suppose that λ, μ and ν are partitions such that the length of λ (resp. μ, resp. ν) is at most $\min(n, m+\ell)$ (resp. $\min(n, m)$, resp. $\min(n, \ell)$). Then the Littlewood-Richardson coefficient

$$c^{\lambda}_{\mu\nu} = \dim \operatorname{Hom}_{GL_n}(F^{\lambda}_{(n)}, F^{\mu}_{(n)} \otimes F^{\nu}_{(n)})$$

while at the same time,

$$c^{\lambda}_{\mu\nu} = \dim \operatorname{Hom}_{GL_m \times GL_\ell}(F^{\mu}_{(m)} \otimes F^{\nu}_{(\ell)}, F^{\lambda}_{(m+\ell)}).$$

Thus, for fixed λ, μ and ν, we have two distinct interpretations of the Littlewood-Richardson coefficients for sufficiently large n, m and ℓ.

Consider another example: branching from GL_n to O_n. If we let these groups act on $\mathcal{P}(M_{n,m})$, we get the see-saw pairs $(O_n, \mathfrak{sp}_{2m})$ and (GL_n, GL_m). The branching coefficients

d^μ_λ from GL_n to O_n can be described as follows:

$$F^\lambda_{(n)} \mid_{O_n} = \sum_\mu d^\mu_\lambda \; E^\mu_{(n)}$$

where

$$\begin{aligned} d^\mu_\lambda &= \dim \operatorname{Hom}_{O_n}(E^\mu_{(n)}, F^\lambda_{(n)}) \\ &= \dim \operatorname{Hom}_{GL_m}\left(F^\lambda_{(m)}, F^\mu_{(m)} \otimes \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m})\right) \\ &= \dim \operatorname{Hom}_{GL_m}\left(F^\lambda_{(m)}, F^\mu_{(m)} \otimes \mathcal{S}(S^2\mathbb{C}^m)\right) \end{aligned}$$

is independent of n, if $n \geq m$, and only depends on the diagrams λ and μ. This allows one to create a theory of "stable characters" for O_n. Similar considerations apply to GL_n and Sp_{2n} and this idea has been actively pursued by [KT90], amongst others.

These are all instances of stability laws. The well-known one-step branching from GL_n to GL_{n-1} is another instance. More precisely, the dominant weights of GL_n (resp. GL_{n-1}) may be indexed by partitions as in Section 3.1. Relative to this parametrization, the only requirement is that n be sufficiently large when compared to the lengths of the partitions. In other words, a single partition indexes a highest weight of a representation of GL_n for all sufficiently large n. Thus, this branching can be described entirely by diagrams, with no mention of the size n, if n is large. Iteration of this branching also shows that when n is large, the weight multiplicities of dominant weights of an irreducible GL_n representation are independent of n, in a similar sense. See [BBL90] for the other classical groups, which don't share this stability property.

In the last two sections that follow, we will illustrate the simplifications that occur in the stable range, highlighting certain specific see-saw pairs. In all these cases, we show that the branching algebras associated to symmetric pairs can all be described by use of suitable branching algebras associated to the general linear groups. Thus, if we can have control of the solution in the general linear group case, we will have some control of the other classical groups. The other non-trivial examples will be important extensions of this work, and we will see them in further papers, for example, [HTW05a], [HTW05b], [HL07], [HL06a] and [HL06b].

8. Stability for Branching from GL_n to O_n

We begin with a detailed discussion of the case of (GL_n, O_n) and $(\mathfrak{sp}_{2m}, GL_m)$. Here we have already encountered the stable range, without the name. It is when $n > 2m$. Several things happen in the stable range:

(a) The representations $E^\mu_{(n)}$ of the orthogonal group remain irreducible when restricted to the special orthogonal group SO_n, and furthermore, no two of them are equivalent.

(b) Recall the algebra $\mathcal{J}_{n,m}$ of O_n-invariant polynomials on $M_{n,m}$ generated by the quadratic invariants, which is the abelian subalgebra $\mathfrak{sp}^{(2,0)}_{2m}$ of $\mathfrak{sp}_{2m}$. In the stable range (in fact it holds true whenever $n \geq m$), the natural surjective homomorphism

$$\mathcal{S}(\mathfrak{sp}^{(2,0)}_{2m}) \to \mathcal{J}_{n,m}$$

is an isomorphism. See Theorem 3.3(a).

(c) In the stable range, the multiplication map

$$\mathcal{H}_{n,m} \otimes \mathcal{J}_{n,m} \simeq \mathcal{H}_{n,m} \otimes \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)}) \to \mathcal{P}(M_{n,m})$$

is also an isomorphism of $O_n \times GL_m$-modules. See Theorem 3.3(a) and 3.3(b).

Of course, the subspace $\mathcal{H}_{n,m}$ of harmonic polynomials is not an algebra – it is not closed under multiplication. This is quite clear, since $\mathcal{H}_{n,m}$ contains all the linear functions, which generate the whole polynomial ring. However, to form the reciprocity algebra associated to the symmetric see-saw pairs (GL_n, O_n) and $(\mathfrak{sp}_{2m}, GL_m)$, we need to take the U_{O_n}-invariants. Thus, our reciprocity algebra is a subalgebra of

$$(\mathcal{H}_{n,m} \otimes \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)}))^{U_{O_n}} = \mathcal{H}_{n,m}^{U_{O_n}} \otimes \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)}) \simeq \left(\bigoplus_{\mu} (E_{(n)}^{\mu})^{U_{O_n}} \otimes F_{(m)}^{\mu}\right) \otimes \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)}). \quad (8.1)$$

Theorem 8.1. *When $n > 2m$, the space $\mathcal{H}_{n,m}^{U_{O_n}}$ is a subalgebra of $\mathcal{P}(M_{n,m})$. Hence, the algebra $\mathcal{P}(M_{n,m})^{U_{O_n}}$ is isomorphic to a tensor product*

$$\mathcal{P}(M_{n,m})^{U_{O_n}} \simeq \mathcal{H}_{n,m}^{U_{O_n}} \otimes \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$$

of the algebras $\mathcal{H}_{n,m}^{U_{O_n}}$ and $\mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$. Furthermore, the algebra $\mathcal{H}_{n,m}^{U_{O_n}}$ is isomorphic (as a representation) to the subalgebra $\mathcal{R}^+(GL_m/U_m)$ of $\mathcal{R}(GL_m/U_m)$ defined by the polynomial representations.

Proof. Note that $\mathcal{H}_{n,m}^{U_{O_n}}$ can be identified with a subalgebra $\mathcal{R}^+(GL_m/U_m)$ of $\mathcal{R}(GL_m/U_m)$ defined by the polynomial representations, from our discussion in §3.2. Consider the space of polynomials belonging to the sum in the last expression of equation (8.1). Let $\{x_{jk} \mid j = 1, \ldots, n, k = 1, \ldots, m\}$ be the standard matrix entries on $M_{n,m}$. In order to make the unipotent group U_{O_n} of O_n maximally compatible with (in fact, contained in) the unipotent subgroup U_n of GL_n, we choose the inner product on $\mathbb{C}^n$ as in Section 3.3. By this choice, joint $O_n \times GL_m$ harmonic highest weight vectors are monomials in the determinants

$$\delta_j = \det \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1j} \\ x_{21} & x_{22} & \ldots & x_{1j} \\ \vdots & \vdots & \vdots & \vdots \\ x_{j1} & x_{j2} & \ldots & x_{jj} \end{bmatrix} \qquad \text{for } j = 1, \ldots, m.$$

From this, we can see that the space $\sum_{\mu} (E_{(n)}^{\mu})^{U_{O_n}} \otimes F_{(m)}^{\mu}$ is spanned by the monomials in the determinants

$$\det \begin{bmatrix} x_{1,b_1} & x_{1,b_2} & \ldots & x_{1,b_j} \\ x_{2,b_1} & x_{2,b_2} & \ldots & x_{2,b_j} \\ \vdots & \vdots & \vdots & \vdots \\ x_{j,b_1} & x_{j,b_2} & \ldots & x_{j,b_j} \end{bmatrix} \quad (8.2)$$

as $\{b_1, b_2, b_3, \ldots, b_j\}$ ranges over all j-tuples of integers from 1 to m. Indeed, the span of such monomials is clearly invariant under $\mathfrak{gl}_m$, and consists of highest weight vectors for O_n. Finally, we see that these monomials will all be harmonic, because the partial Laplacians spanning $\mathfrak{sp}_{2m}^{(0,2)}$ have the form

$$\Delta_{ab} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_{j,a} \partial x_{n+1-j,b}}.$$

Since every term of Δ_{ab} involves differentiating with respect to a variable x_{jk} with $j > n/2$, and the determinants (8.2) do not depend on these variables, we see that they will be annihilated by the Δ_{ab}, which means that they are harmonic. This shows that $\mathcal{H}_{n,m}^{U_{O_n}}$ is a subalgebra of $\mathcal{P}(M_{n,m})$.

We have thus completed the proof of the theorem. $\square$

We can use the description in Theorem 8.1 of $\mathcal{P}(M_{n,m})^{U_{O_n}}$ to relate the branching algebra $\mathcal{P}(M_{n,m})^{U_{O_n}\times U_m}$ to the tensor product algebra for GL_m. As a GL_m-module, the space $\mathfrak{sp}_{2m}^{(2,0)}$ is isomorphic to $\mathcal{S}^2\mathbb{C}^m$, the space of symmetric $m \times m$ matrices. It is well known that the symmetric algebra $\mathcal{S}(\mathcal{S}^2\mathbb{C}^m)$ is multiplicity-free as a representation of GL_m, and decomposes into a sum of one copy of each polynomial representation corresponding to a diagram with rows of even length (or a partition of even parts):

$$\mathcal{S}(\mathcal{S}^2\mathbb{C}^m) \simeq \bigoplus_{\nu} F_{(m)}^{2\nu}.$$

(Note that this result is in several places in the literature. See [GW98] and [How95] for example.)

As a GL_m-module, $\mathcal{S}(\mathcal{S}^2\mathbb{C}^m)$ could be embedded in $\mathcal{R}(GL_m/U_m)$, but the algebra structures on these two algebras are quite different.

Using the dominance filtration (see §3.2), we have a canonical $\widehat{A}^+$-algebra filtration on $\mathcal{S}(\mathcal{S}^2\mathbb{C}^m)$. If we form the associated graded algebra, then Theorem 3.2 says that it will be isomorphic to the subalgebra of $R(GL_m/U_m)$ spanned by the representations attached to diagrams with even length rows.

Let us denote the associated graded algebra of $\mathcal{S}(\mathcal{S}^2\mathbb{C}^m)$ by $\mathrm{Gr}_{\widehat{A}_m^+}\mathcal{S}(\mathcal{S}^2\mathbb{C}^m)$. Let us denote the subalgebra of $\mathcal{R}(GL_m/U_m)$ spanned by the representations attached to diagrams with even length rows by $\mathcal{R}^{+2}(GL_m/U_m)$.

We can filter the tensor product $\mathcal{H}_{n,m}^{U_{O_n}} \otimes \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$ by means of the filtration on $\mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$. The associated graded algebra will then be $\mathcal{H}_{n,m}^{U_{O_n}} \otimes \mathrm{Gr}_{\widehat{A}_m^+}\mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$. This discussion has indicated that the following result holds.

Theorem 8.2. *When $n > 2m$, the associated graded algebra of $\mathcal{P}(M_{n,m})^{U_{O_n}}$ with respect to the dominance filtration on the factor $\mathcal{J}_{n,m}$ is isomorphic to the tensor product of the graded subalgebras $\mathcal{R}^+(GL_m/U_m)$ and $\mathcal{R}^{+2}(GL_m/U_m)$ of $\mathcal{R}(GL_m/U_m)$:*

$$Gr_{\widehat{A}_m^+}(\mathcal{P}(M_{n,m})^{U_{O_n}}) \simeq \mathcal{R}^+(GL_m/U_m) \otimes \mathcal{R}^{+2}(GL_m/U_m).$$

Of course, $\mathrm{Gr}_{\widehat{A}_m^+}(\mathcal{P}(M_{n,m})^{U_{O_n}})$ is isomorphic as a GL_m-module to $\mathcal{P}(M_{n,m})^{U_{O_n}}$ in an obvious way, by construction. Also $\mathrm{Gr}_{\widehat{A}_m^+}(\mathcal{P}(M_{n,m})^{U_{O_n}})$ inherits the $\widehat{A}_{O_n}^+$ grading from $\mathcal{P}(M_{n,m})^{U_{O_n}}$ – it becomes identified with the $\widehat{A}_m^+$ grading on the first factor $\mathcal{R}^+(GL_m/U_m)$ in the tensor product of Theorem 8.2. On the other hand, the second factor is also $\widehat{A}_m^+$-graded in the obvious way, since it is the factor which defines the associated graded. When we take the U_m invariants, we get another grading by $\widehat{A}_m^+$, associated to the A_m action on the U_m invariants. This triply $\widehat{A}_m^+$-graded algebra is evidently a total subalgebra of the tensor product algebra of GL_m.

On the other hand, we could take the U_m invariants inside $\mathcal{P}(M_{n,m})^{U_{O_n}}$, and then pass to the associated graded. It is not hard to convince oneself that these two processes commute with each other. Hence, we finally have:

Corollary 8.3 *When $n > 2m$, the associated graded algebra of U_m invariants in $\mathcal{P}(M_{n,m})^{U_{O_n}}$,*

$$Gr_{\widehat{A}_m^+}\left(\left(\mathcal{P}(M_{n,m})^{U_{O_n}}\right)^{U_m}\right) \simeq \left(Gr_{\widehat{A}_m^+}(\mathcal{P}(M_{n,m})^{U_{O_n}})\right)^{U_m}$$
$$\simeq \left(\mathcal{R}^+(GL_m/U_m) \otimes \mathcal{R}^{+2}(GL_m/U_m)\right)^{U_m}$$

is a triply-graded total subalgebra of the tensor product algebra of GL_m. The restrictions on the gradings which define $Gr_{\widehat{A}_m^+}\left(\left(\mathcal{P}(M_{n,m})^{U_{O_n}}\right)^{U_m}\right)$ are:

(a) *the weight on the first factor of $(\mathcal{R}(GL_m/U_m) \otimes \mathcal{R}(GL_m/U_m))^{U_m}$ should correspond to a partition (i.e., it should be a polynomial weight), and*
(b) *the weight on the second factor should correspond to a partition with even parts.*

Remark: The content of Corollary 8.3 in terms of multiplicities is the Littlewood Restriction Formula [EW04], [HTW05a]; see formula (2.4.1), [Kin71]; see (5.7) with (4.19), [KT90]; see Theorem 1.5.3 and 2.3.1, [Lit44] and [Lit40]. With this result it is possible to compute a basis of the reciprocity algebra for (GL_n, O_n) using [HTW05a]; see [HL06a].

9. Tensor Products for O_n

According to Theorem 6.1, we can compute tensor products for the orthogonal group via the algebra

$$\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}_{n,m}^+)\right)^{U_m} \otimes \left(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}_{n,\ell}^+)\right)^{U_\ell}\right)^{U_{O_n}}.$$

Here the stable range is $n > 2(m+\ell)$. Then we have

$$\mathcal{P}(M_{n,m+\ell}) \simeq \mathcal{H}_{n,m+\ell} \otimes \mathcal{S}(\mathfrak{sp}_{2(m+\ell)}^{(2,0)}).$$

Furthermore,

$$\mathcal{S}(\mathfrak{sp}_{2(m+\ell)}^{(2,0)}) = \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)} \oplus \mathfrak{sp}_{2\ell}^{(2,0)} \oplus (\mathbb{C}^m \otimes \mathbb{C}^\ell))$$
$$\simeq \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)}) \otimes \mathcal{S}(\mathfrak{sp}_{2\ell}^{(2,0)}) \otimes \mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^\ell)$$

Since $\mathcal{J}_{n,m} \simeq \mathcal{S}(\mathfrak{sp}_{2m}^{(2,0)})$ and $\mathcal{J}_{n,\ell} \simeq \mathcal{S}(\mathfrak{sp}_{2\ell}^{(2,0)})$, we see that

$$\begin{aligned}\mathcal{P}(M_{n,m})/I(\mathcal{J}_{n,m}^+) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}_{n,\ell}^+) &\simeq \mathcal{P}(M_{n,m} \oplus M_{n,\ell})/I(\mathcal{J}_{n,m}^+ \oplus \mathcal{J}_{n,\ell}^+) \\ &\simeq \mathcal{H}_{n,m+\ell} \otimes \mathcal{S}(\mathfrak{sp}_{2(m+\ell)}^{(2,0)})/I(\mathfrak{sp}_{2m}^{(2,0)} \oplus \mathfrak{sp}_{2\ell}^{(2,0)}) \\ &\simeq \mathcal{H}_{n,m+\ell} \otimes \mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^\ell).\end{aligned} \tag{9.1}$$

Thus, using equation (9.1), we see that

$$\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}_{n,m}^+) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}_{n,\ell}^+)\right)^{U_{O_n}}$$
$$\simeq \left(\mathcal{H}_{n,m+\ell} \otimes \mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^\ell)\right)^{U_{O_n}} \simeq \mathcal{H}_{n,m+\ell}^{U_{O_n}} \otimes \mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^\ell)$$
$$\simeq \left(\bigoplus_\lambda E_{(n)}^\lambda \otimes F_{(m+\ell)}^\lambda\right)^{U_{O_n}} \otimes \mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^\ell)$$

$$\simeq \left(\bigoplus_{\lambda} (E^{\lambda}_{(n)})^{U_{O_n}} \otimes F^{\lambda}_{(m+\ell)} \right) \otimes \mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^{\ell})$$

$$\simeq \left(\bigoplus_{\lambda} (F^{\lambda}_{(n)})^{U_n} \otimes F^{\lambda}_{(m+\ell)} \right) \otimes \mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^{\ell}).$$

Note that $F^{\lambda}_{(n)}$ is the GL_n representation generated by the highest weight of the O_n representation $E^{\lambda}_{(n)}$ and both $(F^{\lambda}_{(n)})^{U_n}$ and $(E^{\lambda}_{(n)})^{O_n}$ are one dimensional.

Hence, finally we get

$$\begin{aligned}
&\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^{+}_{n,m})\right)^{U_m} \otimes \left(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}^{+}_{n,\ell})\right)^{U_\ell} \right)^{U_{O_n}}. \\
&\simeq \left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^{+}_{n,m}) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}^{+}_{n,\ell})\right)^{U_{O_n}} \right)^{U_m \times U_\ell} \\
&\simeq \left(\left(\bigoplus_{\lambda} (F^{\lambda}_{(n)})^{U_n} \otimes F^{\lambda}_{(m+\ell)} \right) \otimes \mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^{\ell}) \right)^{U_m \times U_\ell}. \qquad (9.2)
\end{aligned}$$

We can interpret this algebra in term of tensor product algebras for general linear groups. Now consider the (polynomial) tensor product algebras

$$(\mathcal{R}^{+}(GL_k/U_k) \otimes \mathcal{R}^{+}(GL_k/U_k))^{U_k} \simeq \bigoplus_{\lambda,\mu} \left(F^{\lambda}_{(k)} \otimes F^{\mu}_{(k)} \right)^{U_k}$$

for $k = n, m$ and ℓ. If we form the tensor product of these, we get

$$\begin{aligned}
&(\mathcal{R}^{+}(GL_n/U_n) \otimes \mathcal{R}^{+}(GL_n/U_n))^{U_n} \otimes (\mathcal{R}^{+}(GL_m/U_m) \otimes \mathcal{R}^{+}(GL_m/U_m))^{U_m} \\
&\qquad \otimes (\mathcal{R}^{+}(GL_\ell/U_\ell) \otimes \mathcal{R}^{+}(GL_\ell/U_\ell))^{U_\ell} \\
&\simeq \bigoplus_{\alpha,\beta,\delta,\lambda,\mu,\nu} \left(F^{\alpha}_{(n)} \otimes F^{\beta}_{(n)} \right)^{U_n} \otimes \left(F^{\delta}_{(m)} \otimes F^{\lambda}_{(m)} \right)^{U_m} \otimes \left(F^{\mu}_{(\ell)} \otimes F^{\nu}_{(\ell)} \right)^{U_\ell}
\end{aligned}$$

Let us denote this algebra by $\mathbb{T}_{n,m,\ell}$. The algebra $\mathbb{T}_{n,m,\ell}$ is $(\widehat{A}^{+}_{n})^3 \times (\widehat{A}^{+}_{m})^3 \times (\widehat{A}^{+}_{\ell})^3$-graded. If we require that $\lambda = \alpha$, or that $\mu = \beta$, or that $\nu = \delta$, then we obtain total subalgebras of $\mathbb{T}_{n,m,\ell}$. If $\delta = \alpha$, we will denote it by $\Delta_{1,3}\mathbb{T}_{n,m,\ell}$, and so forth. The subalgebra obtained by requiring that all three diagonal conditions occur at once will be denoted by using all three Δ's. Thus we will write

$$\begin{aligned}
&\Delta_{1,3}\Delta_{2,5}\Delta_{4,6}\mathbb{T}_{n,m,\ell} \\
&= \sum_{\alpha,\beta,\delta} \left(F^{\alpha}_{(n)} \otimes F^{\beta}_{(n)} \right)^{U_n} \otimes \left(F^{\alpha}_{(m)} \otimes F^{\delta}_{(m)} \right)^{U_m} \otimes \left(F^{\beta}_{(\ell)} \otimes F^{\delta}_{(\ell)} \right)^{U_\ell}
\end{aligned}$$

It takes a similar argument by expanding (9.2) (as in §8) to see that $\Delta_{1,3}\Delta_{2,5}\Delta_{4,6}\mathbb{T}_{n,m,\ell}$ and

$$\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^{+}_{n,m}) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}^{+}_{n,\ell})\right)^{U_{O_n}} \right)^{U_m \times U_\ell}$$

are isomorphic as multigraded vector spaces. They may not be isomorphic as algebras, because

$$\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}^{+}_{n,m}) \otimes \mathcal{P}(M_{n,\ell})/I(\mathcal{J}^{+}_{n,\ell})\right)^{U_{O_n}} \right)^{U_m \times U_\ell}$$

is not graded, while we see that $\Delta_{1,3}\Delta_{2,5}\Delta_{4,6}\mathbb{T}_{n,m,\ell}$ is. However, if we pass to the associated graded of $\mathcal{S}(\mathbb{C}^m \otimes \mathbb{C}^\ell)$, then the two algebras do become isomorphic. We record this fact.

Theorem 9.1. *Assume the stable range $n > 2(m+\ell)$. We have the following isomorphisms of $(\widehat{A}_n^+)^3 \times (\widehat{A}_m^+)^3 \times (\widehat{A}_\ell^+)^3$-graded algebras:*

$$Gr_{(\widehat{A}_n^+)^3\times(\widehat{A}_m^+)^3\times(\widehat{A}_\ell^+)^3}\left(\left(\left(\mathcal{P}(M_{n,m})/I(\mathcal{J}_{n,m}^+)\right)^{U_m} \otimes \left(\mathcal{P}(M_{n,\ell})/I(\mathcal{J}_{n,\ell}^+)\right)^{U_\ell}\right)^{U_{O_n}}\right) \simeq \Delta_{1,3}\Delta_{2,5}\Delta_{4,6}\mathbb{T}_{n,m,\ell}.$$

Remark: The content of Theorem 9.1 in terms of multiplicities can be found in [HTW05a]; see formula (2.1.2), [Kin90]; see Theorem 4.1 and [New51].

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Department of Mathematics, Yale University, Box 208283 Yale Station, New Haven, CT 06520, USA

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore

Department of Mathematical Sciences, University of Wisconsin at Milwaukee, Milwaukee, WI 53201, USA