

STABLE HILBERT SERIES OF $\mathcal{S}(\mathfrak{g})^K$ FOR CLASSICAL GROUPS

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ABSTRACT. Given a classical symmetric pair, (G, K) , with $\mathfrak{g} = \text{Lie}(G)$, we provide descriptions of the Hilbert series of the algebra of K -invariant vectors in the associated graded algebra of $\mathcal{U}(\mathfrak{g})$ viewed as a K -representation under restriction of the adjoint representation. The description illuminates a certain stable behavior of the Hilbert series, which is investigated in a case-by-case basis. We note that the stable Hilbert series of one symmetric pair often coincides with others. Also, for the case of the real form $U(p, q)$ we derive a closed expression for the Hilbert series when $\min(p, q) \rightarrow \infty$.

1. INTRODUCTION.

The adjoint representation of a reductive group G on its Lie algebra \mathfrak{g} induces an action on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The structure of $\mathcal{U}(\mathfrak{g})$ as a representation of G follows from Kostant's theory of harmonic polynomials (see [Kos63]) on \mathfrak{g} . We approach the problem of understanding the restriction of the induced $Ad(G)$ -action on $\mathcal{U}(\mathfrak{g})$ to a symmetric subgroup K . In particular, we focus on the K -invariant subalgebra, $\mathcal{U}(\mathfrak{g})^K$. It is well established that $\mathcal{U}(\mathfrak{g})^K$ has an extraordinarily complicated ring structure, while at the same time, of fundamental importance to the structure of admissible representations of real reductive groups. We present results on the stability of the Hilbert series of the associated graded K -invariant subalgebra. Namely, we provide results that the stable Hilbert series of one symmetric pair coincides with that of another. As a consequence of the PBW theorem, we have $\mathcal{U}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})$ (the associated graded algebra) as a G -representation. Note that here $\mathcal{S}(\cdot)$ denotes the symmetric algebra.

In light of this fact we will consider the algebra $\mathcal{S}(\mathfrak{g})^K$, which even though commutative, also evades a simple presentation in terms of generators and relations in most examples. Specifically, we consider classical symmetric pairs. Equivalently, we consider the situation where \mathfrak{g} is the complexification of the Lie algebra of the real form, G_0 , of a classical linear algebraic group over \mathbb{C} . The group K will be the complex group corresponding to the maximal compact subgroup of G_0 . The ten families that we consider are described in Section 1.1.

This paper consists of four sections. In this first section we set up basic background material and notation in order to state the main result in Section 1.7. In Section 1.8 we interpret the main theorem as a "stable limit". In Section 2 we recall the necessary results to establish the main theorem, which is proved in Section 3. In Section 4 we show how one

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can obtain a closed expression for the Hilbert series of $\mathcal{U}(\mathfrak{g})^K$ in the case of $U(p, q)$ when $\min(p, q) \rightarrow \infty$.

In order to state the main theorem we set up some standard notation and review relevant results. These included: Symmetric pairs and real forms (1.1), Hilbert series (1.2), Partitions (1.3), Irreducible representations of $GL_n(\mathbb{C})$ (1.4), Multiplicity (1.5), and Littlewood-Richardson coefficients (1.6).

1.1. Notation for symmetric pairs and the real forms of the classical groups. We work in the context of linear algebraic groups over the complex numbers, but the motivation for the problem comes from the real forms of these groups. To avoid any possible ambiguity, we recall the relevant background material, which can be found in [GW98], Chapters 1, 11 and 12. To begin with, a linear algebraic group, G , is a Zariski closed subgroup of $GL_k(\mathbb{C})$. We let $\mathcal{O}(G)$ denote the algebra of regular functions on G . In this paper, the most general such groups we consider are (products of) the three classical groups $GL_n(\mathbb{C})$, $O_n(\mathbb{C})$ and $Sp_{2m}(\mathbb{C})$. Unless otherwise stated we take the forms for $O_n(\mathbb{C})$ and $Sp_{2m}(\mathbb{C})$ to be the usual $(x, y) = xy^t$ and $(x, y) = xJ_my^t$ respectively, where:

$$J_m = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix},$$

with I_m being the $m \times m$ identity matrix.

By a *symmetric pair* we mean an ordered pair (G, K) such that G is reductive complex linear algebraic group and K consists of the fixed points of a *regular* involution, θ , of G . Consequently, the group K will be a complex reductive linear algebraic group.

If G is a complex algebraic group, a *complex conjugation* on G is an abstract group involution τ , of G such that $f^\tau \in \overline{\mathcal{O}(G)}$ for all $f \in \mathcal{O}(G)$, where f^τ is the (complex valued) function on G defined by $f^\tau(g) = \overline{f(\tau(g))}$, for all $g \in G$ (see [GW98] Section 1.4.2).

A *real form* of a complex reductive linear algebraic group G is a subgroup $G_0 \subseteq G$ such that there exists a complex conjugation, τ , such that $G_0 = \{g \in G | \tau(g) = g \text{ for all } g \in G\}$. In general, a real form does not have the structure of a complex linear algebraic group, however it is a standard fact that the real forms of G correspond to the symmetrically embedded subgroups of G (see [GW98]). If G_0 is a real form of G , then G is the *complexification* of G_0 , while K is the complexification of a maximal compact subgroup, K_0 .

For our purposes, we will only be interested in the relevant symmetric pair corresponding to a real form G_0 . In order to set up a convenient notation, we will refer to the symmetric pairs by “standard” notation for the corresponding real form as in the three tables below:

G_0	G	K	$\theta : G \rightarrow G$
$U(p, q)$	$GL_{p+q}(\mathbb{C})$	$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$	$g \mapsto I_{p,q} g I_{p,q}$
$GL(n, \mathbb{R})$	$GL_n(\mathbb{C})$	$O_n(\mathbb{C})$	$g \mapsto (g^{-1})^t$
$GL(m, \mathbb{H})$	$GL_{2m}(\mathbb{C})$	$Sp_{2m}(\mathbb{C})$	$g \mapsto -J_m (g^{-1})^t J_m$

where

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.$$

In the first column in the above table are three real forms of the complex general linear group. We have indicated the corresponding symmetric pair in the second and third columns. In the fourth column we have indicated the involution θ that defines the embedding of K in G .

The groups $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ and $GL(n, \mathbb{H})$ denote the groups of $n \times n$ invertible matrices with entries from the set of real numbers, complex numbers, and quaternions respectively. We denote the indefinite unitary groups by $U(p, q)$ as usual. Next we have:

G_0	G	Defining Form	K	θ
$SO^*(2m)$	$SO_{2m}(\mathbb{C})$	D_m	$GL_m(\mathbb{C})$	$g \mapsto -J_m(g^{-1})^t J_m$
$Sp(m, \mathbb{R})$	$Sp_{2m}(\mathbb{C})$	J_m	$GL_m(\mathbb{C})$	$g \mapsto D_m(g^{-1})^t D_m$
$O(p, q)$	$O_{p+q}(\mathbb{C})$	I_{p+q}	$O_p(\mathbb{C}) \times O_q(\mathbb{C})$	$g \mapsto I_{p,q} g I_{p,q}$
$Sp(p, q)$	$Sp_{2(p+q)}(\mathbb{C})$	C_{p+q}	$Sp_{2p}(\mathbb{C}) \times Sp_{2q}(\mathbb{C})$	$g \mapsto I_{2p,2q} g I_{2p,2q}$

where

$$D_m = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \text{ and } C_m = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_1 \end{bmatrix} \text{ (} m \text{ copies of } J_1 \text{).}$$

(Recall: $J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.) The defining form for G is $(x, y) = xMy^t$ with M the square matrix denoted in the table.

Note the following notation convention in the case of the symplectic group. The subscript n of $Sp_n(\mathbb{C})$ refers to the dimension of the defining representation, while the m in $Sp(m)$ refers to the rank (ie: dimension of a maximal torus). We have $n = 2m$. We use this convention since it is used in [GW98]. In our notation, $Sp(m, \mathbb{R})$ will denote the split real form of $Sp_{2m}(\mathbb{C})$. We denote the compact form as $Sp(m)$ (without the \mathbb{R}).

Among the four groups in the above table are $Sp(m, \mathbb{R})$, the indefinite orthogonal, $O(p, q)$ and symplectic groups, $Sp(p, q)$, and the group $SO^*(2m)$, which is defined as:

$$SO^*(2m) = \{g \in SO_{2m}(\mathbb{C}) \mid -J_m \bar{g} J_m = g\}$$

where $g \mapsto \bar{g}$ denotes complex conjugation of the matrix entries of g . Note that this definition (as well as all the others) can be found in a new version of Chapter 1 of [GW98] (See: <http://www.math.ucsd.edu/~nwallach/newch1-g-w.pdf>).

Lastly, we will consider:

G_0	G	K
$GL(n, \mathbb{C})$	$GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$	$GL_n(\mathbb{C})$
$O(n, \mathbb{C})$	$O_n(\mathbb{C}) \times O_n(\mathbb{C})$	$O_n(\mathbb{C})$
$Sp(m, \mathbb{C})$	$Sp_{2m}(\mathbb{C}) \times Sp_{2m}(\mathbb{C})$	$Sp_{2m}(\mathbb{C})$

where the involution in all three cases is $(g_1, g_2) \mapsto (g_2, g_1)$ with $(g_1, g_2) \in G$.

The complex groups $GL_n(\mathbb{C})$, $O_n(\mathbb{C})$ and $Sp_{2m}(\mathbb{C})$ are simultaneously linear algebraic groups and real Lie groups. The fact that they may be viewed in these two situations could cause a certain ambiguity which we resolve as follows: When we write $GL_n(\mathbb{C})$ (resp. $O_n(\mathbb{C})$, $Sp_{2m}(\mathbb{C})$) we refer to the complex algebraic group, while $GL(n, \mathbb{C})$ (resp. $O(n, \mathbb{C})$, $Sp(m, \mathbb{C})$) will refer to the real Lie group. Certainly they are the same as an abstract groups, but the paper is more easily understood if we keep track of the categories that contain these objects.

To summarize, we will refer to these ten real forms of the classical groups. In seven non-complex cases, the complex conjugation defining G_0 may be taken as $\tau(g) = \theta((\bar{g}^t)^{-1})$, while in the three complex cases $\tau(g_1, g_2) = ((\bar{g}_2^t)^{-1}, (\bar{g}_1^t)^{-1})$.

1.2. **Hilbert Series.** In Section 1.8 it will be convenient to state an interpretation of the main result in terms of Hilbert series, so we introduce the following notation:

Definition. Given a symmetric pair (G, K) , let $\mathfrak{g} = \text{Lie}(G)$ and set:

$$\text{HS}(\mathfrak{g}, K; t) := \sum_{d=0}^{\infty} h_d(\mathfrak{g}, K) t^d \text{ where: } h_d(\mathfrak{g}, K) := \dim [\mathcal{S}^d(\mathfrak{g})]^K \text{ (for } d \in \mathbb{N}\text{)}.$$

When convenient we will also write $\text{HS}(G_0; t)$ for $\text{HS}(\mathfrak{g}, K; t)$ where G_0 is the corresponding real form.

For any one of the ten families addressed in this paper, it is a consequence of the main theorem (1.7) that for fixed d the numbers $h_d(\mathfrak{g}, K)$ stabilize for sufficiently large defining parameters. We set up some notation to more plainly indicate this phenomenon.

In the case of (say) $GL(n, \mathbb{C})$ it is convenient to write:

$$\lim \text{HS}(GL(n, \mathbb{C}); t) := \sum_{d=0}^{\infty} \left[\lim_{n \rightarrow \infty} h_d(GL(n, \mathbb{C})) \right] t^d$$

as the term-by-term limit. We will see that these limits exist from Equation 1.7.3. The resulting power series obtained in this way will be referred to as the “stable limit”. We use the same (analogous) notation for the other groups. In the cases where the group G_0 is indexed by two parameters (ie: $U(p, q)$, $O(p, q)$, $Sp(p, q)$) the stable limit corresponds to $\min(p, q) \rightarrow \infty$.

Using this notation, we can express the following corollary to the main theorem that we explain in Section 1.8:

$$\begin{aligned} \lim \text{HS}(GL(n, \mathbb{R})) &= \lim \text{HS}(Sp(m, \mathbb{R})) = \lim \text{HS}(GL(n, \mathbb{H})) = \lim \text{HS}(SO^*(2m)) \\ \lim \text{HS}(O(n, \mathbb{C})) &= \lim \text{HS}(Sp(p, q)) = \lim \text{HS}(Sp(m, \mathbb{C})) = \lim \text{HS}(O(p, q)) \\ \lim \text{HS}(GL(n, \mathbb{C})) &= \lim \text{HS}(U(p, q)) \end{aligned}$$

Establishing the above equalities are one of the purposes of this paper.

1.3. **Notation for partitions.** A partition, λ , with k parts, is a positive integer sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$. The number of parts of a partition is called the *length* of the partition, while the sum of the parts is called the *size* of the partition. In general, we use the same notation for partitions as is done in standard references such as [Mac95, Sta97, Sta99]. For example, we write $\ell(\lambda)$ to denote the length (or number of parts) of a partition, $|\lambda|$ for the size of a partition (i.e., $|\lambda| = \sum_i \lambda_i$). If λ has size m then we will write $\lambda \vdash m$. Also, λ' denotes the transpose (or conjugate) of λ (i.e., $(\lambda')_i = |\{\lambda_j : \lambda_j \geq i\}|$). Note that $\ell(\lambda') = \lambda_1$.

We implicitly identify a partition with its Young diagram. Thus, many terms (such as the conjugate partition) have clear meanings. For example, we will say that λ has *even rows* if it is of the form: $2\delta_1 \geq 2\delta_2 \geq \dots \geq 2\delta_k$, which we denote by: $\lambda = 2\delta$. In the same way, we say that λ has *even columns* if $\lambda = (2\delta)'$ for some partition δ . These partitions play a fundamental role for us.

It will be very important to keep track of the length of the partitions that arise in this paper. We shall see that one innocent, but key fact is that $\ell(\lambda) \leq |\lambda|$. Also, if λ is of the form 2δ then $\ell(\lambda) = \ell(\delta)$, while $|\lambda| = 2|\delta|$. Thus $\ell(\lambda) \leq 2|\delta|$. On the other hand, if λ is of the form $(2\delta)'$ then $\ell(\lambda) \leq 2|\delta|$, but we cannot say anything about the relationship between $\ell(\lambda)$ and $\ell(\delta)$.

If λ denotes a partition of length k and $n \geq k$, then λ defines an n -tuple of non-negative integers by padding out with zeros on the right end (ie: $(\lambda_1, \lambda_2, \dots, \lambda_k, 0, 0, \dots, 0)$). In this way, we have an injection of the set of partitions of length at most n into the lattice \mathbb{Z}^n .

Another, slightly non-standard notation that we will use involves embedding *pairs* of partitions into \mathbb{Z}^n . For a positive integer n , let (μ, ν) denote the n -tuple determined by partitions μ and ν by:

$$(1.3.1) \quad (\mu, \nu) := \underbrace{(\mu_1, \mu_2, \dots, \mu_p, 0, \dots, 0, -\nu_q, \dots, -\nu_1)}_n$$

where we assume that $\ell(\mu) = p$, $\ell(\nu) = q$ and $p + q \leq n$. It is clear that every weakly decreasing n -tuple of integers corresponds uniquely to an ordered pair of partitions in this way.

1.4. Notation for the irreducible finite dimensional representations. We now fix a notation for irreducible finite dimensional representations of $GL_n(\mathbb{C})$. To be precise, we will view these groups in the category of linear algebraic groups over \mathbb{C} with morphisms being homomorphisms of the groups which are regular maps of the underlying affine varieties. In particular, we consider only regular representations.¹

For connected groups, such representations are parameterized by dominant integral characters of a maximal torus using the theorem of the highest weight. In the classical cases, weights are parameterized in the standard coordinates (see [GW98]) using n -tuples of integers.

If a representation of $GL_n(\mathbb{C})$ has polynomial matrix coefficients then the n -tuple indexing the highest weight of the representation has non-negative integer components. Therefore, we may index this n -tuple by a partition λ such that $\ell(\lambda) \leq n$.

In general, the matrix coefficients of a representation of $GL_n(\mathbb{C})$ are not polynomials, but rather, are rational functions in the matrix entries. For these representations, the highest weight is indexed by an n -tuple with negative components. In light of this, we set up the following notation which will be used in Section 4.

Given non-negative integers p and q such that $n \geq p + q$ and non-negative integer partitions α and β with p and q parts respectively, let $F_{(n)}^{(\alpha, \beta)}$ denote the irreducible rational representation of GL_n with highest weight given by the n -tuple (α, β) (see Equation 1.3.1). If $\beta = (0)$ (as is the case for polynomial representations) then we will write $F_{(n)}^\alpha$ for $F_{(n)}^{(\alpha, \beta)}$. This will be our notation for representations with polynomial matrix coefficients.

Note that $(F_{(n)}^\alpha)^*$ is equivalent to $F_{(n)}^{(0, \alpha)}$. In general, the dual of the representation corresponding to (α, β) has a highest weight which is given by the n -tuple (β, α) in the standard coordinates.

If we choose the standard basis for \mathbb{C}^n we obtain a representation of $GL_n(\mathbb{C})$ given by the identity map $GL_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$, which we will call the *standard representation*, and is denoted $F_{(n)}^{(1, (0))}$ in our notation. We will refer to “the” standard representation later on in the paper – this is what we mean.

¹In other sources, a *regular representation* is referred to as a *rational representation*, but with the same meaning as we indicate here.

If $G = GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$ then representatives of the irreducible representations of G are of the form $F_{(n)}^{(\mu_1, \mu_2)} \widehat{\otimes} F_{(m)}^{(\nu_1, \nu_2)}$. Note that we use the symbol $\widehat{\otimes}$ to denote the “outer tensor product”, which we view as a representation of G . If $k := n = m$ then we may restrict this representation to the diagonal $GL_k(\mathbb{C})$; we denote this representation $F_{(k)}^{(\mu_1, \mu_2)} \otimes F_{(k)}^{(\nu_1, \nu_2)}$ where the \otimes denotes the “inner” tensor product of the $GL_k(\mathbb{C})$ -representations.

1.5. Notation for multiplicities. Given completely reducible representations, V_1 and V_2 of complex algebraic groups G_1 and G_2 respectively, together with an embedding $G_1 \hookrightarrow G_2$, we let

$$[V_1, V_2] = \dim \operatorname{Hom}_{G_1}(V_1, V_2)$$

where V_2 is regarded as a representation of G_1 by restriction. If V_1 is irreducible, then $[V_1, V_2]$ is the multiplicity of V_1 in V_2 . This cardinal may of course be infinite if V_1 or V_2 is infinite dimensional. For most of the paper, the restriction to G_1 will be implicit, but we mention it here to be precise.

If G is a reductive algebraic group over \mathbb{C} , let $(V^\lambda)_{\lambda \in \widehat{G}}$ denote representatives of irreducible regular representations of G . If V is a completely reducible representation of G , then set $m_\lambda := [V^\lambda, V]$. In this paper, we will always have $m_\lambda < \infty$, and we will indicate the decomposition of V into irreducible representations of G (with multiplicity) by the expression:

$$V \cong \bigoplus m_\lambda V^\lambda,$$

which of course is shorthand for:

$$V \cong \bigoplus_{\lambda \in \widehat{G}} \underbrace{V^\lambda \oplus V^\lambda \oplus \dots \oplus V^\lambda}_{m_\lambda \text{ copies.}}$$

1.6. Littlewood-Richardson coefficients. In [Mac95] the Littlewood-Richardson coefficients are defined as the structure constants for multiplication in the ring of symmetric polynomials at the Schur basis. That is,

$$s_\mu s_\nu = \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda$$

where s_γ denotes the Schur function indexed by the partition γ .

For most of [Mac95] one works in a ring having infinitely many variables, so there is no restriction on the number of parts of λ , μ and ν . If one passes to finitely many variables the only change that needs to be made is to make sure that all partitions involved do not have more parts than the number of variables. Keeping track of this caveat, one can interpret the Schur function as the character of a representation of $GL_n(\mathbb{C})$. We obtain from this interpretation:

Proposition 1.1. *If we regard the irreducible $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ representation $F_{(n)}^\mu \widehat{\otimes} F_{(n)}^\nu$ as a representation of the diagonally embedded $GL_n(\mathbb{C})$ by restriction then the decomposition into irreducibles is given by:*

$$F_{(n)}^\mu \otimes F_{(n)}^\nu \cong \bigoplus c_{\mu\nu}^\lambda F_{(n)}^\lambda$$

where the sum is over all partitions λ such that $|\lambda| = |\mu| + |\nu|$ and $\ell(\lambda) \leq n$.

In [GW98] and [How95] the following standard result is proved, which we will need in the proof of the main theorem.

Proposition 1.2. *If we regard the irreducible $GL_{n+m}(\mathbb{C})$ -representation as a $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$ -representation by restriction under the embedding:*

$$\left[\begin{array}{cc} GL_n(\mathbb{C}) & 0 \\ 0 & GL_m(\mathbb{C}) \end{array} \right] \subset GL_{n+m}(\mathbb{C}),$$

then we have:

$$F_{(n+m)}^\lambda \cong \bigoplus_{\mu, \nu} c_{\mu\nu}^\lambda F_{(n)}^\mu \widehat{\otimes} F_{(m)}^\nu,$$

where the sum is over all partitions μ and ν such that $\ell(\mu) \leq n$ and $\ell(\nu) \leq m$.

1.7. Main Theorem. We proceed in cases:

CASE $G_0 = U(p, q)$:

For $\mathfrak{g} = \mathfrak{gl}_{p+q}(\mathbb{C})$ and $K = GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$,

$$(1.7.1) \quad h_d(\mathfrak{g}, K) = \sum (c_{\mu\nu}^\lambda)^2$$

where the sum is over all partitions λ , μ , and ν such that: $|\lambda| = |\mu| + |\nu| = d$, $\ell(\lambda) \leq p + q$, $\ell(\mu) \leq p$ and $\ell(\nu) \leq q$.

CASE $G_0 = GL(n, \mathbb{R})$:

For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $K = O_n(\mathbb{C})$,

$$(1.7.2) \quad h_d(\mathfrak{g}, K) = \sum c_{\mu\mu}^{2\lambda}$$

where the sum is over all partitions λ , μ such that: $|\lambda| = |\mu| = d$ and $\ell(\lambda), \ell(\mu) \leq n$.

CASE $G_0 = GL(n, \mathbb{C})$:

For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $K = GL_n(\mathbb{C})$,

$$(1.7.3) \quad h_d(\mathfrak{g}, K) = \sum (c_{\mu\nu}^\lambda)^2$$

where the sum is over all partitions λ , μ , and ν such that: $|\lambda| = |\mu| + |\nu| = d$, $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$.

CASE $G_0 = GL(m, \mathbb{H})$:

For $\mathfrak{g} = \mathfrak{gl}_{2m}(\mathbb{C})$ and $K = Sp_{2m}(\mathbb{C})$,

$$(1.7.4) \quad h_d(\mathfrak{g}, K) = \sum c_{\mu\mu}^{(2\lambda)'}$$

where the sum is over all partitions λ , μ such that: $|\lambda| = |\mu| = d$ and $\ell((2\lambda)'), \ell(\mu) \leq 2m$.

CASE $G_0 = SO^*(2m)$:

For $\mathfrak{g} = \mathfrak{so}_{2m}(\mathbb{C})$ and $K = GL_m(\mathbb{C})$,

$$(1.7.5) \quad h_d(\mathfrak{g}, K) = \sum c_{\mu\mu}^{(2\lambda)'}$$

where the sum is over all partitions λ , μ such that: $|\lambda| = |\mu| = d$ and $\ell((2\lambda)'), \ell(\mu) \leq 2m$, and $\ell(\mu) \leq m$.

CASE $G_0 = Sp(m, \mathbb{R})$:

For $\mathfrak{g} = \mathfrak{sp}_{2m}(\mathbb{C})$ and $K = GL_m(\mathbb{C})$,

$$(1.7.6) \quad h_d(\mathfrak{g}, K) = \sum c_{\mu\mu}^{2\lambda}$$

where the sum is over all partitions λ, μ such that: $|\lambda| = |\mu| = d$ and $\ell(\lambda) \leq 2m$, and $\ell(\mu) \leq m$.

CASE $G_0 = O(n, \mathbb{C})$:

For $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C}) \oplus \mathfrak{so}_n(\mathbb{C})$ and $K = O_n(\mathbb{C})$,

$$(1.7.7) \quad h_d(\mathfrak{g}, K) = \sum c_{(2\mu)'}^{2\lambda} c_{(2\nu)'}$$

where the sum is over all partitions λ, μ , and ν such that: $|\lambda| = |\mu| + |\nu| = d$ and $\ell(\lambda), \ell((2\mu)'), \ell((2\nu)') \leq n$.

CASE $G_0 = Sp(m, \mathbb{C})$:

For $\mathfrak{g} = \mathfrak{sp}_{2m}(\mathbb{C}) \oplus \mathfrak{sp}_{2m}(\mathbb{C})$ and $K = Sp_{2m}(\mathbb{C})$,

$$(1.7.8) \quad h_d(\mathfrak{g}, K) = \sum c_{2\mu 2\nu}^{(2\lambda)'}$$

where the sum is over all partitions λ, μ such that: $|\lambda| = |\mu| + |\nu| = d$ and $\ell((2\lambda)'), \ell(\mu), \ell(\nu) \leq 2m$.

CASE $G_0 = O(p, q)$:

For $\mathfrak{g} = \mathfrak{so}_{p+q}(\mathbb{C})$ and $K = O_p(\mathbb{C}) \times O_q(\mathbb{C})$,

$$(1.7.9) \quad h_d(\mathfrak{g}, K) = \sum c_{2\mu 2\nu}^{(2\lambda)'}$$

where the sum is over all partitions λ, μ , and ν such that: $|\lambda| = |\mu| + |\nu| = d$ and $\ell((2\lambda)') \leq p + q, \ell(\mu) \leq p, \ell(\nu) \leq q$.

CASE $G_0 = Sp(p, q)$:

For $\mathfrak{g} = \mathfrak{sp}_{2p+2q}(\mathbb{C})$ and $K = Sp_{2p}(\mathbb{C}) \times Sp_{2q}(\mathbb{C})$,

$$(1.7.10) \quad h_d(\mathfrak{g}, K) = \sum c_{(2\mu)'}^{2\lambda} c_{(2\nu)'}$$

where the sum is over all partitions λ, μ , and ν such that: $|\lambda| = |\mu| + |\nu| = d$ and $\ell(\lambda) \leq 2(p + q), \ell((2\mu)') \leq 2p, \ell((2\nu)') \leq 2q$.

Proof. See Section 3 for a case-by-case analysis. □

Remark 1.3. Note that the cases $GL(n, \mathbb{R})$ and $GL(n, \mathbb{H})$ are also addressed in [Wil00] and then more thoroughly in [Wil01].

1.8. Stability properties. In each of the ten cases addressed in the main theorem the summation range involves a condition on both the *size* and the *length* of the partitions involved. The parameter describing the bound for the length is independent of the parameter for the size. For fixed size, as the length parameter goes to infinity the condition on the length is automatic due to the fact that for a partition λ , $\ell(\lambda) \leq |\lambda|$. This innocent combinatorial fact allows us to conclude the following stable behavior for a fixed d :

- Examining Equations 1.7.2 and 1.7.6 we see that:

$$(1.8.1) \quad h_d(GL(n, \mathbb{R})) = h_d(Sp(m, \mathbb{R}))$$

provided $d \leq \min(n, m)$;

- Examining Equations 1.7.4 and 1.7.5 we see that:

$$(1.8.2) \quad h_d(GL(n, \mathbb{H})) = h_d(SO^*(2m))$$

provided $d \leq \min(n, m)$;

- Examining Equations 1.7.7 and 1.7.10 we see that:

$$(1.8.3) \quad h_d(O(n, \mathbb{C})) = h_d(Sp(p, q))$$

provided $d \leq \min(\frac{n}{2}, p, q)$;

- Examining Equations 1.7.8 and 1.7.9 we see that:

$$(1.8.4) \quad h_d(Sp(m, \mathbb{C})) = h_d(O(p, q))$$

provided $d \leq \min(m, \frac{p}{2}, \frac{q}{2})$;

- Examining Equations 1.7.3 and 1.7.1 we see that:

$$(1.8.5) \quad h_d(GL(n, \mathbb{C})) = h_d(U(p, q))$$

provided $d \leq \min(n, p, q)$.

A direct consequence of the above equalities are the following stable limits:

$$(1.8.6) \quad \lim HS(GL(n, \mathbb{R})) = \lim HS(Sp(m, \mathbb{R}))$$

$$(1.8.7) \quad \lim HS(GL(n, \mathbb{C})) = \lim HS(U(p, q))$$

$$(1.8.8) \quad \lim HS(GL(n, \mathbb{H})) = \lim HS(SO^*(2m))$$

$$(1.8.9) \quad \lim HS(O(n, \mathbb{C})) = \lim HS(Sp(p, q))$$

$$(1.8.10) \quad \lim HS(Sp(m, \mathbb{C})) = \lim HS(O(p, q))$$

Certainly, one might expect that these formulas follow for other arguments involving, for example, theory related to non-compact dual pairs (see [How89a, How89b]). Such an approach would indeed be very interesting.

From the point of view presented here, we can exploit combinatorial facts concerning the Littlewood-Richardson coefficients. For example, in [Mac95] and [Sta99] the following well know fact is presented:

Proposition 1.4. *For any partitions α, β, γ we have: $c_{\alpha\beta}^{\gamma} = c_{\alpha'\beta'}^{\gamma'}$.*

Consequently, we obtain that the sums in Equations 1.7.2, 1.7.4, 1.7.5, and 1.7.6 are all equal for sufficiently large defining parameters. Moreover, we see another equality between Equations 1.7.7, 1.7.8, 1.7.9, 1.7.10. Thus, the ten cases represent only three:

$$\lim HS(GL(n, \mathbb{R})) = \lim HS(Sp(m, \mathbb{R})) = \lim HS(GL(n, \mathbb{H})) = \lim HS(SO^*(2m))$$

$$\lim HS(O(n, \mathbb{C})) = \lim HS(Sp(p, q)) = \lim HS(Sp(m, \mathbb{C})) = \lim HS(O(p, q))$$

$$\lim HS(GL(n, \mathbb{C})) = \lim HS(U(p, q))$$

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2. BACKGROUND FOR THE PROOFS.

Interestingly, we require relatively little machinery to prove the main theorem. In fact, the only results required are multiplicity free spaces for $GL_n(\mathbb{C})$ and (related) standard results concerning the Littlewood-Richardson coefficients. We state the necessary results here for reference.

2.1. **The Cartan-Helgason theorem.** One of the tools that we will use for the proof of the main theorem is the Cartan-Helgason theorem (see [GW98] or [How95]). This fact amounts to the assertion that if (G, K) is a symmetric pair then the G decomposition of regular functions on G/K is multiplicity free. The irreducible regular G -representations that occur in $\mathcal{O}(G/K)$ are precisely those representations with a K -invariant vector. We describe here four classical instances of this theorem, the first of which we describe as a remark since it is a consequence of Schur's lemma:

Remark 2.1. For any reductive group K , we may symmetrically embed K in $G := K \times K$ as $\{(k, k) | k \in K\}$. The irreducible regular representations of G may be taken as $V_1 \hat{\otimes} V_2$ where (ρ_1, V_1) and (ρ_2, V_2) are irreducible representations of K . We then obtain:

$$\dim (V_1 \otimes V_2)^K = \begin{cases} 1, & \text{if } V_1 \cong V_2^*; \\ 0, & \text{otherwise,} \end{cases}$$

by noting that $\text{Hom}_K(V_1, V_2) \cong (V_1^* \otimes V_2)^K$ and invoking Schur's lemma.

2.1.1. *Symmetric pair: $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$.* An irreducible representation of $GL_n(\mathbb{C})$ may be regarded as a representation of $O_n(\mathbb{C})$ by restriction. In so doing, we have the following

Theorem. *For an integer partition λ with $\ell(\lambda) \leq n$:*

$$\dim (F_{(n)}^\lambda)^{O_n(\mathbb{C})} = \begin{cases} 1, & \lambda_i \text{ even for all } i; \\ 0, & \text{otherwise.} \end{cases}$$

2.1.2. *Symmetric pair: $(GL_{2m}(\mathbb{C}), Sp_{2m}(\mathbb{C}))$.* As in the last case, an irreducible representation of $GL_{2m}(\mathbb{C})$ may be regarded as a representation of $Sp_{2m}(\mathbb{C})$ by restriction. We have the following

Theorem. For an integer partition λ with $\ell(\lambda) \leq 2m$:

$$\dim (F_{(2m)}^\lambda)^{Sp_{2m}(\mathbb{C})} = \begin{cases} 1, & (\lambda)_i \text{ even for all } i; \\ 0, & \text{otherwise.} \end{cases}$$

2.1.3. *Symmetric pair:* $(GL_{p+q}(\mathbb{C}), GL_p(\mathbb{C}) \times GL_q(\mathbb{C}))$. As in the last cases, an irreducible representation of $GL_{p+q}(\mathbb{C})$ may be regarded as a representation of $GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ by restriction where we embed $GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ in $GL_{p+q}(\mathbb{C})$ as:

$$\begin{bmatrix} GL_p(\mathbb{C}) & 0 \\ 0 & GL_q(\mathbb{C}) \end{bmatrix} \subset GL_{p+q}(\mathbb{C}).$$

We have the following

Theorem. Given $n = p + q$, let μ and ν be partitions with at most p and q parts respectively. Then,

$$\dim (F_{(n)}^{(\mu, \nu)})^{GL_p \times GL_q} = \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

2.2. **Classical Multiplicity Free Spaces for $GL_n(\mathbb{C})$.** We recall, in our notation, three celebrated multiplicity free representations for $GL_n(\mathbb{C})$. See for [GW98] or [How95] for proofs.

2.2.1. *The $k \times m$ Matrices.*

We define the following action of $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$ on the algebra of complex valued polynomial function on the $k \times m$ complex matrices. For $(g, h) \in GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$, $X \in M_{k,m}$ and $f \in \mathcal{P}(M_{k,m})$, set: $(g, h) \cdot f(X) = f(g^t X h)$. Under this action we can identify: $\mathcal{P}(M_{k,m}) \cong \mathcal{S}(\mathbb{C}^k \widehat{\otimes} \mathbb{C}^m)$ as a representation of the group $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$. For our purposes, the decomposition into irreducible representations is of particular interest:

Theorem 2.2. As a $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$ -representation,

$$\mathcal{S}^d(\mathbb{C}^k \widehat{\otimes} \mathbb{C}^m) \cong \bigoplus_{\substack{\lambda: |\lambda|=d \\ \ell(\lambda) \leq \min(k,m)}} F_{(n)}^\lambda \widehat{\otimes} F_{(m)}^\lambda.$$

We also state the following dual version of the above theorem. Define the action of $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$ on $\mathcal{P}(M_{k,m})$ by: for $(g, h) \in GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$, $X \in M_{k,m}$ and $f \in \mathcal{P}(M_{k,m})$, set: $(g, h) \cdot f(X) = f(g^{-1} X h)$. Under this action we can identify: $\mathcal{P}(M_{k,m}) \cong \mathcal{S}((\mathbb{C}^k)^* \widehat{\otimes} \mathbb{C}^m)$ as a representation of the group $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$.

Corollary 2.3. As a $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$ -representation,

$$\mathcal{S}\left((\mathbb{C}^k)^* \widehat{\otimes} \mathbb{C}^m\right) \cong \bigoplus_{\substack{\lambda: |\lambda|=d \\ \ell(\lambda) \leq \min(k,m)}} (F_{(k)}^\lambda)^* \widehat{\otimes} F_{(m)}^\lambda.$$

2.2.2. *The Symmetric Matrices.*

We define the following action of $GL_n(\mathbb{C})$ on the algebra of complex valued polynomial function on the symmetric $n \times n$ complex matrices, SM_n . For $g \in GL_n(\mathbb{C})$, $X \in SM_n$ and $f \in \mathcal{P}(SM_n)$, set: $g \cdot f(X) = f(g^t X g)$. Under this action we can identify: $\mathcal{P}(SM_n) \cong \mathcal{S}(\mathcal{S}^2 \mathbb{C}^n)$ as a representation of the group $GL_n(\mathbb{C})$. We have the multiplicity free decomposition:

Theorem. *As a $GL_n(\mathbb{C})$ -representation,*

$$\mathcal{S}^d(\mathcal{S}^2\mathbb{C}^n) \cong \bigoplus_{\substack{\lambda: |\lambda|=d \\ \ell(\lambda) \leq n}} F_{(n)}^{2\lambda}.$$

2.2.3. The Skew-Symmetric Matrices.

We define an analogous action of $GL_n(\mathbb{C})$ on the algebra of complex valued polynomial function on the skew-symmetric $n \times n$ complex matrices, AM_n . For $g \in GL_n(\mathbb{C})$, $X \in AM_n$ and $f \in \mathcal{P}(AM_n)$, set: $g \cdot f(X) = f(g^t X g)$. Under this action we can identify: $\mathcal{P}(AM_n) \cong \mathcal{S}(\wedge^2\mathbb{C}^n)$ as a representation of the group $GL_n(\mathbb{C})$. We have the multiplicity free decomposition:

Theorem. *As a $GL_n(\mathbb{C})$ -representation,*

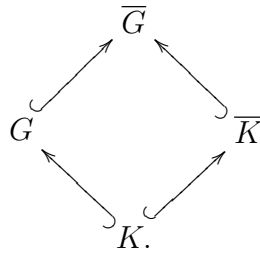
$$\mathcal{S}^d(\wedge^2\mathbb{C}^n) \cong \bigoplus_{\substack{\lambda: |\lambda|=d \\ \ell((2\lambda)') \leq n}} F_{(n)}^{(2\lambda)'}$$

3. PROOF OF THE MAIN THEOREM.

We now proceed to prove the main theorem. We do this in case-by-case analysis, however each is presented in the following manner: We start with the adjoint representation, \mathfrak{g} , of G , which gives rise to a representation of G on the degree d symmetric tensors, $\mathcal{S}^d(\mathfrak{g})$. Our goal is to compute the dimension of invariants when this representation is restricted to K ,

$$\begin{array}{c} G \rightarrow GL(\mathcal{S}^d(\mathfrak{g})) \\ \cup \\ K \end{array}$$

This representation may be directly analyzed using branching rules from G to K , but in general this is not the best route to the result. Instead, it will be convenient to realize that K is symmetrically embedded in another group \overline{K} which also acts on $\mathcal{S}^d(\mathfrak{g})$. Furthermore, there exists yet another group, \overline{G} in which both G and \overline{K} are symmetrically embedded. The picture is:



The idea is to exploit the existence of a certain graded, \overline{G} -multiplicity free representation, $\overline{\mathcal{S}} = \bigoplus \overline{\mathcal{S}}[d]$ which as a graded G -representation is equivalent to $\mathcal{S}(\mathfrak{g})$. This representation is of the form $\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W})$ for some \overline{G} -representation \widetilde{W} .

Understanding the graded K -module structure of $\mathcal{S}(\mathfrak{g})$ will be equivalent to understanding the graded K -module structure of $\overline{\mathcal{S}}$. We will see that it is technically much easier to first decompose relative to the \overline{K} action and then use a classical instance of the Cartan-Helgason theorem to determine the dimension of the space of K -invariants.

3.1. **CASE:** $G_0 = U(p, q)$. Let $n := p + q$.

$$\boxed{G = GL_n(\mathbb{C}) \quad | \quad K \cong GL_p(\mathbb{C}) \times GL_q(\mathbb{C})}$$

The group K is symmetrically embedded in G in the standard way as:

$$K := \begin{bmatrix} GL_p(\mathbb{C}) & 0 \\ 0 & GL_q(\mathbb{C}) \end{bmatrix} \subset G$$

We then define \overline{G} and \overline{K} as:

$$\begin{array}{ccc} \overline{G} := G \times G & \supset & \overline{K} := K \times K \\ \cup & & \cup \\ G & \supset & K \end{array}$$

with the obvious vertical diagonal embeddings.

Let $W = \mathbb{C}^n$ denote the standard representation of G . Under the adjoint representation of G we have: $\mathfrak{g} \cong W^* \otimes W$. This representation is the restriction of the irreducible \overline{G} -representation $\widehat{W} \cong W^* \widehat{\otimes} W$. Let $\mathcal{S} := \mathcal{S}(\widehat{W})$. By Corollary 2.3, we have:

$$(3.1.1) \quad \mathcal{S}^d(\widehat{W}) \cong \bigoplus_{\substack{\lambda: |\lambda|=d \\ \ell(\lambda) \leq n}} (F_{(n)}^\lambda)^* \widehat{\otimes} F_{(n)}^\lambda$$

under the action of \overline{G} . We will exploit this fact by restricting to the subgroup, $\overline{K} \subseteq \overline{G}$. We then apply Proposition 1.2 to each occurrence of $F_{(p+q)}^\lambda$ to obtain the following decomposition of the $\mathcal{S}^d(\widehat{W})$ as a \overline{K} -representation,

$$\mathcal{S}^d(W^* \widehat{\otimes} W) \cong \bigoplus \left(c_{\mu\nu}^\lambda F_{(p)}^\mu \widehat{\otimes} F_{(q)}^\nu \right)^* \otimes \left(c_{\alpha\beta}^\lambda F_{(p)}^\alpha \widehat{\otimes} F_{(q)}^\beta \right)$$

where the above sum is over α, β, μ, ν and λ such that:

$$(3.1.2) \quad \begin{array}{l} \ell(\mu), \ell(\alpha) \leq p \\ \ell(\nu), \ell(\beta) \leq q \\ \ell(\lambda) \leq n, \text{ and } |\lambda| = d. \end{array}$$

Schur's lemma (see Section 2.1) implies that we will have a K -invariant exactly when $\mu = \alpha$ and $\nu = \beta$. And so,

$$\dim \mathcal{S}^d(\mathfrak{g})^K = \sum (c_{\mu\nu}^\lambda)^2 \dim \left[\left(F_{(p)}^\mu \widehat{\otimes} F_{(q)}^\nu \right)^* \widehat{\otimes} F_{(p)}^\mu \widehat{\otimes} F_{(q)}^\nu \right]^K$$

where the sum is over all partitions λ, μ , and ν as in Line 3.1.2. Again by Schur's lemma we see that:

$$\dim \left[\left(F_{(p)}^\mu \widehat{\otimes} F_{(q)}^\nu \right)^* \widehat{\otimes} F_{(p)}^\mu \widehat{\otimes} F_{(q)}^\nu \right]^K = 1. \text{ Equation 1.7.1 follows.}$$

3.2. **CASE:** $G_0 = GL(n, \mathbb{R})$.

$$\boxed{G = GL_n(\mathbb{C}) \quad | \quad K \cong O_n(\mathbb{C})}$$

The group K is symmetrically embedded in G in the standard way as:

$$K := \{g \in G \mid g^{-1} = g^t\} \subset G.$$

We then define \overline{G} and \overline{K} as:

$$\begin{array}{lcl} \overline{G} & := & G \times G & \supset & \overline{K} & := & \{(g, g) | g \in G\} \\ \cup & & & & \cup & & \\ G' & := & \{((g^{-1})^t, g) | g \in G\} & \supset & K' & := & \{(k, k) | k \in K\}. \end{array}$$

Note that the primes are indicated to describe the symmetric embeddings of G and K precisely. In particular, note that we have embedded $GL_n(\mathbb{C})$ in \overline{G} in two distinct ways:

$$GL_n(\mathbb{C}) \cong \overline{K} := \{(g, g) | g \in GL_n(\mathbb{C})\} \subset \overline{G}$$

and

$$GL_n(\mathbb{C}) \cong G := \{((g^{-1})^t, g) | g \in GL_n(\mathbb{C})\} \subset \overline{G}.$$

As before, let $W = \mathbb{C}^n$ denote the standard representation of $GL_n(\mathbb{C})$.

Under the adjoint representation of G we have $\mathfrak{g} \cong W^* \otimes W$, which is the restriction of the irreducible representation $W^* \widehat{\otimes} W$ of \overline{G} . Note that under restriction to K , $W \cong W^*$ since the regular representations of $O_n(\mathbb{C})$ are self-dual. Let $\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W})$ where $\widetilde{W} := W \widehat{\otimes} W$. As a representation of K , $\mathcal{S}(\mathfrak{g}) \cong \overline{\mathcal{S}}$. By Theorem 2.2 we have:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus_{\substack{\mu: |\mu|=d \\ \ell(\mu) \leq n}} F_{(n)}^\mu \widehat{\otimes} F_{(n)}^\mu \text{ as a } \overline{G}\text{-representation.}$$

We now restrict to the diagonal subgroup of \overline{K} . In doing this, the irreducible representations $F_{(n)}^\mu \widehat{\otimes} F_{(n)}^\mu$ decompose into irreducible representations of \overline{K} according to the Littlewood-Richardson rule (see Proposition 1.1). For a partition ν , the $GL_m(\mathbb{C})$ -irrep. $F_{(n)}^\nu$ occurs in $\mathcal{S}(W \otimes W)$ with multiplicity $c_{\mu\mu}^\nu$. Thus,

$$\mathcal{S}^d(\mathfrak{g})^K \cong \mathcal{S}^d(W \otimes W)^{O_n(\mathbb{C})} \cong \bigoplus c_{\mu\mu}^\nu (F_{(n)}^\nu)^{O_n(\mathbb{C})}$$

where μ and ν are partitions with $\ell(\mu), \ell(\nu) \leq n$ and $|\nu| = 2|\mu| = 2d$. The Equation 1.7.2 follows as an application of an instance of the Cartan-Helgason Theorem (see Section 2.1.1). That is, we obtain a K -invariant for each ν of the form 2λ where λ is a partition with length at most n .

3.3. CASE: $G_0 = GL(n, \mathbb{C})$.

$G = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$	$K \cong GL_n(\mathbb{C})$
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The group K is diagonally embedded in G . We define \overline{G} and the embedding of G as follows:

$$\begin{array}{lcl} \overline{G} & := & G \times G \cong \times_{i=1}^4 GL_n(\mathbb{C}) \\ \cup & & \\ G' & := & \{(g, g, h, h) | g, h \in GL_n(\mathbb{C})\} \cong G. \end{array}$$

We embed a group \overline{K} in \overline{G} as:

$$\begin{array}{lcl} \overline{G} & \supset & \overline{K} := \{(g, h, g, h) | g, h \in GL_n(\mathbb{C})\} \\ & & \cup \\ K & \cong & K' := \{(k, k, k, k) | k \in GL_n(\mathbb{C})\}. \end{array}$$

Let W denote the standard representation of $GL_n(\mathbb{C})$. The adjoint representation of $GL_n(\mathbb{C})$ is $\mathfrak{gl}_n(\mathbb{C}) = W^* \otimes W$, and so the adjoint representation of $G = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ is

$$\begin{aligned} \mathfrak{g} &= (\mathfrak{gl}_n(\mathbb{C}) \widehat{\otimes} 1_{\mathbb{C}}) \oplus (1_{\mathbb{C}} \widehat{\otimes} \mathfrak{gl}_n(\mathbb{C})) \\ &\cong ((W^* \otimes W) \widehat{\otimes} 1_{\mathbb{C}}) \oplus (1_{\mathbb{C}} \widehat{\otimes} (W^* \otimes W)) && \text{as a } G\text{-representation;} \\ &\cong (W^* \otimes W) \oplus (W^* \otimes W) && \text{as a } K\text{-representation.} \end{aligned}$$

where $1_{\mathbb{C}}$ denotes the trivial representation of $GL_n(\mathbb{C})$.

We obtain a representation, $\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W})$ of \overline{G} where:

$$\widetilde{W} := (W^* \widehat{\otimes} W \widehat{\otimes} 1_{\mathbb{C}} \widehat{\otimes} 1_{\mathbb{C}}) \oplus (1_{\mathbb{C}} \widehat{\otimes} 1_{\mathbb{C}} \widehat{\otimes} W^* \widehat{\otimes} W).$$

As in the other cases, $\overline{\mathcal{S}} \cong \mathcal{S}(\mathfrak{g})$ as a graded G -representation.

Using Corollary 2.3 on each summand of \widetilde{W} we obtain the following multiplicity free decomposition:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus \left(F_{(n)}^\mu \right)^* \widehat{\otimes} F_{(n)}^\mu \widehat{\otimes} \left(F_{(n)}^\nu \right)^* \widehat{\otimes} F_{(n)}^\nu$$

where the sum is over all partitions μ and ν with $\ell(\mu), \ell(\nu) \leq n$ and $|\mu| + |\nu| = d$. When we restrict to \overline{K} structure is obtained from the decomposing the inner tensor products, $F_{(n)}^\mu \otimes F_{(n)}^\nu$ and $\left(F_{(n)}^\mu \right)^* \otimes \left(F_{(n)}^\nu \right)^*$. The \overline{K} -decomposition is:

$$\begin{aligned} \mathcal{S}^d(\widetilde{W}) &\cong \bigoplus_{\mu, \nu} \left(\bigoplus_{\alpha} c_{\mu\nu}^\alpha F_{(n)}^\alpha \right)^* \widehat{\otimes} \left(\bigoplus_{\beta} c_{\mu\nu}^\beta F_{(n)}^\beta \right) \\ &\cong \bigoplus_{\alpha, \beta} \left(\sum_{\mu, \nu} c_{\mu\nu}^\alpha c_{\mu\nu}^\beta \right) \left[\left(F_{(n)}^\alpha \right)^* \widehat{\otimes} F_{(n)}^\beta \right] \end{aligned}$$

where the sums are over α, β such that $\ell(\alpha), \ell(\beta) \leq n$ and $|\alpha| = |\beta| = |\mu| + |\nu|$. Now, we restrict to K ,

$$\dim \left[\mathcal{S}^d(\widetilde{W}) \right]^K = \sum_{\alpha, \beta} \left(\sum_{\mu, \nu} c_{\mu\nu}^\alpha c_{\mu\nu}^\beta \right) \dim \left[\left(F_{(n)}^\alpha \right)^* \widehat{\otimes} F_{(n)}^\beta \right]^K$$

We obtain K -invariants exactly when $\alpha = \beta$. We denote the common value of α and β by λ . Equation 1.7.3 follows.

3.4. CASE: $G_0 = GL(m, \mathbb{H})$. Let $n = 2m$.

$$\boxed{G = GL_{2m}(\mathbb{C}) \quad K = Sp_{2m}(\mathbb{C})}$$

The group K is symmetrically embedded in G in the standard way as:

$$K := \{g \in G \mid g^{-1} = -J_m g^t J_m\} \subset G$$

where: J_m is as before. We then define \overline{G} and \overline{K} as:

$$\begin{array}{lcl} \overline{G} & := & G \times G & \supset & \overline{K} & := & \{(g, g) \mid g \in G\} \\ \cup & & & & \cup & & \\ G' & := & \{(-J(g^{-1})^t J, g) \mid g \in G\} & \supset & K' & := & \{(k, k) \mid k \in K\}. \end{array}$$

As before, the primes are indicated to describe precisely the embeddings. In particular, note that we have embedded $GL_{2m}(\mathbb{C})$ in \overline{G} in two distinct ways.

This case is almost identical to the case in Subsection 3.2, so we truncate the full discussion. There are a few minor differences such as the use Theorem 2.1.2 instead of Theorem 2.1.1.

As before, $\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W})$, where $\widetilde{W} = W \widehat{\otimes} W$ with W the standard representation of G . We obtain:

$$\mathcal{S}^d(\mathfrak{g})^K \cong \mathcal{S}^d(W \otimes W)^{Sp_{2m}(\mathbb{C})} \cong \bigoplus c_{\mu\nu}^\nu (F_{(2m)}^\nu)^{Sp_{2m}(\mathbb{C})}$$

where μ and ν are partitions with $\ell(\mu), \ell(\nu) \leq 2m$ and $|\nu| = 2|\mu| = 2d$. In the present case, we obtain a K -invariant exactly when $\nu = (2\lambda)'$, for some λ . Equation 1.7.4 follows.

3.5. CASE: $G_0 = SO^*(2m)$.

$G = SO_{2m}(\mathbb{C})$	$K \cong GL_m(\mathbb{C})$
---------------------------	----------------------------

The group K is embedded in G , as:

$$K \cong K' := \left\{ \begin{bmatrix} g & 0 \\ 0 & (g^{-1})^t \end{bmatrix} : g \in K \right\} \subset G$$

Note that we choose the form of G using the matrix D_m as in Section 1.1.

We define \overline{G} and the embedding of G as follows:

$$\begin{aligned} \overline{G} &:= GL_{2m}(\mathbb{C}) \\ \cup \\ G' &:= SO_{2m}(\mathbb{C}) \cong G \end{aligned}$$

with the standard embedding of SO_{2m} .

Let \overline{K}_L and \overline{K}_R denote isomorphic copies of $GL_m(\mathbb{C})$. Let W_L and W_R denote the respective standard representations of these groups. We set $\overline{K} := \overline{K}_L \times \overline{K}_R$, and embed in \overline{G} as:

$$\begin{aligned} \overline{G} \supset \overline{K}' &:= \left\{ \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} : g \in \overline{K}_L, h \in \overline{K}_R \right\} \cong \overline{K} \\ \cup \\ K &\cong K'. \end{aligned}$$

Let \overline{W} be the standard representation of \overline{G} , which we also regard as the standard representation of G . When restricted to \overline{K} , $\overline{W} \cong W_L \widehat{\otimes} 1_{\mathbb{C}} \oplus 1_{\mathbb{C}} \widehat{\otimes} W_R$, and when restricted to K , $\overline{W} \cong W \oplus W^*$, where W is the standard representation of K .

As a representation of G , $\mathfrak{g} \cong \wedge^2(\overline{W})$. Let $\widetilde{W} := \wedge^2(\overline{W})$ and $\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W})$. We view the latter as a graded, representation of \overline{G} . Using Theorem 2.2.3, we have a multiplicity free \overline{G} -decomposition:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus F_{(2m)}^{(2\lambda)'}$$

where the sum is over all partitions λ such that $|\lambda| = d$ and $\ell((2\lambda)') \leq 2m$.

We now decompose relative to the action of \overline{K} using Proposition 1.2:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus c_{\mu\nu}^{(2\lambda)'} F_{(m)}^\mu \widehat{\otimes} F_{(m)}^\nu$$

where μ and ν are partitions with $|\mu| + |\nu| = 2|\lambda|$ and $\ell(\mu), \ell(\nu) \leq m$.

When we restrict to K the decomposition becomes:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus c_{\mu\nu}^{(2\lambda)'} F_{(m)}^\mu \otimes (F_{(m)}^\nu)^*.$$

Clearly, we obtain a K invariant exactly when $\mu = \nu$. Equation 1.7.5 follows.

3.6. **CASE:** $G_0 = Sp(m, \mathbb{R})$.

$$\boxed{G = Sp_{2m}(\mathbb{C}) \quad | \quad K = GL_m(\mathbb{C})}$$

The group K is diagonally embedded in G , as:

$$K \cong K' := \left\{ \left[\begin{array}{cc} g & 0 \\ 0 & (g^{-1})^t \end{array} \right] : g \in K \right\} \subset G$$

We define \overline{G} and the embedding of G as follows:

$$\begin{aligned} \overline{G} &:= GL_{2m}(\mathbb{C}) \\ \cup \\ G' &:= Sp_{2m}(\mathbb{C}) \cong G \end{aligned}$$

with the standard embedding of Sp_{2m} .

Let $\overline{K}_L, \overline{K}_R, W_L, W_R$ be as in the $SO^*(2m)$ case. We set $\overline{K} := \overline{K}_L \times \overline{K}_R$, and embed in \overline{G} as:

$$\begin{aligned} \overline{G} \supset \overline{K} &:= \left\{ \left[\begin{array}{cc} g & 0 \\ 0 & h \end{array} \right] : g \in \overline{K}_L, h \in \overline{K}_R \right\} \\ \cup \\ K \cong K' &. \end{aligned}$$

Let \overline{W} be the standard representation of \overline{G} , which we also regard as the standard representation of G . When restricted to \overline{K} , $\overline{W} \cong W_L \widehat{\otimes} 1_{\mathbb{C}} \oplus 1_{\mathbb{C}} \widehat{\otimes} W_R$, and when restricted to K , $\overline{W} \cong W \oplus W^*$, where W is the standard representation of K .

As a representation of G , $\mathfrak{g} \cong \mathcal{S}^2(\overline{W})$. Let $\overline{\mathcal{S}} := \mathcal{S}(\overline{W})$ where $\mathcal{S}^2(\overline{W})$, which we view as a graded, representation of \overline{G} . Using Theorem 2.2.2, we have a multiplicity free \overline{G} -decomposition:

$$\mathcal{S}^d(\overline{W}) \cong \bigoplus F_{(2m)}^{2\lambda}$$

where the sum is over all partitions λ such that $|\lambda| = d$ and $\ell(\lambda) \leq 2m$.

We now decompose relative to the action of \overline{K} using Proposition 1.2:

$$\mathcal{S}^d(\overline{W}) \cong \bigoplus c_{\mu\nu}^{2\lambda} F_{(m)}^{\mu} \widehat{\otimes} F_{(m)}^{\nu}$$

where μ and ν are partitions with $|\mu| + |\nu| = 2|\lambda|$ and $\ell(\mu), \ell(\nu) \leq m$. When we restrict to K ,

$$\mathcal{S}^d(\overline{W}) \cong \bigoplus c_{\mu\nu}^{2\lambda} F_{(m)}^{\mu} \otimes (F_{(m)}^{\nu})^*.$$

We obtain a K invariant exactly when $\mu = \nu$ as before. Equation 1.7.6 follows.

3.7. **CASE:** $G_0 = O(n, \mathbb{C})$.

$$\boxed{G = O_n(\mathbb{C}) \times O_n(\mathbb{C}) \quad | \quad K \cong O_n(\mathbb{C})}$$

The group K is diagonally embedded in G , as will be the case in the $G_0 = Sp(m, \mathbb{C})$ case. We define \overline{G} and the embedding of G as follows:

$$\begin{aligned} \overline{G} &:= GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \\ \cup \\ G' &:= \{(g, h) | g, h \in O_n(\mathbb{C})\} \cong G \end{aligned}$$

with the standard embedding of $O_n(\mathbb{C})$ in $GL_n(\mathbb{C})$. Define: $\overline{K} := \{(g, g) | g \in GL_n(\mathbb{C})\} \subset \overline{G}$.

Set $\widetilde{W} := \wedge^2(W) \widehat{\otimes} 1_{\mathbb{C}} \oplus 1_{\mathbb{C}} \widehat{\otimes} \wedge^2(W)$, where W is the standard representation of $GL_n(\mathbb{C})$. We view \widetilde{W} as a representation of \overline{G} , but as a representation of G , $\mathfrak{g} \cong \widetilde{W}$. Define:

$$\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W}) \cong \mathcal{S}(\wedge^2(W)) \widehat{\otimes} \mathcal{S}(\wedge^2(W)),$$

which is a graded representation of \overline{G} . Using Theorem 2.2.2, we have a multiplicity free \overline{G} -decomposition:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus F_{(n)}^{(2\mu)'} \widehat{\otimes} F_{(n)}^{(2\nu)'}$$

where the sum is over all partitions μ and ν such that $|\mu| + |\nu| = d$ and $\ell((2\mu)'), \ell((2\nu)') \leq n$.

We now decompose relative to the action of \overline{K} using Proposition 1.1:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus c_{(2\mu)'}^{\gamma} F_{(n)}^{\gamma}$$

where μ and ν are as before and γ is such that $|\gamma| = 2(|\mu| + |\nu|)$ and $\ell(\gamma) \leq n$. This is the \overline{K} -decomposition. By Theorem 2.1.1 we see that we obtain a K -invariant exactly when γ is of the form 2λ for some λ . Equation 1.7.7 follows.

3.8. **CASE:** $G_0 = Sp(m, \mathbb{C})$.

$G = Sp_{2m}(\mathbb{C}) \times Sp_{2m}\mathbb{C}$	$K \cong Sp_{2m}(\mathbb{C})$
--	-------------------------------

The group K is diagonally embedded in G , as:

$$K \cong K' := \{(k, k) | k \in K\}.$$

We define \overline{G} and the embedding of G as follows:

$$\begin{aligned} \overline{G} &:= GL_{2m}(\mathbb{C}) \times GL_{2m}(\mathbb{C}) \\ \cup \\ G' &:= \{(g, h) | g, h \in Sp_{2m}(\mathbb{C})\} \cong G \end{aligned}$$

with the standard embedding of $Sp_{2m}(\mathbb{C})$ in $GL_{2m}(\mathbb{C})$. Define: $\overline{K} := \{(g, g) | g \in GL_{2m}(\mathbb{C})\} \subset \overline{G}$.

Set $\widetilde{W} := \mathcal{S}^2(\widetilde{W}) \widehat{\otimes} 1_{\mathbb{C}} \oplus 1_{\mathbb{C}} \widehat{\otimes} \mathcal{S}^2(\widetilde{W})$, where W is the standard representation of $GL_{2m}(\mathbb{C})$. We view \widetilde{W} as a representation of \overline{G} , but as a representation of G , $\mathfrak{g} \cong \widetilde{W}$. Define:

$$\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W}) \cong \mathcal{S}(\mathcal{S}^2(W)) \widehat{\otimes} \mathcal{S}(\mathcal{S}^2(W)),$$

which is a graded representation of \overline{G} . Using Theorem 2.2.2, we have a multiplicity free \overline{G} -decomposition:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus F_{(2m)}^{2\mu} \widehat{\otimes} F_{(2m)}^{2\nu}$$

where the sum is over all partitions μ and ν such that $|\mu| + |\nu| = d$ and $\ell(\mu), \ell(\nu) \leq 2m$.

We now decompose relative to the action of \overline{K} using Theorem 1.1:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus c_{2\mu 2\nu}^{\gamma} F_{(2m)}^{\gamma}$$

where μ and ν are as before and γ is such that $|\gamma| = 2(|\mu| + |\nu|)$ and $\ell(\gamma) \leq 2m$. This is the \overline{K} -decomposition. By Theorem 2.1.2 we see that we obtain a K -invariant exactly when γ is of the form $(2\lambda)'$ for some λ . Equation 1.7.8 follows.

3.9. **CASE:** $G_0 = O(p, q)$. Set $n := p + q$.

$$\boxed{G = O_n(\mathbb{C}) \quad | \quad K \cong O_p(\mathbb{C}) \times O_q(\mathbb{C})}$$

The group K is embedded in G as a direct sum in the same way as the $G_0 = U(p, q)$ case.

We define $\overline{G} := GL_n(\mathbb{C})$ and embed G in the standard way. We set

$$\begin{aligned} \overline{K} &:= \left\{ \left[\begin{array}{cc} g & 0 \\ 0 & h \end{array} \right] \middle| g \in GL_p(\mathbb{C}), h \in GL_q(\mathbb{C}) \right\} \\ \cup \\ K' &:= \left\{ \left[\begin{array}{cc} g & 0 \\ 0 & h \end{array} \right] \middle| g \in O_p(\mathbb{C}), h \in O_q(\mathbb{C}) \right\} \cong K, \end{aligned}$$

where K' is defined to describe the embedding of K in \overline{K} . Note that $O_p(\mathbb{C})$ (resp. $O_q(\mathbb{C})$) is embedded in $GL_p(\mathbb{C})$ (resp. $GL_q(\mathbb{C})$) in the standard way.

Set $\widetilde{W} := \wedge^2(W)$, where W is the standard representation of \overline{G} . As a representation of G , $\mathfrak{g} \cong \widetilde{W}$. Define $\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W})$, which we view a graded representation of \overline{G} . Using Theorem 2.2.3, we have a multiplicity free \overline{G} -decomposition:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus F_{(n)}^{(2\lambda)'}$$

where the sum is over all partitions λ such that $|\lambda| = d$ and $\ell((2\lambda)') \leq n$.

We now decompose relative to the action of \overline{K} using Theorem 1.2:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus c_{\alpha\beta}^{(2\lambda)'} F_{(p)}^\alpha \widehat{\otimes} F_{(q)}^\beta$$

where α and β are such that $d = 2|\lambda| = |\alpha| + |\beta|$ and $\ell(\alpha) \leq p$, $\ell(\beta) \leq q$. This is the \overline{K} -decomposition. By Theorem 2.1.1 we see that we obtain a K -invariant exactly when α (resp. β) is of the form 2μ (resp. 2ν) for some μ (resp. ν). Equation 1.7.9 follows.

3.10. **CASE:** $G_0 = Sp(p, q)$. Set $m := p + q$, and $n = 2m$.

$$\boxed{G = Sp_{2m}(\mathbb{C}) \quad | \quad K \cong Sp_{2p}(\mathbb{C}) \times Sp_{2q}(\mathbb{C})}$$

The group K is embedded in G as a direct sum in the same way as the $G_0 = U_{p,q}$ and $G_0 = O(p, q)$ cases. Note that we take form defining G to be the one given by the matrix C_m as in Section 1.1.

We define $\overline{G} := GL_{2m}(\mathbb{C})$ and embed G in the standard way. We set

$$\begin{aligned} \overline{K} &:= \left\{ \left[\begin{array}{cc} g & 0 \\ 0 & h \end{array} \right] \middle| g \in GL_{2p}(\mathbb{C}), h \in GL_{2q}(\mathbb{C}) \right\} \\ \cup \\ K' &:= \left\{ \left[\begin{array}{cc} g & 0 \\ 0 & h \end{array} \right] \middle| g \in Sp_{2p}(\mathbb{C}), h \in Sp_{2q}(\mathbb{C}) \right\} \cong K, \end{aligned}$$

where K' is defined to describe the embedding of K in \overline{K} , as in the previous case.

Set $\widetilde{W} := \mathcal{S}^2(W)$, where W is the standard representation of \overline{G} . As a representation of G , $\mathfrak{g} \cong \widetilde{W}$. Define $\overline{\mathcal{S}} := \mathcal{S}(\widetilde{W})$, which we view a graded representation of \overline{G} . Using Theorem 2.2.2, we have a multiplicity free \overline{G} -decomposition:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus F_{(n)}^{2\lambda}$$

where the sum is over all partitions λ such that $|\lambda| = d$ and $\ell(\lambda) \leq n$.

We now decompose relative to the action of \overline{K} using Proposition 1.2:

$$\mathcal{S}^d(\widetilde{W}) \cong \bigoplus c_{\alpha\beta}^{2\lambda} F_{(2p)}^\alpha \widehat{\otimes} F_{(2q)}^\beta$$

where α and β are such that $d = 2|\lambda| = |\alpha| + |\beta|$ and $\ell(\alpha) \leq 2p$, $\ell(\beta) \leq 2q$. This is the \overline{K} -decomposition. By Theorem 2.1.2 we see that we obtain a K -invariant exactly when α (resp. β) is of the form $(2\mu)'$ (resp. $(2\nu)'$) for some μ (resp. ν). Equation 1.7.10 follows.

4. CLOSED EXPRESSIONS

The purpose of this section is to show that in six of the ten cases, we can actually close the sums involving Littlewood-Richardson coefficients. Since the ten cases naturally break into three disjoint subsets, we are only required to address three of the sums. Of these three cases, one has a simple and rather elegant expression, another is considerably more complicated while it remains an open problem to close the third.

This latter case, corresponds to the groups $O(n, \mathbb{C})$, $Sp(m, \mathbb{C})$, $Sp(p, q)$ and $O(p, q)$. This is a subject of further study, but is not likely to close to a simple formula. The problem would amount to closing:

$$\sum_{\lambda, \mu, \nu} c_{(2\mu)'(2\nu)'}^{2\lambda} t^{|\lambda|}$$

which, of course, is the same as:

$$\sum_{\lambda, \mu, \nu} c_{2\mu 2\nu}^{(2\lambda)'} t^{|\lambda|}.$$

Probably, obtaining a simple formula for either of the above expressions would be almost as hard as:

$$\sum_{\lambda, \mu, \nu} c_{\mu\nu}^\lambda t^{|\lambda|},$$

which is a listed open problem on Richard Stanley's web page:

<http://www-math.mit.edu/~rstan/ec/ch7supp.pdf>

Also on Richard Stanley's web page is:

$$\sum_{\lambda, \mu} c_{\mu\mu}^{2\lambda} t^{|\lambda|} = \prod_{i \geq 1} \frac{1}{\sqrt{1-t^i}} \cdot \prod_{j \geq 1} \frac{1}{(1-t^{2j})^{2j-2}}.$$

Stanley's proof of this impressive formula involves some rather technical manipulations of q -series, which we omit. We do however note:

$$\sum_{\lambda, \mu} c_{\mu\mu}^{(2\lambda)'} t^{|\lambda|} = \sum_{\lambda, \mu} c_{\mu\mu}^{2\lambda} t^{|\lambda|}.$$

From our point of view, the significance of the above expressions are that they close the sums of Littlewood-Richardson coefficients that appear in the $GL(n, \mathbb{R})$, $GL(m, \mathbb{H})$, $SO^*(2n)$, and $Sp(m, \mathbb{R})$ cases. For a combinatorial interpretation of the coefficients see [Wil01].

Now, as a final result, we will close the sum of Littlewood-Richardson coefficients occurring in the $U(p, q)$ case, which we have seen is the same as closing the sum in the $GL(n, \mathbb{C})$ case. The formula we obtain also appears on the above web page, but we present here a representation theoretic proof as it may be of independent interest.

Theorem 4.1.

$$\sum (c_{\mu\nu}^\lambda)^2 t^{|\lambda|} = \prod_{k=1}^{\infty} \frac{1}{1 - 2t^k}.$$

In order to prove the above theorem, we will need to introduce some additional notation. Let $\bar{x} = (x_1, x_2, \dots)$, $\bar{y} = (y_1, y_2, \dots)$, and $\bar{z} = (z_1, z_2, \dots)$ be three sets of countably infinite indeterminates. We now expand the following symmetric product into Schur functions:

$$(4.0.1) \quad \prod_{i,j,k=1}^{\infty} \left(\frac{1}{1 - x_i y_j z_k} \right) = \sum_{\lambda, \mu, \nu} g_{\lambda\mu\nu} s_\mu(\bar{x}) s_\nu(\bar{y}) s_\lambda(\bar{z}).$$

The coefficients, $g_{\lambda\mu\nu}$, are non-negative integers which can be interpreted as the tensor product multiplicities for the symmetric groups (see [Mac95, Sta99]). That is to say, for partitions λ , μ and ν of size m ,

$$U_\mu \otimes U_\nu \cong \bigoplus_{\lambda: |\lambda|=m} g_{\lambda\mu\nu} U_\lambda$$

where U_γ is the irreducible S_m -representation indexed by γ (with $|\gamma| = m$) as in [GW98]. Note that we have $g_{\lambda\mu\nu} = g_{\alpha\beta\gamma}$ where $\alpha\beta\gamma$ is a permutation of $\lambda\mu\nu$. This fact is a consequence of the fact that the representations of S_m are self dual.

By specialization $z_k = t^k$ in Equation 4.0.1 we obtain a formal series in \bar{x} and \bar{y} . Define the coefficient of $s_\mu(\bar{x}) s_\nu(\bar{y})$ to be:

$$G_{\mu\nu}(t) := \sum_{\lambda} g_{\lambda\mu\nu} s_\lambda(t, t^2, t^3, \dots).$$

We will now provide an interpretation of $G_{\mu\nu}(t)$ involving the representation theory of $GL_n(\mathbb{C})$ as $n \rightarrow \infty$.

Proposition 4.2. *For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ we have:*

$$(4.0.2) \quad \sum_{d=0}^{\infty} \left(\lim_{n \rightarrow \infty} \left[F_{(n)}^{(\mu, \nu)}, \mathcal{S}^d(\mathfrak{g}) \right] \right) t^d = \frac{G_{\mu\nu}(t)}{\prod_{k=1}^{\infty} (1 - t^k)}$$

Proof. This result was first proved in [Sta84]. Please see Remark 4.5 for an outline of the proof in the notation of this paper. \square

The fact that the denominator of the right side of Equation 4.0.2 formally looks like the familiar η -function has a clear interpretation. Given a connected, reductive, linear algebraic group G (over \mathbb{C}) with Lie algebra \mathfrak{g} , Kostant's theorem asserts $\mathcal{S}(\mathfrak{g}) \cong \mathcal{S}(\mathfrak{g})^G \otimes \mathcal{H}(\mathfrak{g})$, where $\mathcal{H}(\mathfrak{g})$ denotes the space of harmonic polynomials on \mathfrak{g} (see [Kos63]). We will see a reflection of this celebrated theorem in the following discussion where we take $G = GL_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$.

Set $n = p + q$ and embed $K = GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ in $G = GL_n(\mathbb{C})$ as in the proof of Equation 1.7.1. We will now identify the distribution of the K -invariants in the harmonic polynomials on $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Define:

$$h_{p,q}(d) := \dim \left(\mathcal{H}^d(\mathfrak{g}) \right)^{GL_p \times GL_q}$$

and

$$f_{p,q}(d) := \dim \left(\mathcal{S}^d(\mathfrak{g}) \right)^{GL_p \times GL_q}.$$

From Equation 1.7.1 we see that $f_{p,q}(d) = f_{p,q}(d_0)$ for all $d \geq d_0$ where $d_0 = \min(p, q)$. As we remarked before, this fact justifies the definition

$$f(d) := \lim_{p,q \rightarrow \infty} f_{p,q}(d).$$

We also set $F(t) = \sum_{d=0}^{\infty} f(d)t^d$. Proposition 1.7.1 implies that:

$$F(t) = \sum (c_{\mu\nu}^{\lambda})^2 t^{|\lambda|}$$

where the sum is over all triples of non-negative integer partitions λ, μ and ν .

Let $I_n(t) = \sum_{d=0}^{\infty} \dim \mathcal{S}^d(\mathfrak{g})^{GL_n(\mathbb{C})} t^d$. It is a standard fact due to Chevalley (see [Che55]) that $\mathcal{S}(\mathfrak{g})^G$ is a polynomial ring. For the case of $GL_n(\mathbb{C})$, we can choose generators for this ring in degrees $1, 2, \dots, n$. Thus $I_n(t) = \prod_{k=1}^n \frac{1}{1-t^k}$. Note that as $n \rightarrow \infty$ the coefficients of the product stabilize. In light of this fact, we set $I(t) := \prod_{k=1}^{\infty} \left(\frac{1}{1-t^k} \right)$.

A combination of this fact and Kostant's theorem imply that $F(t) = I(t)H(t)$, where $H(t)$ is a formal series in t with non-negative integer coefficients. Theorem 4.1 follows from:

Proposition 4.3.

$$H(t) = \prod_{k=0}^{\infty} \left(\frac{1-t^k}{1-2t^k} \right)$$

Proof. For $d \geq 0$, define $h(d)$ by $H(t) = \sum_{d=0}^{\infty} h(d)t^d$. From the stability properties of $I_n(t)$ and $f_{p,q}(d)$ it is easy to see that $h(d) = \lim_{p,q \rightarrow \infty} h_{p,q}(d)$.

Applying the Cartan-Helgason Theorem from Section 2.1.3 and Proposition 4.2 we obtain $H(t) = \sum_{\mu} G_{\mu\mu}(t)$. Therefore, we will consider the symmetric function

$$\sum_{\lambda, \mu} g_{\lambda\mu\mu} s_{\lambda}(\bar{z})$$

and evaluate at the point $\bar{z} = (t, t^2, t^3, \dots)$ thus obtaining $H(t)$. We now interpret the above sum in term of the representation theory of the symmetric group.

As a representation of $S_m \times S_m$ under left and right multiplication we have:

$$\mathbb{C}[S_m] \cong \bigoplus_{\mu \vdash m} U_{\mu} \otimes U_{\mu},$$

(where $\mathbb{C}[S_m]$ is the group algebra of the symmetric group). If we restrict this action to the diagonal subgroup we obtain: $\mathbb{C}[S_m] \cong \bigoplus_{\lambda \vdash m} \left(\sum_{\mu} g_{\lambda\mu\mu} \right) U_{\lambda}$. Observe that the diagonal action of S_m on $\bigoplus U_{\mu} \otimes U_{\mu}$ is, as a representation, isomorphic to the conjugation action of S_m on $\mathbb{C}[S_m]$.

Given a representation $\rho : S_m \rightarrow GL(V)$, denote the Frobenius characteristic (see [Mac95, Sta99]) of V by $\text{ch}(V)$. That is,

$$\text{ch}(V)(\sigma) = \frac{1}{m!} \sum_{\sigma \in S_m} \chi_{\rho}(\sigma) p_{\lambda(\sigma)}(\bar{x}),$$

where $p_{\mu}(\bar{x})$ denotes the power symmetric function, which for a partition μ , is defined as $p_{\mu}(\bar{x}) = p_{\mu_1}(\bar{x}) p_{\mu_2}(\bar{x}) \cdots$ with $p_k(\bar{x}) = x_1^k + x_2^k + \cdots$, and $\lambda(\sigma)$ is a partition denoting the shape of σ in its disjoint cycle notation (see [Mac95]).

Suppose $V = \mathbb{C}[S_m]$ with S_m acting by conjugation. For a permutation σ , let $C_{S_m}(\sigma)$ denote the centralizer of σ in S_m . Then,

$$\text{ch}(\mathbb{C}[S_m]) = \frac{1}{m!} \sum_{\sigma \in S_m} |C_{S_m}(\sigma)| p_{\lambda(\sigma)}(\bar{x}) = \sum_{\mu \vdash m} z_\mu \frac{p_\mu(\bar{x})}{z_\mu} = \sum_{\mu \vdash m} p_\mu(\bar{x}).$$

where $z_\mu = 1^{a_1} 2^{a_2} \cdots a_1! a_2! \cdots$ with μ having a_i cycles of size i .

Summing these symmetric functions together we obtain:

$$\sum_{m=0}^{\infty} \text{ch}(\mathbb{C}[S_m]) = \sum_{\mu} p_\mu(\bar{x}) = \prod_{k=1}^{\infty} \frac{1}{1 - p_k(\bar{x})}.$$

On the other hand, note that $\text{ch}(U_\lambda) = s_\lambda(\bar{x})$ and therefore we have shown that:

$$\sum_{\lambda, \mu} g_{\lambda \mu} s_\lambda(\bar{x}) = \prod_{k=1}^{\infty} \frac{1}{1 - p_k(\bar{x})}.$$

And so,

$$\begin{aligned} H(t) &= \left(\prod_{k=1}^{\infty} \frac{1}{1 - p_k} \right) (t, t^2, t^3, \dots) = \prod_{k=1}^{\infty} \frac{1}{1 - p_k(t, t^2, t^3, \dots)} = \prod_{k=1}^{\infty} \frac{1}{1 - (t^k + t^{2k} + \dots)} \\ &= \prod_{k=1}^{\infty} \frac{1}{1 - \frac{t^k}{1 - t^k}} = \prod_{k=1}^{\infty} \frac{1 - t^k}{1 - 2t^k}. \end{aligned}$$

□

Remark 4.4. Note that the coefficient of t^n in $\prod_{k=1}^{\infty} \frac{1-t^k}{1-qt^k}$ is the number of conjugacy classes in the finite group $GL(n, F)$ where F is the field with q elements (see [BFH86]).

Remark 4.5. We briefly describe the proof of Proposition 4.2 from a point of view closer to the representation theory of classical groups as described in [HTW05]. First, we note from Corollary 2.3 that we have:

$$\mathcal{S}^d(\mathfrak{g}) \cong \bigoplus_{\substack{\rho: |\rho|=d \\ \ell(\rho) \leq n}} \left(F_{(n)}^\rho \right)^* \otimes F_{(n)}^\rho$$

as a representation of $GL_n(\mathbb{C})$. In our notation, $\left(F_{(n)}^\rho \right)^* \cong F_{(n)}^{(0, \rho)}$.

Using the Theorem 2.1.1 of [HTW05] we obtain:

$$(4.0.3) \quad \lim_{n \rightarrow \infty} \left[F_{(n)}^{(0, \rho)} \otimes F_{(n)}^{(\rho, 0)}, F_{(n)}^{(\mu, \nu)} \right] = \sum c_{\lambda \mu}^\rho c_{\lambda \nu}^\rho$$

where the sum is over all partitions λ . (Note that the sum is easily seen to be finite after taking into account the support of the Littlewood-Richardson coefficients.) This formula perhaps first appeared [Kin71] as (4.6) with (4.15).

The partitions ρ , μ and ν are fixed on the right side of Equation 4.0.3. We will put together a formal sum of Schur functions to encode these numbers combinatorially as:

$$S(\bar{x}, \bar{y}; t) = \sum_{\mu, \nu, \rho} \left(\sum_{\lambda} c_{\lambda \mu}^\rho c_{\lambda \nu}^\rho \right) s_\mu(\bar{x}) s_\nu(\bar{y}) t^{|\rho|} = \sum_{\rho, \lambda} s_{\rho/\lambda}(\bar{x}) s_{\rho/\lambda}(\bar{y}) t^{|\rho|}$$

where we define the “Skew Schur” function as in [Mac95, Sta99]:

$$s_{\alpha/\beta}(\bar{x}) = \sum_{\gamma} c_{\beta\gamma}^{\alpha} s_{\gamma}(\bar{x}).$$

Proposition 4.2 now follows from the identity:

$$\sum_{\rho, \lambda} s_{\rho/\lambda}(\bar{x}) s_{\rho/\lambda}(\bar{y}) t^{|\rho|} = \prod_k \left[\left(\frac{1}{1-t^k} \right) \prod_{i,j} \left(\frac{1}{1-x_i y_j} \right) \right],$$

which can be found in [Mac95] p. 94 (28a).

Lastly, we point out that several analogous skew identities in [Mac95, Sta99] may be interpreted in the context of the generalization of Kostant’s result in [KR71] as in [Wil02].

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