The cubic, the quartic, and the exceptional group G_2

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Abstract We study an example first addressed in a 1949 paper of J. A. Todd, where the author obtains a complete system of generators for the covariants in the polynomial functions on the eight dimensional space of the double binary form of degree (3,1), under the action of $SL_2 \times SL_2$. We reconsider Todd's result by examining the complexified Cartan complement corresponding to the maximal compact subgroup of simply connected split G_2. A result of this analysis involves a connection with the the branching rule from the rank two complex symplectic Lie algebra to a principally embedded $\mathfrak{sl}_2$-subalgebra. Special cases of this branching rule are related to covariants for cubic and quartic binary forms.[1]

Key words: binary form, branching rule, double binary form, G_2, harmonic polynomials, principal $\mathfrak{sl}_2$, symmetric space

1 Introduction

Let F^k denote the irreducible $k+1$ dimensional representation of SL_2, over the field $\mathbb{C}$. If V is a complex vector space then we will denote the complex algebra of polynomial functions on V by $\mathbb{C}[V]$, which is graded by degree.

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That is, for a non-negative integer k denote

$$\mathbb{C}[V]_k = \left\{ f \in \mathbb{C}[V] | f(tv) = t^k f(v) \text{ for all } t \in \mathbb{C}, v \in V \right\}.$$

Thus, $\mathbb{C}[V] = \bigoplus \mathbb{C}[V]_k$. We will identify $F^k = \mathbb{C}[V]_k$ where $V = \mathbb{C}^2$ is the defining representation of SL_2 with the usual action defined by $g \cdot f(v) = f(vg)$ for $g \in \mathrm{SL}_2$, $v \in V$ and $f \in F^k$.

Let x and y denote the standard coordinate functionals on $\mathbb{C}^2$, so that $\mathbb{C}[V]$ may be identified with the polynomial algebra $\mathbb{C}[x,y]$. Thus, $F^k = \mathbb{C}[x,y]_k$ for each non-negative integer k. In particular, we have

$$F^2 = \left\{ ax^2 + bxy + cy^2 | a, b, c \in \mathbb{C} \right\}.$$

The SL_2-invariant subalgebra of $\mathbb{C}[F^2]$ is generated by $\delta = b^2 - 4ac$ (i.e. the discriminant), which defines a non-degenerate symmetric bilinear form on F^2. The image of $\mathfrak{sl}_2$ in $\mathrm{End}(F^2)$ is contained in $\mathfrak{so}_3$ with respect to this symmetric form. Define

$$\Delta = \frac{\partial^2}{\partial b^2} - 4\frac{\partial^2}{\partial a \partial c},$$

which is in turn a generator of the constant coefficient SL_2-invariant differential operators on $\mathbb{C}[F^2]$. The spherical harmonic polynomials are denoted by

$$\ker \Delta = \left\{ f \in \mathbb{C}[F^2] | \Delta f = 0 \right\}.$$

It is well known that we have a "separation of variables", $\mathbb{C}[F^2] = \mathbb{C}[\delta] \otimes \ker \Delta$. Furthermore, as a representation of SL_2, the graded components of $\ker \Delta$ are irreducible SL_2-representations. Specifically, the degree k harmonic polynomials is equivalent to F^{2k}. Thus, the theory of spherical harmonics provides a graded decomposition of $\mathbb{C}[F^2]$ into irreducible representations of SL_2.

One desires a similar decomposition of $\mathbb{C}[F^k]$ for $k > 2$ (see [7]), which we present for $d \leq 4$. Specifically, our goal in this article is to point out a connection between the (infinite dimensional) representation theory of the simply connected split real form of the exceptional group G_2 with the invariant theory for the cubic ($k = 3$, see Corollary 1) and quartic ($k = 4$, see Corollary 2) binary forms.

An important ingredient to the background of this paper concerns the various embedding of $\mathfrak{sl}_2$ into a given Lie algebra. Specifically, a subalgebra $\mathfrak{l}$ of a semisimple Lie algebra $\mathfrak{g}$ is said to be a *principal* $\mathfrak{sl}_2$*-subalgebra* if $\mathfrak{l} \cong \mathfrak{sl}_2$ and contains a regular nilpotent element of $\mathfrak{g}$ ([4], [6], [9]). These subalgebras are conjugate, so we sometimes speak of "the" principal $\mathfrak{sl}_2$-subalgebra. There is a beautiful connection between the principal $\mathfrak{sl}_2$-subalgebra and the cohomology of the corresponding simply connected compact Lie group (see [9]). There is a nice discussion of this theory in [3].

A principal $\mathfrak{sl}_2$ subalgebra of $\mathfrak{sl}_n$ is given by the image of $\mathfrak{sl}_2$ in the n-dimensional irreducible representation of $\mathfrak{sl}_2$ (over $\mathbb{C}$). For $n \geq 4$ these are not maximal. Rather, for even n the image is contained in the standard symplectic subalgebra of $\mathfrak{sl}_n$, while for odd n the image is contained in the standard orthogonal subalgebra of $\mathfrak{sl}_n$.

Since we are interested in the cubic (n=4) and quartic (n=5), it is worthwhile to recall the Lie algebra isomorphism $\mathfrak{sp}_4 \cong \mathfrak{so}_5$. As we shall see, we will exploit an instance Howe duality for $\mathfrak{sp}_4$. Nonetheless, an exposition of the results presented here could equally well be cast for $\mathfrak{so}_5$. Along these lines, the referee has pointed out to us that another proof of the main theorem may be obtained from the theory of spinors (following [14]).

Central to our treatment is the observation that the complex rank two symplectic Lie algebra $\mathfrak{sp}_4(\cong \mathfrak{so}_5)$ acts on the cubic and quartic forms. In particular, these are the two fundamental representations of $\mathfrak{sp}_4$ of dimension four and five respectively. From this point of view the problem of decomposing finite dimensional representations of $\mathfrak{sp}_4$ when restricted to the principally embedded $\mathfrak{sl}_2$-subalgebra generalizes the invariant theory of the cubic and quartic.

The unitary dual of simply connected split G_2 was determined in [18]. This important example was a outgrowth of decades of work emerging from the study of Harish-Chandra modules. More recently, several authors have taken on the task of classification of the *generalized Harish-Chandra modules*. Specifically, we find motivation from [13].

Since $F^k \cong (F^k)^*$ as an SL_2-representation, we have an SL_2-invariant form in $\mathrm{End}(F^k)$. For even k, this invariant is symmetric and for odd k the form is skew-symmetric. The cubic forms, F^3, define a four dimensional representation of $\mathfrak{sl}_2$, which is symplectic. That is,

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_4 \hookrightarrow \mathrm{End} F^3,$$

where $\mathfrak{sp}_4$ is the subalgebra of $\mathrm{End}(F^3)$ preserving the degree two skew-symmetric form.

On the other hand, the quartic forms, F^4, define a five dimensional representation of $\mathfrak{sl}_2$, which is orthogonal. That is,

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_5 \hookrightarrow \mathrm{End} F^4,$$

where $\mathfrak{so}_5$ is the subalgebra of $\mathrm{End} F^4$ preserving the degree two symmetric form.

Upon a fixed choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sp}_4$, and Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{sp}_4$ we obtain a choice of root system, Φ with positive roots, Φ^+, and simple roots denoted Π. Let ω_1 and ω_2 denote the corresponding fundamental weights, and denote the irreducible finite dimensional representation indexed by $\lambda = a\omega_1 + b\omega_2 \in \mathfrak{h}^*$ for non-negative integer a and b by

$$L(\lambda) = L(a, b),$$

where we let $\dim L(1, 0) = 4$ and $\dim L(0, 1) = 5$.

The key point here is that upon restriction to $\mathfrak{sl}_2$, the space of the $\mathfrak{sp}_4$-representation $L(1,0)$ is the space of cubic forms, while $L(0,1)$ is the space of quartic forms. This unification of cubic and quartic forms was first systematically studied in [11] and [12].

In general, if $\mathcal{V}$ denotes a representation of of a group (resp. Lie algebra) and $\mathcal{W}$ denotes an irreducible representation of a subgroup (resp. subalgebra) H then upon restriction to H let

$$[\mathcal{W}, \mathcal{V}] = \dim \mathrm{Hom}_{\mathrm{H}}(\mathcal{W}, \mathcal{V})$$

denote the multiplicity.

The present article concerns, in part, the function, $b(k,l,m) = [F^k, L(l,m)]$, for a principally embedded $\mathfrak{sl}_2$ in $\mathfrak{sp}_4$. It is important to note that the numbers $b(k,l,m)$ are not difficult to compute nor are they theoretically mysterious. The point is the relationship to the group G_2, as we shall see.

We relate this situation to the problem of decomposing harmonic polynomials, in the sense of B. Kostant and S. Rallis [10]. Specifically, we let K denote a symmetrically embedded subgroup of the complex exceptional group G_2, such that the Lie algebra, $\mathfrak{k}$, of K is isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Let $\mathfrak{p}$ denote the complexified Cartan complement of $\mathfrak{p}$ in the Lie algebra of G_2. The group K acts on $\mathfrak{p}$ by restricting the adjoint representation. From the general theory of Kostant and Rallis, we know that the K-invariant subalgebra, $\mathbb{C}[\mathfrak{p}]^{\mathrm{K}}$, is a polynomial ring and $\mathbb{C}[\mathfrak{p}]$ is a free module over $\mathbb{C}[\mathfrak{p}]^{\mathrm{K}}$.

In the instance where $\mathfrak{g}$ is the Lie algebra of G_2 and $\mathrm{K} = \mathrm{SL}_2 \times \mathrm{SL}_2$, we have $\mathbb{C}[\mathfrak{p}]^{\mathrm{K}} = \mathbb{C}[\delta_1, \delta_2]$ with δ_i degree 2 and 6 respectively. Let Δ_i $(i = 1, 2)$ denote the corresponding constant coefficient K-invariant differential operators and set

$$\mathcal{H}_{\mathfrak{p}} = \{ f \in \mathbb{C}[\mathfrak{p}] | \Delta_i(f) = 0 \text{ for } i = 1, 2 \} .$$

to be the harmonic polynomials together with their gradation by degree. For each $d = 0, 1, 2, \cdots$ the group K acts linearly on $\mathcal{H}_{\mathfrak{p}}^d$. See [19] for general results on the decomposition of $\mathcal{H}_{\mathfrak{p}}^d$ into irreducible K-representations.

Motivation for considering $\mathcal{H}_{\mathfrak{p}}$ for the symmetric pair corresponding to split G_2 comes from considering the (underlying Harish-Chandra module) of spherical principal series, which are isomorphic to $\mathcal{H}_{\mathfrak{p}}$ as representations of K.

The finite-dimensional irreducible representations of $\mathrm{K} \cong \mathrm{SL}_2 \times \mathrm{SL}_2$ with polynomial matrix coefficients are of the form $F^k \otimes F^l$. In the G_2 case when $\mathrm{K} = \mathrm{SL}_2 \times \mathrm{SL}_2$, we have $\mathfrak{p} = F^3 \otimes F^1$. We organize the graded K-multiplicities in a formal power series in q defined by

$$p_{k,l}(q) = \sum_{d=0}^{\infty} [F^k \otimes F^l, \mathcal{H}_{\mathfrak{p}}^d] q^d.$$

We set $p(k,l;d) = [F^k \otimes F^l, \mathcal{H}_{\mathfrak{p}}^d]$ for non-negative integers k, l and d.

It turns out that $p_{k,l}(q)$ is a polynomial in q. That is, K-irreps occur with finite multiplicity in $\mathcal{H}_{\mathfrak{p}}$. In the last section we recall the results from [10] concerning this multiplicity. In the special case of G_2 falls into the literature on double binary (3,1) forms studied by J. A. Todd (see [16]).

From our point of view, we find the unification of the cubic and quartic seen in [12] via $\mathfrak{sp}_4$-representations in parallel with the work presented in [16] on (3,1)-forms. Neither of these references mention the group G_2. Since the group G_2 plays an important role in invariant theory (see [1]), we would like to point out:

Theorem 1. *For non-negative integers k, l and m we have*

$$b(k,l,m) = \sum_{j\geq 0} p(k,l;2m+l-6j)$$

We prove this theorem in the next section, and point out special cases, including the decomposition of the polynomial functions on the cubic and quartic. The proof uses an instance of R. Howe's theory of dual pairs (see [5]) applied to the problem of computing branching multiplicities (see [8]). In the final section we recall the result of Todd, and provide a picture of the Brion polytope (see [2, 15]) associated with this example.

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Finally, we would like to thank the referee for many helpful suggestions, and corrections.

2 Proof of the main theorem

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ is an integer partition, then let F_n^λ denote the irreducible finite dimensional representation of GL_n (resp. $\mathfrak{gl}_n$) with highest weight indexed by λ as in [5]. Note that these representations restrict irreducibly to SL_n (resp. $\mathfrak{sl}_n$). In the case $n = 2$ we will use the notation $F^d = F_2^{(d)}$.

For positive integers r and c, let $M_{r,c}$ denote the $r \times c$ complex matrices. Throughout this section we set $\mathfrak{p} = F^3 \otimes F^1$, which as a vector space may be identified with $M_{4,2}$. The group $\mathrm{GL}_4 \times \mathrm{GL}_2$ acts on $\mathbb{C}[M_{4,2}]$ by the action $(g_1, g_2) \cdot f(X) = f(g_1^T X g_2)$ for $g_1 \in \mathrm{GL}_4$, $g_2 \in \mathrm{GL}_2$, $f \in \mathbb{C}[M_{4,2}]$, and $X \in M_{4,2}$. (Here g^T denotes the transpose of a matrix g.) Under this action, we have a multiplicity free decomposition

$$\mathbb{C}[M_{4,2}]_d \cong \bigoplus F_4^\mu \otimes F_2^\mu$$

where the sum is over all partitions $\mu = (\mu_1 \geq \mu_2 \geq 0)$ with at most two parts and of size d (i.e. $\mu_1 + \mu_2 = d$).

This, multiplicity free, decomposition of $\mathbb{C}[M_{4,2}]_d$ into irreducible $\mathrm{GL}_4 \times \mathrm{GL}_2$-representations is the starting point. We will proceed as follows: (1) restrict to the group $\mathrm{Sp}_4 \times \mathrm{GL}_2 \subset \mathrm{GL}_4 \times \mathrm{GL}_2$ and decompose by using $(\mathrm{Sp}_4, \mathfrak{so}_4)$-Howe duality, then (2) restrict further to the principally embedded SL_2 in Sp_4 on the left, and the $\mathrm{SL}_2 \subset \mathrm{GL}_2$ on the right. Lastly, we compare this decomposition into irreducibles as predicted by Kostant-Rallis theory.

With this plan in mind, we will set

$$H(k,l) = \sum_{d=0}^{\infty} \left[F^k \otimes F^l, \mathbb{C}[M_{4,2}]_d\right] q^d$$

for non-negative integers k and l. We will compute two expressions for $H(k,l)$. For the first, we start with $(\mathrm{GL}_4, \mathrm{GL}_2)$-Howe duality,

$$H(k,l) = \sum_{\lambda_1 \geq \lambda_2 \geq 0} \left[F^k \otimes F^l, F_4^\lambda \otimes F_2^\lambda\right] q^{\lambda_1 + \lambda_2},$$

where we restrict the GL_4 and GL_2 irreps to (the principally embedded) SL_2. The key idea is to, first, restrict the left acting GL_4 to Sp_4, then restrict from Sp_4 to SL_2.

2.1 Symplectic-orthogonal Howe duality

The $(\mathrm{Sp}_{2k}, \mathfrak{so}(2n))$ instance of Howe duality concerns two commuting actions on $\mathbb{C}[M_{2k,n}]$. The first is given by left multiplication of Sp_{2k}, while the second is given by a Lie algebra of Sp_{2k}-invariant polynomial coefficient differential operators. The operators in this second action are generated a set spanning a Lie algebra isomorphic to $\mathfrak{so}(2n)$ (with respect to the commutator bracket). We have a multiplicity free decomposition:

Theorem 2. *Given integers k and n,*

$$\mathbb{C}[M_{2k,n}] \cong \bigoplus V_{2k}^\mu \otimes V_\mu^{2n}$$

where the sum is over $\mu = (\mu_1 \geq \cdots \geq \mu_l > 0)$ – a non-negative integer partition with $l \leq \min(k,n)$.

The modules V_{2k}^μ are finite dimensional irreducible representations of Sp_{2k}, while V_μ^{2n} are, in general ($n > 1$), infinite dimensional irreducible highest weight representations of the Lie algebra $\mathfrak{so}(2n)$.

Using the results of [8] one is able to obtain, in a stable range ($k \geq n$), a concrete combinatorial description of the branching rule from SL_{2n} to the symmetric subgroup Sp_{2n}. These results come from translating the branching problem to one of decomposing the $\mathfrak{so}_{2n}$ irreps under the actin of $\mathfrak{gl}_n$.

In the stable range (see [8]), the $\mathfrak{so}_{2n}$-representations V_μ^{2n} are (generalized) Verma modules. This fact implies that upon restriction to the symmetrically embedded $\mathfrak{gl}_n$ subalgebra we have

$$V_\mu^{2n} = \mathbb{C}[\wedge^2\mathbb{C}^n] \otimes F_n^\mu,$$

as a $\mathfrak{gl}_n$-representation.

If $n = 2$ then $\wedge^2\mathbb{C}^2$ is invariant for SL_2. Thus, it is very easy to see the decomposition of these $\mathfrak{so}_4$ irreps into $\mathfrak{sl}_2$-irreps. Consequently, in the very special case of decomposing SL_4 representations into Sp_4-irreps, we obtain a multiplicity free branching rule. This is no surprise since SL_4 is isomorphic to Spin(6) and Sp_4 is isomorphic to Spin(5), and the (Spin(6), Spin(5)) branching rule is well known to be multiplicity free (see [5]).

In any case, we have that, upon restriction, the GL_4-representations,

$$F_4^\mu \cong V_4^\mu \oplus V_4^{\mu-(1,1)} \oplus V_4^{\mu-(2,2)} \oplus \cdots \oplus V^{(\mu_1-\mu_2,0)},$$

as Sp_4-representations.

The fact that the partition $(1,1)$ has size 2 means that there is a degree 2 $Sp_4 \times SL_2$-invariant, f_1, in $\mathbb{C}[M_{4,2}]$. This polynomial, obviously, remains invariant under the subgroup $SL_2 \times SL_2$. Moreover, the algebra $\mathbb{C}[M_{4,2}]$ is free as a module over $\mathbb{C}[f_1]$. Since the degree of f_1 is two, we have the following:

$$H(k,l) = \frac{1}{1-q^2} \sum_\mu \left[F^k \otimes F^l, V^\mu \otimes F^{\mu_1-\mu_2}\right] q^{\mu_1+\mu_2}$$

where the sum is over all $\mu = (\mu_1 \geq \mu_2 \geq 0)$ with $|\mu| = \mu_1 + \mu_2$. Thus, $l = \mu_1 - \mu_2$.

Let $m = \mu_2$ so that $\mu_1 + \mu_2 = 2m + l$. That is, $V^\mu = L(l,m)$ and we have

$$b(k,l,m) = \left[F^k, L(l,m)\right].$$

Restricting from Sp_4 to the principally embedded SL_2 we obtain

$$H(k,l) = \frac{1}{1-q^2} \sum_{m=0}^{\infty} b(k,l,m) q^{2m+l}. \tag{1}$$

2.2 The symmetric pair $(\mathrm{G}_2, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$

An ordered pair of groups, (G, K), is said to be a *symmetric pair* if G is a connected reductive linear algebraic group (over $\mathbb{C}$), and K is an open subgroup of the fixed points of a regular involution on G, . The Lie algebra of G, denoted $\mathfrak{g}$, contains the Lie algebra of K, denoted $\mathfrak{k}$. The differential of the involution, denoted θ has -1 eigenspace

$$\mathfrak{p} = \{X \in \mathfrak{g} | \theta(X) = -X\}.$$

Let $\mathfrak{a}$ denote a maximal toral[2] subalgebra contained in $\mathfrak{p}$. Let M denote the centralizer of $\mathfrak{a}$ in K.

Upon restricting the adjoint representation of G on $\mathfrak{g}$, we obtain a linear group action of K on $\mathfrak{p}$. From [10], we know that $\mathbb{C}[\mathfrak{p}]^K$ is freely generated by $\dim \mathfrak{a}$ elements, as an algebra. The same is true about the constant coefficient K-invariant differential operators, denoted $\mathcal{D}(\mathfrak{p})^{\mathrm{K}}$. Define

$$\mathcal{H}_{\mathfrak{p}} = \{f \in \mathbb{C}[\mathfrak{p}] | \partial f = 0 \text{ for all } \partial \in \mathcal{D}(\mathfrak{p})^{\mathrm{K}}\}.$$

These are the K-harmonic polynomials on $\mathfrak{p}$. Set $\mathcal{H}_{\mathfrak{p}}^d = \mathcal{H}_{\mathfrak{p}} \cap \mathbb{C}[\mathfrak{p}]_d$. As a K-representation we have a gradation, $\mathcal{H}_{\mathfrak{p}} = \bigoplus_{d=0}^{\infty} \mathcal{H}_{\mathfrak{p}}^d$.

From [10], $\mathcal{H}_{\mathfrak{p}} \cong \mathrm{Ind}_M^K 1$. That is, as a representation of K,

$$\mathcal{H} \cong \{f \in \mathbb{C}[K] | f(mk) = f(k) \text{ for all } m \in \mathrm{M} \text{ and } k \in \mathrm{K}\}.$$

By Frobenius reciprocity, each irreducible K-representation, F, occurs with multiplicity $\dim F^M$ in $\mathcal{H}_{\mathfrak{p}}$.

In the present article, we consider the case when $\mathfrak{g}$ is the Lie algebra of G_2 and $\mathfrak{k} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ in detail. We have remarked that $\mathfrak{p} \cong F^3 \otimes F^1$. The group M is isomorphic to the eight element quaternion group,

$$M \cong \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}.$$

An embedding of M into K is given by

$$i \mapsto \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$
$$j \mapsto \left(\begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}, \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix} \right)$$
$$k \mapsto \left(\begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \right)$$

In [5], this example is treated in Chapter 12. It is pointed out that $\mathbb{C}[\mathfrak{p}]^{\mathrm{K}} = \mathbb{C}[f_1, f_2]$ where $\deg f_1 = 2$ and $\deg f_2 = 6$. The fact that $\mathbb{C}[\mathfrak{p}]$ is free over

[2] A *toral* subalgebra is an abelian subalgebra consisting of semisimple elements.

$\mathbb{C}[f_1, f_2]$ leads to,

$$H(k,l) = \frac{p_{k,l}(q)}{(1-q^2)(1-q^6)} = \frac{1}{(1-q^2)(1-q^6)} \sum_{d=0}^{\infty} p(k,l;d) q^d$$
$$= \frac{1}{1-q^2} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} p(k,l;d) q^{d+6j}.$$

We introduce a new parameter m, and re-index with $d + 6j = l + 2m$ we obtain

$$H(k,l) = \frac{1}{1-q^2} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} p(k,l;l+2m-6j) q^{l+2m}. \qquad (2)$$

The main result follows by comparing Equations 1 and 2.

2.3 Special Cases

The insolvability of the quintic motivates special attention to forms of degree less than five. The invariant theory of quadratic forms reduces to the theory of spherical harmonics (i.e. Kostant-Rallis theory for the pair (SO(4),SO(3)).) mentioned at the beginning of this article.

The structure of G_2 is intimately related to the cubic (see [1]). The complexified Cartan decomposition of split G_2 has $\mathfrak{p}$ being two copies of the cubic. However, there is no symmetric pair in which $\mathfrak{p}$ consists of exactly on copy of the cubic form. Nonetheless, the algebra of polynomial functions on the cubic forms is free over the invariant subalgebra (see Chapter 12 of [5]).

From our point of view, the multiplicity of an irreducible SL_2-representation in the polynomial function on the cubic is given by reduces to:

Corollary 1 (The Cubic).

$$\left[F^k, \mathbb{C}[F^3]_d\right] = \sum_{j \geq 0} p(k,d;d-6j)$$

Proof. The irreducible representation of $\mathfrak{sp}_4$ of the form $L[d,0] \cong \mathbb{C}[V]_d$ where $V = \mathbb{C}^4$ is the defining representation. Upon restriction to a principal $\mathfrak{sl}_2$ subalgebra $V \cong F^3$. The result follows by taking $m = 0$ in the main theorem.

The polynomial function on the quartic forms are a free module over the invariants. This fact is a special case of Kostant-Rallis theory for (SL_3, SO_3), where $\mathfrak{p}$ consists of the five dimensional space of trace zero 3×3 symmetric matrices, denoted $SM(3)_0$. As a representation of SL_2 (locally SO_3), $\mathfrak{p} = SM(3)_0$ is the quartic under the conjugation action of SO_3.

The relationship between the quartic and G_2 is less transparent. In a nutshell, since Sp_4 is isomorphic to Spin(5), and the five dimensional defining representation of Spin(5) restricts irreducibly to the quartic with respect to the principally embedded SL_2.

However, it follows from the split G_2 example by specializing the main theorem that:

Corollary 2 (The Quartic). *Let $\mathcal{H}(F^4)$ denote the space of spherical harmonics with respect to the orthogonal representation, F^4. For each d, we have $L(0, d) = \mathcal{H}^d(F^4)$, and*

$$\left[F^k, \mathcal{H}^d(F^4)\right] = \sum_{j \geq 0} p(k, 0; 2d - 6j).$$

Proof. Let $l = 0$ from the main theorem.

There is a third obvious special case describing the $\mathfrak{sl}_2$-invariants in an arbitrary $\mathfrak{sp}_4$-representation with respect to the principal embedding. That is:

Corollary 3 (principal invariants).

$$\dim \left(L(l, m)\right)^{SL_2} = \sum_{j \geq 0} p(0, l; 2m - 6j)$$

Proof. Let $k = 0$ in the main theorem.

The pair $(\mathfrak{sp}_4, \mathfrak{sl}_2)$ is not a symmetric pair. In particular, there is no instance of the Cartan-Helgason theorem describing $\mathfrak{sp}_4$-irreps. with a $\mathfrak{sl}_2$-invariant vector. In fact, the $\mathfrak{sl}_2$-invariant subspace does not have to be one dimensional.

We mention this special case since it suggests a relationship between the $(\mathfrak{sp}_4, \mathfrak{sl}_2)$ generalized Harish-Chandra modules and the unitary dual of split G_2.

3 Todd's covariants for double binary (3,1) forms, and the Brion polytope

We conclude this article with the remark that the $(G_2, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ instance of Kostant-Rallis theory is seen in the double binary (3,1) forms, as studied by J. A. Todd in 1949. Although there is no mention of the group G_2, a generating set of covariants is provided. That is, for each $d = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ a list of highest weight vectors (given as polynomials) is provided. The weights are summarized in the following.

3.1 Table: $\mathrm{SL}_2 \times \mathrm{SL}_2$*-covariants in* $\mathbb{C}[F^3 \otimes F^1]$

The $\mathrm{SL}_2 \times \mathrm{SL}_2$ highest weight vectors in $\mathbb{C}[F^3 \otimes F^1]$ span a graded subalgebra – the covariants. In the table below, we denote Todd's generator for the highest weight of $F^k \otimes F^l$ by $[k, l]$. The degrees are indicated.

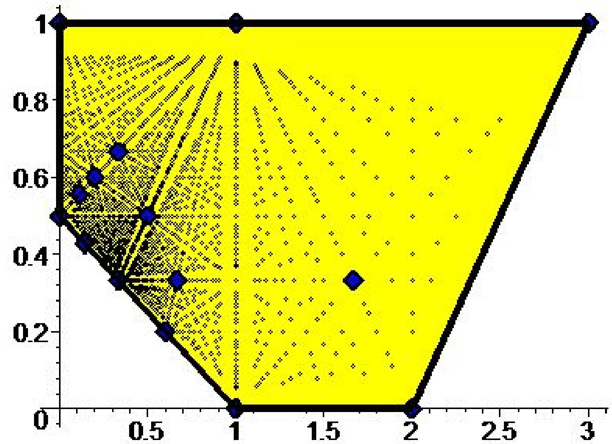
degree	weight			
1	$[3,1]$			
2	$[4,0]$	$[2,2]$	$[0,0]$	
3	$[5,1]$	$[3,3]$	$[1,1]$	
4	$[4,0]$	$[2,2]$	$[0,4]$	
5	$[3,1]$	$[1,3]$		
6	$[6,0]$	$[4,2]$	$[2,4]$	$[0,0]$
7	$[1,3]$			
8	$[0,4]$			
9	$[1,5]$			
12	$[0,6]$			

Not included in [16] are the *relations* between these covariants, nor is any information about the vector space dimension of their span in each multi-graded component. Note that, modulo the degree 2 and 6 invariants, the multiplicity of a given covariant may be computed by the dimension of the M-fixed vectors. Thus, implicitly the eight element quaternion group plays role in this example.

A convenient way to "visualize" this example comes from the theory of the *Brion polytope* as described in [15] (see also [2]). That is for each occurrence of $F^k \otimes F^l$ in $\mathcal{H}_{\mathfrak{p}}^d$, with $d > 0$, plot the point

$$(\frac{k}{d}, \frac{l}{d})$$

on the xy-plane. The closure of these points is a polytope. In the special case corresponding to G_2, this polytope is two dimensional and can be visualized as



The large dots in the above picture are the Todd covariants. The smaller dots correspond to specific K-types. We refer the reader to [17] for a detailed account of how this polytope relates to the $(\mathfrak{sp}_4, \mathfrak{sl}_2)$ branching rule, asymptotically.

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