

SMALL SEMISIMPLE SUBALGEBRAS OF SEMISIMPLE LIE ALGEBRAS

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To Roger Howe, with friendship and admiration.

ABSTRACT. Let \mathfrak{g} denote a semisimple Lie algebra with the property that none of its simple factors is of type A_1 . Suppose that $\mathfrak{k} \subseteq \mathfrak{g}$ is a Lie subalgebra isomorphic to \mathfrak{sl}_2 . The goal of this paper is to prove the existence of a positive integer $b(\mathfrak{k}, \mathfrak{g})$ such that if V is any irreducible finite dimensional \mathfrak{g} -module then when restricted to \mathfrak{k} , the decomposition of V will contain some irreducible \mathfrak{k} -module with dimension less than $b(\mathfrak{k}, \mathfrak{g})$. Beyond proving the theorem, we show how it may be generalized by introducing the notion of a *small* subalgebra.

1. INTRODUCTION

The main goal of this paper is to provide a proof of the following:

Theorem (see [18]). *Let \mathfrak{k} be an \mathfrak{sl}_2 -subalgebra of a semisimple Lie algebra \mathfrak{g} , none of whose simple factors is of type A_1 . Then there exists a positive integer $b(\mathfrak{k}, \mathfrak{g})$, such that for every irreducible finite dimensional \mathfrak{g} -module V , there exists an injection of \mathfrak{k} -modules $W \rightarrow V$, where W is an irreducible \mathfrak{k} -module of dimension less than $b(\mathfrak{k}, \mathfrak{g})$.*

Note that the above statement (announced in [18]) is not asserting that *every* irreducible representation of \mathfrak{k} of dimension less than $b(\mathfrak{k}, \mathfrak{g})$ occurs. Rather, the theorem asserts a *uniform* upper bound on the dimension of the smallest irreducible representation of \mathfrak{k} that occurs. One might say we consider the *lowest* highest weight of a \mathfrak{g} -module relative to the restricted action of \mathfrak{k} .

More precisely, let W_i denote the irreducible $i + 1$ dimensional representation of \mathfrak{sl}_2 . If V is an irreducible finite dimensional \mathfrak{g} -module then, upon restriction to a fixed subalgebra isomorphic to \mathfrak{sl}_2 we have a decomposition:

$$V \cong \bigoplus_{i=1}^{\infty} \underbrace{W_i \oplus \cdots \oplus W_i}_{m_i(V)}$$

with $m_i(V) \geq 0$ being the multiplicity of W_i in V . The above theorem asserts that the number:

$$\max \left\{ \min_{m_i(V) \neq 0} i \mid V \text{ an irreducible finite dimensional } \mathfrak{g}\text{-module} \right\}$$

is finite provided that \mathfrak{g} has no simple factor of type A_1 .

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In fact, the result is more general than $\mathfrak{k} \cong \mathfrak{sl}_2$. In Section 3.5.2 (and then in Theorem 4.0.11) we discuss the situation when \mathfrak{k} is replaced by a general semisimple Lie algebra. The general result involves an analogous hypothesis that the simple factors of \mathfrak{g} not belong to a certain finite set of simple Lie algebras determined by the subalgebra \mathfrak{k} . The proof in the general situation uses essentially the same machinery as the \mathfrak{sl}_2 case. We emphasize the \mathfrak{sl}_2 case in some non-trivial examples.

As the title suggests, we introduce the terminology that a semisimple Lie algebra \mathfrak{k} is said to be a *small* subalgebra of a semisimple Lie algebra \mathfrak{g} if there exists a bound $b(\mathfrak{k}, \mathfrak{g})$ such that for every irreducible finite dimensional \mathfrak{g} -module V , there exists an injection of \mathfrak{k} -modules $W \rightarrow V$, where W is an irreducible \mathfrak{k} -module having dimension less than $b(\mathfrak{k}, \mathfrak{g})$. For example, the above theorem asserts that any \mathfrak{sl}_2 -subalgebra of (say) E_8 is small in E_8 . Note that the rank n symplectic Lie algebra is not small in \mathfrak{sl}_{2n} , under the standard embedding. This can be seen by observing that the symmetric powers of the standard representation of sl_{2n} remain irreducible when restricted to sp_{2n} .

Although technically different, we remark that it would not significantly harm the content of the result if we regard $b(\mathfrak{k}, \mathfrak{g})$ as a bound on a norm of the highest weight of the representation of \mathfrak{k} rather than its dimension. Clearly, in the \mathfrak{sl}_2 -case these are essentially the same notion.

We consider the results of this paper interesting in their own right. However, we should note that the original motivation stems from a study of generalized Harish-Chandra modules as described in [18]. We recall the situation here.

Let \mathfrak{k} denote a reductive subalgebra of a semisimple Lie algebra \mathfrak{g} . By a $(\mathfrak{g}, \mathfrak{k})$ -module M we mean a \mathfrak{g} -module such that \mathfrak{k} acts locally finitely. That is, for all $v \in M$, $\dim \mathcal{U}(\mathfrak{k})v < \infty$ (here $\mathcal{U}(\mathfrak{k})$ is the universal enveloping algebra of \mathfrak{k}). We say that the $(\mathfrak{g}, \mathfrak{k})$ -module is *admissible* if every irreducible finite dimensional representation of \mathfrak{k} occurs with finite multiplicity in M .

The question arises: Given an infinite dimensional, admissible $(\mathfrak{g}, \mathfrak{k})$ -module M , does there exist a semisimple subalgebra $\mathfrak{k}' \subset \mathfrak{k}$ such that M is an admissible $(\mathfrak{g}, \mathfrak{k}')$ -module? Indeed if \mathfrak{k}' is a small subalgebra of \mathfrak{k} then the answer is *no*. To see this, let W_1, W_2, \dots, W_m denote all of the irreducible \mathfrak{k}' -modules with dimension less than $b(\mathfrak{k}', \mathfrak{k})$ (it is an easy exercise using the Weyl dimension formula to see that there are finitely many). Next, consider any infinite set S of pairwise inequivalent irreducible finite dimensional \mathfrak{k} -modules occurring in M . As \mathfrak{k}' -modules, each $V \in S$ will have an irreducible constituent equivalent to at least one W_i . By the pigeonhole principle there must exist j (with $1 \leq j \leq m$) such that *infinitely* many elements of S contain W_j as a \mathfrak{k}' -submodule. Thus M has an irreducible \mathfrak{k}' -module with infinite multiplicity.

Another source of motivation comes from the theory of induced representations. Let G and K denote a linear algebraic group over \mathbb{C} with Lie algebra \mathfrak{g} and \mathfrak{k} respectively. Assume that $K \subseteq G$. If V is a G -representation then we may restrict the G -action to K to obtain a representation of K . Restriction is in fact a functor, denoted Res , from the category of regular G -representations to regular K -representations. The right adjoint of Res is the induction functor, denoted Ind . The relationship between these two functors is the *Frobenius*

Reciprocity theorem asserting the linear isomorphism,

$$\mathrm{Hom}_G(V, \mathrm{Ind}_K^G W) \cong \mathrm{Hom}_K(\mathrm{Res}_G^K V, W)$$

where V and W are representations of G and K respectively.

As is widely done, results about the restriction of a G -representation may be reinterpreted in terms of induced K -representations. In our context one can ask: Does there exist a *finite dimensional* K -representation, W , such that any irreducible representation of G may be embedded in $\mathrm{Ind}_K^G W$? Using Frobenius reciprocity, one can easily see that the question is answered in the affirmative exactly when \mathfrak{k} is a small subalgebra of \mathfrak{g} by taking W to be the direct sum all irreducible representations of K with dimension less than $b(\mathfrak{k}, \mathfrak{g})$.

Historically, an important variation of this question concerns the theory of *models* (see [1]), which have played a crucial role in representation theory of real groups (see for example [24]). In the setting of compact Lie groups, one seeks a K -representation, W such that every G -representation occurs in $\mathrm{Ind}_K^G W$ with multiplicity one. Note that in this setting one assumes that K is the maximal compact subgroup of the split real form corresponding to G . Thus one is motivated to understand induced representations outside of this setting.

These theorems might be classified in the area of branching rules. However, it is fair to say that they are of a more qualitative nature than other results in the subject. In fact, it is important to note that the standard algorithms relying on the Weyl character formula or Kostant's partition function do not seem to shed light on this type of question. On the other hand, the results presented here may be interpreted combinatorially as statements about characters of representations. For example, let s_λ denotes the Schur polynomial in the variables $x_0, x_1, x_2, \dots, x_n$ indexed by a partition λ as in [17]. We may make the substitution $x_i = t^{n-2i}$ and then expand

$$s_\lambda = \sum_{k \geq 0} m_k \chi^k(t),$$

where $\chi^k(t) = \frac{t^{k+1} - t^{-k-1}}{t - t^{-1}}$. The non-negative integers m_k are combinatorially complicated, although there is a bound $b(n)$ independent of λ such that $m_k > 0$ for some $k < b(n)$. This fact is an application of the above theorem with $\mathfrak{g} = \mathfrak{sl}_n$ and \mathfrak{k} a principally embedded \mathfrak{sl}_2 -subalgebra of \mathfrak{g} .

In this paper we do not obtain a bound on the numbers $b(\mathfrak{k}, \mathfrak{g})$ in the most general situation. We do however, compute the smallest possible value for some specific cases such as when $\mathfrak{g} = G_2$ and \mathfrak{k} is a principal \mathfrak{sl}_2 -subalgebra. Certainly, any example can be dealt with computationally using our methods, and a future direction of research would be to obtain a bound for $b(\mathfrak{k}, \mathfrak{g})$ for large families of pairs $(\mathfrak{g}, \mathfrak{k})$.

The proofs in this paper require the theory of algebraic groups in a fairly strong way. It is important to note that our results ultimately rely on Theorem 2.0.1, which is proved in Section 2. This theorem involves the question of when the coordinate ring of a linear algebraic group has unique factorization.

The geometric nature of the problem leads one to try other approaches involving geometric ideas that have, in other settings, shed light on problems of representation theory. One approach that was suggested to us by Alan Huckleberry was to cast the problem in the context of symplectic geometry. In this setting the theory of the moment map may provides enough information about the support of a \mathfrak{g} -module when restricted to a non-abelian subalgebra

\mathfrak{k} . This approach may be very fruitful in that it may also provide information regarding the multiplicities arising from the branching problem. This will be a future direction of research.

The paper is organized as follows: Section 2 opens with a statement of Theorem 2.0.1, which turns out to be an important step in the proof of the main result. Section 3 introduces the necessary facts about Lie algebras and representation theory, with the goal being the proof of Proposition 3.5.7 (as an application of Theorem 2.0.1), and Proposition 3.3.1. In Section 4 we prove the main theorem, using Propositions 3.3.1 and 3.5.7. In Section 5, we apply the theorem to the special case where \mathfrak{g} is the exceptional Lie algebra G_2 , and \mathfrak{k} is a principal \mathfrak{sl}_2 -subalgebra of \mathfrak{g} . We obtain a sharp estimate of $b(\mathfrak{k}, \mathfrak{g})$ (in this case).

2. INVARIANT THEORY.

The context for this section lies in the theory of algebraic group actions on varieties. A good general reference for our terminology and notation is [21] which contains translations of works, [20] and [22]. For general notation and terminology from commutative algebra and algebraic geometry see [8] and [11]. For the general theory of linear algebraic groups, see [2].

All varieties are defined over \mathbb{C} , although we employ some results that are valid in greater generality. Unless otherwise stated, all groups are assumed to have the structure of a connected linear algebraic group. Of particular interest is the situation where a group, G , acts on an affine or quasiprojective variety. Given a (quasi-) affine variety, X , we denote the ring of regular functions on X , by $\mathbb{C}[X]$, and $\mathbb{C}[X]^G$ denotes the ring of G -invariant functions.

We now turn to the following:

Problem. *Let X be an irreducible (quasi-) affine variety. Let G be a algebraic group acting regularly on X . When is $\mathbb{C}[X]^G \not\cong \mathbb{C}$? That is to say, when do we have a non-trivial invariant?*

The general problem may be too hard in this generality. We begin by investigating a more restrictive situation which we describe next.

For our purposes, a generic orbit, $\mathcal{O} \subset X$, is defined to be an orbit of a point $x \in X$ with minimal isotropy group (ie: $G_{x_0} = \{g \in G \mid g \cdot x_0 = x_0\}$ for $x_0 \in X$).

Theorem 2.0.1. *Assume that X is an irreducible quasiprojective variety with a regular action by a linear algebraic group H such that:*

- (1) *A generic H -orbit, \mathcal{O} , in X has $\dim \mathcal{O} < \dim X$,*
- (2) *$\mathbb{C}[X]$ is factorial, and*
- (3) *H has no rational characters¹.*

Then, $\mathbb{C}[X]^H \not\cong \mathbb{C}$. Furthermore, $\text{trdeg } \mathbb{C}[X]^H = \text{codim } \mathcal{O}$.

This theorem will be used to prove Proposition 3.5.1. In order to provide a proof of the above theorem we require some preparation. Let $\mathbb{C}(X)$ denote the field of complex valued rational functions on X . Our plan will be to first look at the ring of rational invariants, $\mathbb{C}[X]^H$. Given an integral domain, R , let QR denote the quotient field. Clearly $Q(\mathbb{C}[X]^H) \subseteq \mathbb{C}(X)^H$. Under the assumptions of Theorem 2.0.1 we have equality. As is seen by:

Theorem 2.0.2 ([21] p. 165). *Suppose $k(X) = Qk[X]$. If either,*

¹A rational character of H is defined to be a regular function $\chi : H \rightarrow \mathbb{C}^\times$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in H$.

- (a) the group G^0 is solvable², or
- (b) the algebra $k[X]$ is factorial,

then any rational invariant of the action $G : X$ can be represented as a quotient of two integral semi-invariants (of the same weight). If, in addition, G^0 has no nontrivial characters (which in case (a) means it is unipotent), then $k(X)^G = Q(k[X]^G)$.

As one might expect from the hypothesis (2) of Theorem 2.0.1, our applications of Proposition 2.0.2 will involve condition (b). We will then use:

Proposition 2.0.3 ([21] p. 166). *Suppose the variety X is irreducible. The algebra $k[X]^G$ separates orbits in general position if and only if $k(X)^G = Qk[X]^G$, and in this case there exists a finite set of integral invariants that separates orbits in general position and the transcendence degree of $k[X]^G$ is equal to the codimension of an orbit in general position.*

Proof of Theorem 2.0.1. In our situation X is a quas affine variety, thus $k(X) = Qk[X]$. By assumptions (2) and (3) of Theorem 2.0.1 and Theorem 2.0.2 (using part (b)) we have that $k(X)^G = Q(k[X]^G)$. $\dim \mathbb{C}[X]^H = \text{codim } \mathcal{O}$ follows from the irreducibility of X and Proposition 2.0.3. By assumption (1), $\text{codim } \mathcal{O} > 0$. Thus, $\mathbb{C}[X]^H \not\cong \mathbb{C}$. \square

Question. For our purposes, we work within the context of the assumptions of Theorem 2.0.1, but to what extent may we relax the assumptions to keep the conclusion of the theorem?

Consider a triple (G, S, H) such that the following conditions (*) hold³

Conditions (*).

- (1) G is a connected, simply-connected, semisimple linear algebraic group over \mathbb{C} .
- (2) S and H are connected algebraic subgroups of G such that:
 - (a) $S \subseteq G$, is a connected algebraic subgroup with no non-trivial rational characters.
 - (b) $H \subseteq G$, is semisimple (and hence has no non-trivial rational characters).

For our situation we will require the following result (used in Proposition 3.5.1):

Theorem 2.0.4 (Voskresenskiĭ (see [19], [25], [26])). *If G is a connected, simply connected, semisimple linear algebraic group then the ring of regular functions, $\mathbb{C}[G]$, is factorial.*

Remark 2.0.5. From the theory of the big cell, we can easily deduce that a connected algebraic group is always *locally* factorial. One should note however that there are obstructions to the factorial property of $\mathbb{C}[G]$. For example, if G is a connected linear algebraic group then $\mathbb{C}[G]$ is factorial iff the Picard group of G (denoted $\text{Pic}(G)$) is trivial (see [11] Proposition 6.2 and Corollary 6.16). Using this fact we can deduce Theorem 2.0.4 from:

Proposition 2.0.6 (see [13] Proposition 4.6). *Let G be a connected linear algebraic group. Then, there exists a finite covering $G' \rightarrow G$ of algebraic groups such that $\text{Pic}(G') = 0$.*

Remark 2.0.7. If we assume that G is semisimple (and connected), then we can deduce that the dual of the fundamental group of G is isomorphic to $\text{Pic}(G)$ from Proposition 3.2 of [14].

²Notation: As usual, G^0 denotes the connected component of the identity in G .

³Throughout this article, we denote the Lie algebras of G and H by \mathfrak{g} and \mathfrak{h} , but we do not need this notation at present.

We now return to our situation. Later, in the proof of Proposition 3.5.1 we will combine Theorem 2.0.4 with the following result:

Theorem 2.0.8 (see [21], page 176). *If $\mathbb{C}[X]$ is factorial and the group S is connected and has no nontrivial characters, then $\mathbb{C}[X]^S$ is factorial.*

We will apply the above result to the variety $X := G/S$. Regarding the geometric structure of the quotient we refer the reader to the excellent survey in [20] Section 4.7. We briefly summarize the main points: if G is an algebraic group with an algebraic subgroup $L \subseteq G$ then the quotient G/L has the structure of a quasiprojective variety. If G is reductive, then G/L is affine iff L is reductive; G/L is projective iff L is a parabolic subgroup (ie: L contains a Borel subgroup of G).

The condition of G/L being quasiaffine is more delicate, but includes the case where $L = S$ from the conditions (*). More specifically, G/L is quasiaffine when L is quasiparabolic⁴, which will be the situation in Section 3.4 of the present paper.

Remark 2.0.9. In general, a subgroup S of G is called *observable* if G/S is quasiaffine. A theorem of Sukhanov (see [20] page 194) asserts that a connected algebraic subgroup $H \subseteq G$ is observable if and only if it is tamely embedded⁵ in some quasiparabolic subgroup. Also note that by a result of Weisfeiler, any algebraic subgroup of G may be tamely embedded in some parabolic subgroup.

A consequence of Theorems 2.0.4 and 2.0.8 is that the algebra $\mathbb{C}[X]$ is factorial, where $X = G/S$. The group H then acts regularly on G/S by left translation (ie: $x \cdot gS = xgS$, for $g \in G$, $x \in H$). And therefore, we are in a position to apply Theorem 2.0.1, as in the next section.

3. REPRESENTATION THEORY.

This section begins by recalling some basic notation, terminology and results of Lie theory. We refer the reader to [3], [4], [9], [10], and [12] for this material.

3.1. Notions from Lie theory. Let G denote a semisimple, connected, complex algebraic group. We will assume that G is simply connected. T will denote a maximal algebraic torus in G . Let $r = \dim T$. Let B be a Borel subgroup containing T . The unipotent radical, U , of B is a maximal unipotent subgroup of G such that $B = T \cdot U$. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \mathfrak{n}^+$ denote the Lie algebras of G, T, B , and U respectively. Let $W := N_G(T)/T$ denote the Weyl group corresponding to G and T .

The Borel subalgebra, \mathfrak{b} , contains the Cartan subalgebra, \mathfrak{h} and is a semidirect sum⁶ $\mathfrak{b} = \mathfrak{h} \rtimes \mathfrak{n}^+$.

The weights of \mathfrak{g} are the linear functionals $\xi \in \mathfrak{h}^*$. For $\alpha \in \mathfrak{h}^*$, set:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}.$$

⁴A *quasiparabolic* subgroup of G is one of the form $\{g \in P \mid \lambda(g) = 1\}$ where P is parabolic subgroup of G and λ is a dominant character of P .

⁵An algebraic group H is tamely embedded in an algebraic group F if the unipotent radical of H is contained in the unipotent radical of F .

⁶The symbol \rtimes stands for the semidirect sum of Lie algebras. If $\mathfrak{g} = \mathfrak{g}_1 \rtimes \mathfrak{g}_2$ then \mathfrak{g}_2 is an ideal in \mathfrak{g} and $\mathfrak{g}/\mathfrak{g}_2 \cong \mathfrak{g}_1$.

For $0 \neq \alpha \in \mathfrak{h}^*$, we say that α is a root if $\mathfrak{g}_\alpha \neq (0)$. For such α , we have $\dim \mathfrak{g}_\alpha = 1$. Let Φ denote the set of roots. We then have the decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

The choice of B defines a decomposition $\Phi = \Phi^+ \cup -\Phi^+$ so that $\mathfrak{n}^+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. We refer to Φ^+ (resp. $\Phi^- := -\Phi^+$) as the positive (resp. negative) roots. Set: $\mathfrak{n}^- = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$. Let \overline{B} denote the (opposite) Borel subgroup of G with Lie algebra $\mathfrak{h} \oplus \mathfrak{n}^-$. There is a unique choice of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ contained in Φ^+ , such that each $\alpha \in \Phi^+$ can be expressed as a non-negative integer sum of simple roots. Π is a vector space basis for \mathfrak{h}^* . Given $\xi, \eta \in \mathfrak{h}^*$ we write $\xi \preceq \eta$ if $\eta - \xi$ is a non-negative integer combination of simple (equiv. positive) roots. \preceq is the dominance order on \mathfrak{h}^* .

For each positive root α , we may choose a triple: $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{h}$, such that $H_\alpha = [X_\alpha, X_{-\alpha}]$ and $\alpha(H_\alpha) = 2$. Span $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ is then a three dimensional simple (TDS) subalgebra of \mathfrak{g} , and is isomorphic to \mathfrak{sl}_2 .

The adjoint representation, $ad : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ allows us to define the Killing form, $(X, Y) = \text{Trace}(ad X ad Y)$ ($X, Y \in \mathfrak{g}$). The semisimplicity of \mathfrak{g} is equivalent to the non-degeneracy of the Killing form. By restriction, the form defines a non-degenerate form on \mathfrak{h} , also denoted $(,)$. Using this form we may define $\iota : \mathfrak{h} \rightarrow \mathfrak{h}^*$ by $\iota(X)(-) = (X, -)$ ($X \in \mathfrak{h}$), which allows us to identify \mathfrak{h} with \mathfrak{h}^* . Under this identification, we have $\iota(H_\alpha) = \frac{2\alpha}{(\alpha, \alpha)} =: \alpha^\vee$.

By definition, the Weyl group, W acts on T . By differentiating this action we obtain an action on \mathfrak{h} , which is invariant under $(,)$. Via ι , we obtain an action of W on \mathfrak{h}^* . In light of this, we view W as a subgroup of the orthogonal group on \mathfrak{h}^* . W preserves Φ . For each $\alpha \in \Phi$, set $s_\alpha(\xi) = \xi - (\xi, \alpha^\vee)\alpha$ (for $\xi \in \mathfrak{h}^*$) to be the reflection through the hyperplane defined by α^\vee . We have $s_\alpha \in W$. For $\alpha_i \in \Pi$, let $s_i := s_{\alpha_i}$, be the simple reflection defined by α_i . W is generated by the simple reflections. For $w \in W$, let $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ be a reduced expression (ie: an expression for w with shortest length). The number ℓ is independent of the choice of reduced expression. We call $\ell =: \ell(w)$ the length of w . Note that $\ell(w) = |w(\Phi^+) \cap \Phi^-|$. There is a unique longest element of W , denoted w_0 of length $|\Phi^+|$.

The fundamental weights, $\{\omega_1, \dots, \omega_r\}$ are defined by the conditions $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$. We fix the ordering of the fundamental weights to correspond with the usual numbering of the nodes in the Dynkin diagram as in [4]. Set $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^r \omega_i$. A weight $\xi \in \mathfrak{h}^*$ is said to be dominant if $(\xi, \alpha) \geq 0$ for all $\alpha \in \Pi$. The weight lattice $P(\mathfrak{g}) = \{\xi \in \mathfrak{h}^* | (\xi, \alpha^\vee) \in \mathbb{Z}\} = \sum_{i=1}^r \mathbb{Z}\omega_i$. We define the dominant integral weights to be those $\xi \in P(\mathfrak{g})$ such that $(\xi, \alpha) \geq 0$ for all $\alpha \in \Pi$. The set of dominant integral weights, $P_+(\mathfrak{g})$, parameterizes the irreducible finite dimensional representations of \mathfrak{g} (or equivalently, of G). We have ${}^7 P_+(\mathfrak{g}) = \sum_{i=1}^r \mathbb{N}\omega_i$.

3.2. Notions from representation theory. $\mathcal{U}(\mathfrak{g})$ denotes the universal enveloping algebra of \mathfrak{g} . The category of Lie algebra representations of \mathfrak{g} is equivalent to the category of $\mathcal{U}(\mathfrak{g})$ -modules. A \mathfrak{g} -representation (equiv. $\mathcal{U}(\mathfrak{g})$ -module), M , is said to be a weight module if $M = \bigoplus M(\xi)$, where:

$$M(\xi) = \{v \in M | Hv = \xi(H)v \forall H \in \mathfrak{h}\}.$$

⁷As usual, $\mathbb{N} = \{0, 1, 2, \dots\}$ (the non-negative integers).

Among weight modules are the modules admitting a highest weight vector. That is to say, a unique (up to scalar multiple) vector, $v_0 \in M$ such that:

- (1) $\mathbb{C}v_0 = M^{\mathfrak{n}^+} := \{v \in M \mid \mathfrak{n}^+ v = 0\}$,
- (2) $M(\lambda) = \mathbb{C}v_0$ for some $\lambda \in \mathfrak{h}^*$, and
- (3) $\mathcal{U}(\mathfrak{n}^-)v_0 = M$.

Such a module is said to be a highest weight module (equiv highest weight representation). λ is the highest weight of M . Given $\xi \in \mathfrak{h}^*$ with $M(\xi) \neq (0)$ we have $\xi \preceq \lambda$.

For $\lambda \in \mathfrak{h}^*$, we let \mathbb{C}_λ be the 1-dimensional representation of \mathfrak{h} defined by λ , then extended trivially to define a representation of \mathfrak{b} by requiring $\mathfrak{n}^+ \cdot \mathbb{C}_\lambda = (0)$. Let $N(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$ denote the Verma module defined by λ . Let $L(\lambda)$ denote the irreducible quotient of $N(\lambda)$. For $\lambda, \mu \in \mathfrak{h}^*$, $L(\lambda) \cong L(\mu)$ iff $\lambda = \mu$. $L(\lambda)$ (and $N(\lambda)$) are highest weight representations. Any irreducible highest weight representation is equivalent to $L(\lambda)$ for a unique $\lambda \in \mathfrak{h}^*$. The theorem of the highest weight asserts that $\dim L(\lambda) < \infty$ iff $\lambda \in P_+(\mathfrak{g})$.

Each $\mu \in P(\mathfrak{g})$, corresponds to a linear character of T , denoted e^μ . For $\lambda \in P_+(\mathfrak{g})$, the character of $L(\lambda)$ defines a complex valued regular function on T . This character may be expressed as in the following:

Theorem 3.2.1 (Weyl).

$$ch L(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}.$$

It becomes necessary for us to refer to representations of both the group G and the Lie algebra of G (always denoted \mathfrak{g}). By our assumptions on G (in conditions $(*)$), every finite dimensional complex representation of \mathfrak{g} integrates to a regular representation of G . The differential of this group representation recovers the original representation of the Lie algebra. We will implicitly use this correspondence.

For our purposes, of particular importance is the decomposition of the regular ⁸ representation of G . That is,

Theorem 3.2.2. For $f \in \mathbb{C}[G]$, $(g, h) \in G \times G$ define: $(g, h) \cdot f(x) = f(g^{-1}xh)$ ($x \in G$). Under this action we have the classical Peter-Weyl decomposition:

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in P_+(\mathfrak{g})} L(\lambda)^* \otimes L(\lambda), \quad (3.1)$$

as a representation of $G \times G$. Here the superscript $*$ denotes the dual representation. Note that $L(\lambda)^*$ is an irreducible highest weight representation of highest weight $-\omega_0(\lambda)$.

We introduce the following notation:

Definition 3.2.3. Let $\sigma : G \rightarrow GL(V)$ (resp. $\tau : H \rightarrow GL(W)$) be a representation of a group G (resp. H). If $H \subseteq G$ we may regard (σ, V) as a representation of H by restriction. We set:

$$[V, W] := \dim \text{Hom}_H(V, W).$$

If either V or W is infinite dimensional $\text{Hom}_H(V, W)$ may be infinite dimensional. In this case $[V, W]$ should be regarded as an infinite cardinal. (We will not encounter this situation in

⁸We use the word “regular” in two senses, the other being in the context of algebraic geometry.

what is to follow.) If V is completely reducible as an H -representation, and W is irreducible, $[V, W]$ is the multiplicity of W in V (By Schur's lemma).

Note: We will use the same (analogous) notation in the category of Lie algebra representations.

3.3. On the $\chi(T)$ -gradation of $\mathbb{C}[U \backslash G]$. We apply a philosophy taught to us by Roger Howe. As before, U is a maximal unipotent subgroup of G , T a maximal torus (normalizing U).

As in Theorem 3.2.2, $U \times G \subseteq G \times G$ acts on $\mathbb{C}[G]$. We have $\mathbb{C}[G]^U \cong \mathbb{C}[U \backslash G]$. As T normalizes U we have an action of T on $\mathbb{C}[U \backslash G]$ via $t \cdot f(x) = f(t^{-1}x)$ for $t \in T$ and $x \in U \backslash G$. We call this action the *left* action, since the multiplication is on the left. By Theorem 3.2.2, we have:

$$\mathbb{C}[U \backslash G] = \bigoplus_{\lambda \in P_+(\mathfrak{g})} (L(\lambda)^*)^U \otimes L(\lambda). \quad (3.2)$$

A consequence of the theorem of the highest weight is that $\dim(L(\lambda)^*)^U = 1$. We let, $\chi(T) \cong \mathbb{Z}^r$, denote the character group of T . Each $\lambda \in P_+(\mathfrak{g})$ defines a character, e^λ , of T . Set: $\mathbb{C}[U \backslash G]_\lambda := \{f \in \mathbb{C}[U \backslash G] \mid f(t^{-1}x) = e^\lambda(t)f(x) \ \forall x \in U \backslash G, t \in T\}$. G then acts (by right multiplication) on $\mathbb{C}[U \backslash G]$, and under this action we have: $\mathbb{C}[U \backslash G]_\lambda \cong L(\lambda)$. We then obtain a $\chi(T)$ -gradation of the algebra. That is to say, $\mathbb{C}[U \backslash G]_\xi \cdot \mathbb{C}[U \backslash G]_\eta \subseteq \mathbb{C}[U \backslash G]_{\xi+\eta}$. We exploit this phenomenon to obtain:

Proposition 3.3.1. *Let W be an irreducible finite dimensional representation of a reductive subgroup, H , of G . Let $\lambda, \mu \in P_+(\mathfrak{g})$. If $L(\mu)^H \neq (0)$ and $[W, L(\lambda)] \neq 0$ then $[W, L(\lambda + \mu)] \neq 0$.*

Remark 3.3.2. In Proposition 3.3.1, the G -representations $L(\lambda)$ and $L(\lambda + \mu)$ are regarded as H -representations by restriction.

Proof of Proposition 3.3.1. Let $f \in \mathbb{C}[U \backslash G]_\mu^H \cong L(\mu)^H$ and $\tilde{W} \subseteq \mathbb{C}[U \backslash G]_\lambda \cong L(\lambda)$ such that $W \cong \tilde{W}$ as a representation of H . Then $f \cdot W \subseteq \mathbb{C}[U \backslash G]_{\lambda+\mu}$. Under the (right) action of H we have, $f \cdot \tilde{W} \cong W$. Therefore, $[W, \mathbb{C}[U \backslash G]_{\lambda+\mu}] \neq 0$. \square

3.4. The maximal parabolic subgroups of G . A connected algebraic subgroup, P , of G containing a Borel subgroup is said to be parabolic. There exists an inclusion preserving one-to-one correspondence between parabolic subgroups and subsets of Π . We will recall the basic set-up.

Let $\mathfrak{p} = \text{Lie}(P)$ denote the Lie algebra of a parabolic subgroup P . Then $\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha \in \Gamma} \mathfrak{g}_\alpha$, where $\Gamma := \Phi^+ \cup \{\alpha \in \Phi \mid \alpha \in \text{Span}(\Pi')\}$ for a unique $\Pi' \subseteq \Pi$. Set:

$$\mathfrak{l} = \mathfrak{h} \oplus \sum_{\alpha \in \Gamma \cap -\Gamma} \mathfrak{g}_\alpha, \quad \mathfrak{u}^+ = \sum_{\alpha \in \Gamma, \alpha \notin -\Gamma} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{u}^- = \sum_{\alpha \in \Gamma, \alpha \notin -\Gamma} \mathfrak{g}_{-\alpha}.$$

Then we have, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$ and $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{p}$. The subalgebra \mathfrak{l} is the Levi factor of \mathfrak{p} , while \mathfrak{u}^+ is the nilpotent radical of \mathfrak{p} . \mathfrak{l} is reductive and hence $\mathfrak{l} = \mathfrak{l}_{ss} \oplus \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{z}(\mathfrak{l})$ and \mathfrak{l}_{ss} denote the center and semisimple part of \mathfrak{l} respectively.

The following result is well known (see for example, [16] page 196). This form is a slight modification of Exercise 12.2.4 in [10] (p. 532).

Proposition 3.4.1. For $0 \neq \lambda \in P_+(\mathfrak{g})$, let v_λ be a highest weight vector in $L(\lambda)$.

Let $X = G \cdot v_\lambda \subseteq L(\lambda)$ denote the orbit of v_λ and let $G_{v_\lambda} = \{g \in G \mid g \cdot v_\lambda = v_\lambda\}$ denote the corresponding isotropy group. Then, X is a quasiaffine variety stable under the action of \mathbb{C}^\times on $L(\lambda)$ defined by scalar multiplication. This \mathbb{C}^\times -action defines a graded algebra structure on $\mathbb{C}[X] = \bigoplus_{d=0}^{\infty} \mathbb{C}[X]^{(d)}$.

The action of G on $\mathbb{C}[X]$, defined by $g \cdot f(x) = f(g^{-1} \cdot x)$ (for $g \in G$ and $x \in X$) commutes with the \mathbb{C}^\times -action and therefore each graded component of $\mathbb{C}[X]$ is a representation of G . Furthermore, we have:

$$\mathbb{C}[X]^{(n)} \cong L(-nw_0(\lambda))$$

for all $n \in \mathbb{N}$.

Proof. Set $V = L(\lambda)$, and let v_λ be a highest weight vector in V . Let $\mathbb{P}V = \{[v] \mid v \in V\}$ denote the complex projective space on V . G then acts on $\mathbb{P}V$ by $g \cdot [v] = [g \cdot v]$ for $g \in G$ and $v \in V$. The isotropy group of $[v_\lambda]$ contains the Borel subgroup, B and therefore is a parabolic subgroup, $P \subseteq G$. G/P is projective so, $G \cdot [v_\lambda] \subset \mathbb{P}V$ is closed. This means that the affine cone, $\mathbb{A} := \bigcup_{g \in G} [g \cdot v_\lambda] \subset V$ is closed. Let $a \in \mathbb{A}$. Then, $a = z(g \cdot v_\lambda)$ for some $z \in \mathbb{C}$ and $g \in G$. The action of G on V is linear so, $a = g \cdot zv_\lambda$. By the assumption that $\lambda \neq 0$, T acts on $\mathbb{C}v_\lambda$ by a non-trivial linear character. All non-trivial linear characters of tori are surjective (this is because the image of a (connected) algebraic group homomorphism is closed and connected). Therefore, if $z \neq 0$, then $zv_\lambda = t \cdot v_\lambda$ for some $t \in T$. This fact implies⁹ that either $a \in X$ or $a = 0$. And so, $\overline{\mathbb{A}} = X \cup \{0\} = \mathbb{A}$. X is therefore quasiaffine since $X = \mathbb{A} - \{0\}$ ($0 \notin X$ since $\lambda \neq 0$). We have also shown that X is stable under scalar multiplication by a non-zero complex number.

Let $v_\lambda^* \in L(\lambda)^*$ be a highest weight vector. Upon restriction v_λ^* defines a regular function on X in $\mathbb{C}[X]^{(1)}$, which is a highest weight vector for the left action of G on $\mathbb{C}[X]$. Then, $(v_\lambda^*)^n \in \mathbb{C}[X]^{(n)}$. $(v_\lambda^*)^n$ is a highest weight vector, hence, $[\mathbb{C}[X]^{(n)}, L(n\lambda)^*] \neq 0$. That is to say, we have an injective G -equivariant map, $\psi : L(n\lambda)^* \hookrightarrow \mathbb{C}[X]^{(n)}$. It remains to show that ψ is an isomorphism.

Since X is quasiaffine, $\mathbb{C}[X] \cong \mathbb{C}[G]^{G_{v_\lambda}}$. By restriction of the regular representation, $G \times G_{v_\lambda}$ acts on $\mathbb{C}[G]$. And so by, Theorem 3.2.2:

$$\mathbb{C}[X] \cong \bigoplus_{\xi \in P_+(\mathfrak{g})} L(\xi)^* \otimes L(\xi)^{G_{v_\lambda}}$$

Set $\mathcal{L}_\xi := L(\xi)^{G_{v_\lambda}}$. $U \subseteq G_{v_\lambda}$ since v_λ is a highest weight vector. Therefore, $\mathcal{L}_\xi \subseteq L(\xi)^U$ and $\dim \mathcal{L}_\xi \leq 1$. And so, $\mathbb{C}[X]$ is multiplicity free as a representation of G (under the action of left multiplication).¹⁰ We will show that the only possible ξ for which $\dim \mathcal{L}_\xi > 0$ are the non-negative integer multiples of λ . By the fact that $\mathbb{C}[X]$ is multiplicity free we will see that ψ must be an isomorphism.

If $\mathcal{L}_\xi \neq (0)$ then $\mathcal{L}_\xi = L(\xi)^U$ since they are both 1-dimensional. Assume that $\mathcal{L}_\xi \neq (0)$. Choose, $0 \neq v_\xi \in \mathcal{L}_\xi$. Note that v_ξ is a highest weight vector. T acts on $\mathbb{C}v_\xi$ by $t \cdot v_\xi = e^\xi(t)v_\xi$ for all $t \in T$. Set $T_\xi := T \cap G_{v_\lambda}$ and $\mathfrak{h}_\lambda := \text{Lie}(T_\lambda) = \{H \in \mathfrak{h} \mid \lambda(H) = 0\}$.

⁹If $a \neq 0$, $a = g \cdot (t \cdot v_\lambda) = (gt) \cdot v_\lambda \in X$.

¹⁰Alternatively, \overline{B} has a dense orbit in X (since \overline{BU} is dense in G (ie: the big cell)). A dense orbit under a Borel subgroup is equivalent to having a multiplicity free coordinate ring.

$H \in \mathfrak{h}_\lambda$ implies both $H \cdot v_\xi = 0$ and $H \cdot v_\xi = \xi(H)v_\xi$. Hence we have $\xi(H) = 0$ when $H \in \mathfrak{h}_\lambda$. This statement is equivalent to λ and ξ being linearly dependent. Furthermore, we have $n_1\xi = n_2\lambda$ for $n_1, n_2 \in \mathbb{N}$ since ξ and λ are both dominant integral weights.

If $\dim \mathcal{L}_\xi = 1$ then $[\mathbb{C}[X]^{(n)}, L(\xi)^*] \neq 0$ for some $n \in \mathbb{N}$. This forces $[\mathbb{C}[X]^{(nn_1)}, L(n_1\xi)^*] \neq 0$ and therefore, $[\mathbb{C}[X]^{(nn_1)}, L(n_2\lambda)^*] \neq 0$. As before, $[\mathbb{C}[X]^{(n_2)}, L(n_2\lambda)^*] \neq 0$. Using the fact that $\mathbb{C}[X]$ is multiplicity free, we have $nn_1 = n_2$. And so, $\xi = n\lambda$. \square

Remark. The closure of the variety X in Proposition 3.4.1 is called the *highest weight variety* in [23] (see [10]).

If $\Pi' = \Pi - \{\alpha\}$ for some simple root α , then the corresponding parabolic subgroup is maximal (among proper parabolic subgroups). Consequently, the maximal parabolic subgroups of G may be parameterized by the nodes of the Dynkin diagram, equivalently, by fundamental weights of G . Set: $\lambda = -w_0(\omega_k)$. Let $v_k \in L(\lambda)^U$ be a highest weight vector. Define:

$$X_{\mathfrak{g}}^{(k)} := G \cdot v_k \subseteq L(\lambda) \quad (1 \leq k \leq r),$$

the orbit of v_k under the action of G . (When there is no chance of confusion, we write $X^{(k)}$ for $X_{\mathfrak{g}}^{(k)}$.)

We have seen that this orbit has the structure of a quasiaffine variety. It is easy to see that the isotropy group, $S^{(k)}$ of v_k is of the form $S^{(k)} = [P_k, P_k]$, where P_k denotes a maximal parabolic subgroup of G .¹¹ Let $\mathfrak{p} := \text{Lie}(P_k)$ and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$ denote the Levi decomposition. We have $\mathfrak{p} = \mathfrak{l} + n^+$ and $n^+ = [\mathfrak{b}, \mathfrak{b}] \subset [\mathfrak{p}, \mathfrak{p}]$ and $\mathfrak{l}_{ss} = [\mathfrak{l}, \mathfrak{l}] \subseteq [\mathfrak{p}, \mathfrak{p}]$. Therefore, $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{l}_{ss} + n^+ = \mathfrak{l}_{ss} \oplus \mathfrak{u}^+$. (Note that all that is lost is $\mathfrak{z}(\mathfrak{l})$.) We say that P_k is the parabolic subgroup corresponding to the fundamental weight ω_k .

We have $\dim X^{(k)} = \dim \mathfrak{g}/[\mathfrak{p}, \mathfrak{p}]$ and $2 \dim \mathfrak{g}/[\mathfrak{p}, \mathfrak{p}] = \dim(\mathfrak{g}/\mathfrak{l}_{ss}) + 1$ (Note that $\dim \mathfrak{z}(\mathfrak{l}) = 1$ since P_k is maximal.) In light of these facts, we see that the dimension of $X^{(k)}$ may be read off of the Dynkin diagram. The dimension is important for the proof of Corollary 3.5.2. In Section 6, we explicitly compute $\dim \mathfrak{g}/\mathfrak{l}_{ss}$ for the exceptional Lie algebras and low rank classical Lie algebras.

Corollary 3.4.2. *For $X = X^{(k)}$ where $1 \leq k \leq r$ we have:*

$$\mathbb{C}[X] \cong \bigoplus_{n=0}^{\infty} L(n\omega_k) \quad (3.3)$$

as a representation of G .

Proof. The result is immediate from Proposition 3.4.1. \square

3.5. A consequence of Theorem 2.0.1.

The goal of Section 2 was to prove Theorem 2.0.1. We now apply this theorem to obtain the following:

Proposition 3.5.1. *Assume G and H satisfy conditions $(*)$, and we take $S = S^{(k)}$, for some $1 \leq k \leq r$. Set: $X := X^{(k)}$.*

$$\dim H < \dim X \implies \mathbb{C}[X]^H \not\cong \mathbb{C}.$$

¹¹This fact is not true for the orbit of an arbitrary dominant integral weight. In general, $[\mathfrak{p}, \mathfrak{p}] \neq \text{Lie}(G_{v_\lambda})$, for any parabolic subalgebra \mathfrak{p} .

Proof. If \mathcal{O} is a generic H -orbit in X then $\dim \mathcal{O} \leq \dim H < \dim X$. By Theorems 2.0.4 and 2.0.8, the algebra $\mathbb{C}[X]$ is factorial, because $\mathbb{C}[X] = \mathbb{C}[G]^S$. The result follows from Theorem 2.0.1. \square

We now provide a representation theoretic interpretation of this proposition as it relates to \mathfrak{sl}_2 .

3.5.1. The \mathfrak{sl}_2 -case. Consider a triple (G, S, H) such that the following conditions (**) hold:

Conditions ().**

- (1) (G, S, H) satisfy conditions (*), and:
- (2) \mathfrak{g} has no simple factor of Lie type A_1 .
- (3) $S = S^{(k)}$, where $1 \leq k \leq r$. Set: $X := X^{(k)}$ ($:= G/S$).
- (4) $\mathfrak{sl}_2 \cong \text{Lie}(H) =: \mathfrak{k} \subseteq \mathfrak{g}$

Corollary 3.5.2. *Assume conditions (**), and that \mathfrak{g} has no simple factor of Lie type A_2 . Then, $\mathbb{C}[X]^H \not\cong \mathbb{C}$.*

Proof. The statement can be reduced to the case where \mathfrak{g} is simple. For G simple and not of type A_1 or A_2 , we appeal to the classification of maximal parabolic subgroups (see the tables in Section 6) to deduce that $\dim X > 3$. as $H \cong SL_2(\mathbb{C})$ (locally). Hence, $\dim H < \dim X$. We are then within the hypothesis of Proposition 3.5.1. \square

We next address the case when \mathfrak{g} does have a simple factor of Lie type A_2 . For this material we need to analyze the set of \mathfrak{sl}_2 -subalgebras of $\mathfrak{g} = \mathfrak{sl}_3$, up to a Lie algebra automorphism. For general results on the subalgebras of \mathfrak{g} we refer the reader to [6] and [7].

In the case of $\mathfrak{g} = \mathfrak{sl}_3$ there are two such \mathfrak{sl}_2 subalgebras. One being the root \mathfrak{sl}_2 -subalgebra corresponding to any one of the three positive roots of \mathfrak{g} . The other is the famous principal \mathfrak{sl}_2 -subalgebra.

A principal \mathfrak{sl}_2 -subalgebra ([7], [15]) of \mathfrak{g} is a subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ such that $\mathfrak{k} \cong \mathfrak{sl}_2$ and contains a regular nilpotent element. These subalgebras are conjugate, so we sometimes speak of “the” principal \mathfrak{sl}_2 -subalgebra. There is a beautiful connection between the principal \mathfrak{sl}_2 -subalgebra and the cohomology of G , (see [15]). There is a nice discussion of this theory in [5].

Lemma 3.5.3. *Assume conditions (**). Let G and H be such that $\mathfrak{g} \cong \mathfrak{sl}_3$, and $\text{Lie}(H)$ is a principal \mathfrak{sl}_2 -subalgebra of \mathfrak{g} . Then, $\dim \mathbb{C}[X]^H > 0$, for $X = X^{(1)}$ or $X = X^{(2)}$.*

Proof. A principal \mathfrak{sl}_2 -subgroup in \mathfrak{sl}_3 is embedded as a symmetric subalgebra. More precisely, let $H := SO_3(\mathbb{C}) \subseteq G$, then $\text{Lie}(H)$ is a principal \mathfrak{sl}_2 -subalgebra of \mathfrak{sl}_3 .

In general, if $G = SL_n(\mathbb{C})$ and $K = SO_n(\mathbb{C})$ with H embedded in G in the standard way, then the pair (G, H) is symmetric (ie: H is the fixed point set of a regular involution on G). As before, let r denote the rank of G . In order to prove the lemma (for $r = 2$), it suffices to observe that by the Cartan-Helgason theorem (see [10], Chapter 11) we have:

$$\dim L(\lambda)^K = \begin{cases} 1, & \lambda \in \sum_{i=1}^r 2\mathbb{N}\omega_i; \\ 0, & \text{Otherwise.} \end{cases}$$

\square

Lemma 3.5.4. *Assume conditions (**). Let G and H be such that $\mathfrak{g} \cong \mathfrak{sl}_3$, and $Lie(H)$ is a root \mathfrak{sl}_2 -subalgebra of \mathfrak{g} . Then, $\dim \mathbb{C}[X]^H > 0$, for $X = X^{(1)}$ or $X = X^{(2)}$.*

Proof. It is the case that $L(\omega_1)$ and $L(\omega_2)$ both have H -invariants, as they are equivalent to the standard representation of G and its dual respectively. \square

Summarizing we obtain:

Proposition 3.5.5. *Under assumptions (**), $\dim \mathbb{C}[X]^H > 0$.*

Proof. The statement reduces to the case where \mathfrak{g} is simple, because a maximal parabolic subalgebra of \mathfrak{g} must contain all but one simple factor of \mathfrak{g} . Therefore, assume that \mathfrak{g} is simple without loss of generality. For $\mathfrak{g} \not\cong \mathfrak{sl}_3$, apply Corollary 3.5.2. If $\mathfrak{g} \cong \mathfrak{sl}_3$, then $Lie(H)$ is embedded as either a root \mathfrak{sl}_2 -subalgebra, or as a principal \mathfrak{sl}_2 -subalgebra. Apply Lemmas 3.5.4 and 3.5.3 to the respective cases. \square

The following will be of fundamental importance in Section 4.

Definition 3.5.6. For G and H as in conditions (*), we consider the following set of positive integers ¹²:

$$M(G, H, j) := \left\{ n \in \mathbb{Z}^+ \mid \dim [L(n\omega_j)]^H \neq (0) \right\}$$

where j is a positive integer with $1 \leq j \leq r$. Set:

$$m(G, H, j) := \begin{cases} \min M(G, H, j), & \text{if } M(G, H, j) \neq \emptyset; \\ 0, & \text{if } M(G, H, j) = \emptyset. \end{cases}$$

We will also write $m(\mathfrak{g}, \mathfrak{k}, j)$ (resp. $M(\mathfrak{g}, \mathfrak{k}, j)$) where (as before) $\mathfrak{k} = Lie(H)$ and $\mathfrak{g} = Lie(G)$.

Proposition 3.5.7. *For G and H as in conditions (**),*

$$m(G, H, k) > 0 \text{ for all, } 1 \leq k \leq r.$$

Proof. Apply Corollary 3.4.2 and Proposition 3.5.5. \square

Proposition 3.5.1 applies to a much more general situation than $\mathfrak{k} \cong \mathfrak{sl}_2$.

3.5.2. The semisimple case. As it turns out, what we have done for the \mathfrak{sl}_2 -subalgebras can be done for any semisimple subalgebra.

Proposition 3.5.8. *If G and H are as in condition (*), then for each $1 \leq k \leq r$,*

$$\dim H < \dim X^{(k)} \implies m(G, H, k) > 0.$$

Proof. Apply Proposition 3.5.1 and Corollary 3.4.2. \square

In order to effectively apply Proposition 3.5.8 we will want to guarantee that $\dim H < \dim X^{(k)}$ for all $1 \leq k \leq r$. This will happen if all simple factors of \mathfrak{g} have sufficiently high rank. This idea motivates the following definition. Consider a group G (as in condition (*)) and define:

$$e(\mathfrak{g}) := \min_{1 \leq k \leq r} \dim X_{\mathfrak{g}}^{(k)}$$

¹²As always, $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ (the positive integers).

For a semisimple complex Lie algebra \mathfrak{k} , define¹³:

$$E(\mathfrak{k}) = \left\{ \mathfrak{s} \mid \begin{array}{l} (1) \ \mathfrak{s} \text{ is a simple complex Lie algebra} \\ (2) \ \dim \mathfrak{k} \geq e(\mathfrak{s}) \end{array} \right\}.$$

Corollary 3.5.9. *Assume G and H as in conditions (*). If \mathfrak{g} has no simple factor that is in the set $E(\mathfrak{k})$ then $m(G, H, k) > 0$ for all k with $1 \leq k \leq r$.*

Proof. Immediate from Proposition 3.5.8 and the definition of $E(\mathfrak{k})$. □

Example.

$$E(\mathfrak{sl}_3) = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, B_2, B_3, B_4, C_3, C_4, D_4, G_2\}$$

By the tables in Section 6, we can determine:

\mathfrak{s}	A_1	A_2	A_3	A_4	A_5	A_6	A_7	B_2	B_3	B_4	C_3	C_4	D_3	D_4	G_2
$e(\mathfrak{s})$	2	3	4	5	6	7	8	4	6	8	6	8	4	7	6

4. A PROOF OF A THEOREM IN PENKOV-ZUCKERMAN

This section is devoted to the proof of the following theorem:

Theorem 4.0.10. *Let \mathfrak{k} be an \mathfrak{sl}_2 -subalgebra of a semisimple Lie algebra \mathfrak{g} , none of whose simple factors is of type A_1 . Then there exists a positive integer $b(\mathfrak{k}, \mathfrak{g})$, such that for every irreducible finite dimensional \mathfrak{g} -module V , there exists an injection of \mathfrak{k} -modules $W \rightarrow V$, where W is an irreducible \mathfrak{k} -module of dimension less than $b(\mathfrak{k}, \mathfrak{g})$.*

Proof. We assume all of the structure of Sections 3.1 and 3.2 (ie: \mathfrak{h} , Φ , W , etc.). Consider a fixed \mathfrak{k} and \mathfrak{g} . For $k \in \mathbb{N}$, let $V(k)$ denote the irreducible, finite dimensional representation of \mathfrak{k} of dimension $k+1$. Each irreducible representation of \mathfrak{g} may be regarded as a \mathfrak{k} representation by restriction. As before, for $\lambda \in P_+(\mathfrak{g})$, we let $[L(\lambda), V(k)]$ denote the multiplicity of $V(k)$ in $L(\lambda)$. Set:

$$g_0(\lambda) := \min\{\dim V(k) \mid k \in \mathbb{N} \text{ and } [L(\lambda), V(k)] \neq 0\}.$$

For each fundamental weight ω_i ($1 \leq i \leq r$), let $m_i := m(\mathfrak{g}, \mathfrak{k}, i)$ (as in Definition 3.5.6). By Proposition 3.5.7, $m_i \neq 0$ for all i . Set $\delta_i := m_i \omega_i$, and define $\mathcal{C}_0 := \{\sum_{i=1}^r a_i \omega_i \mid 0 \leq a_i < m_i\}$. We set:

$$b(\mathfrak{k}, \mathfrak{g}) := \max\{g_0(\lambda) \mid \lambda \in \mathcal{C}_0\} + 1.$$

For $\mathbf{q} \in \mathbb{N}^r$, let $\mathcal{C}_{\mathbf{q}} := (\sum_{i=1}^r q_i \delta_i) + \mathcal{C}_0$. By the division algorithm, the collection of sets, $\{\mathcal{C}_{\mathbf{q}} \mid \mathbf{q} \in \mathbb{N}^r\}$ partitions $P_+(\mathfrak{g})$. We claim that for every $\mathbf{q} \in \mathbb{N}^r$, $\max\{g_0(\lambda) \mid \lambda \in \mathcal{C}_{\mathbf{q}}\} < b(\mathfrak{k}, \mathfrak{g})$.

The result follows from this claim. Indeed, let $V = L(\lambda)$, for $\lambda \in P_+(\mathfrak{g})$. There exists (a unique) $\mathbf{q} \in \mathbb{N}^r$ such that $\lambda \in \mathcal{C}_{\mathbf{q}}$. Let W be the irreducible \mathfrak{k} -representation of dimension $g_0(\lambda)$. By definition of g_0 , there exists an injection of W into $L(\lambda)$. By the claim, $\dim W = g_0(\lambda) < b(\mathfrak{k}, \mathfrak{g})$. The result follows.

We now will establish the claim by applying Proposition 3.3.1. Let $\lambda' \in \mathcal{C}_{\mathbf{q}}$ with $\mathbf{q} \in \mathbb{N}^r$. Set: $\lambda := \lambda' - \mu$ for $\mu = \sum_{i=1}^r q_i \delta_i$. By definition of $\mathcal{C}_{\mathbf{q}}$, we have $\lambda \in \mathcal{C}_0$, and so $g_0(\lambda) < b(\mathfrak{k}, \mathfrak{g})$. This means that there exists an irreducible \mathfrak{k} -representation, W , such that $\dim W < b(\mathfrak{k}, \mathfrak{g})$ and $[W, L(\lambda)] \neq 0$. By definition, $\lambda' = \lambda + \mu$. We see that since $L(\delta_i)^{\mathfrak{k}} \neq (0)$ for all i , we have $L(\mu)^{\mathfrak{k}} \neq (0)$. Applying Proposition 3.3.1 we see that $[W, L(\lambda')] \neq 0$. This means that $g_0(\lambda') \leq b(\mathfrak{k}, \mathfrak{g})$. The claim follows. □

¹³Explanation for notation: E and e are chosen with the word ‘‘exclusion’’ in mind.

Theorem 4.0.11. *Let \mathfrak{k} be a semisimple subalgebra of a semisimple Lie algebra \mathfrak{g} , none of whose simple factors is in the set $E(\mathfrak{k})$. Then there exists a positive integer $b(\mathfrak{k}, \mathfrak{g})$, such that for every irreducible finite dimensional \mathfrak{g} -module V , there exists an injection of \mathfrak{k} -modules $W \rightarrow V$, where W is an irreducible \mathfrak{k} -module of dimension less than $b(\mathfrak{k}, \mathfrak{g})$.*

Proof. The proof is essentially the same as the proof of Theorem 4.0.10. The only changes are a substitution of Proposition 3.5.8 for Proposition 3.5.7, and we index irreducible representation of \mathfrak{k} by $P_+(\mathfrak{k})$ rather than \mathbb{N} . The result follows from the fact that for a given positive integer d there are only finitely many \mathfrak{k} -modules with dimension equal to d (here we use that \mathfrak{k} is semisimple!).

We leave it to the reader to fill in the details. □

The above theorems begs us to compute the smallest value of $b(\mathfrak{k}, \mathfrak{g})$. This is the subject of the Section 5 for the case when \mathfrak{g} is the exceptional Lie algebra G_2 and \mathfrak{k} is a principal \mathfrak{sl}_2 -subalgebra of \mathfrak{g} . Other examples will follow in future work.

We remark that the number $b(\mathfrak{k}, \mathfrak{g})$ clearly depends on \mathfrak{k} (as the notation suggests). Of course, there are only finitely many \mathfrak{sl}_2 -subalgebras in \mathfrak{g} , up to automorphism of \mathfrak{g} . We can therefore, consider the maximum value of $b(\mathfrak{k}, \mathfrak{g})$ as \mathfrak{k} ranges over this finite set. We will call this number $b(\mathfrak{g})$. With this in mind, one might attempt to estimate $b(\mathfrak{g})$ for a given semisimple Lie algebra \mathfrak{g} .

On the other hand, there is a sense that one could fix \mathfrak{k} to be (say) a principal \mathfrak{sl}_2 -subalgebra in some \mathfrak{g} . We could then consider the question of whether $b(\mathfrak{k}, \mathfrak{g})$ is bounded as \mathfrak{g} varies (among semisimple Lie algebras with no simple A_1 factor).

Even more impressive would be allowing *both* \mathfrak{g} and \mathfrak{k} to vary. It is certainly not clear that a bound would even exist for $b(\mathfrak{k}, \mathfrak{g})$. If it did, we would be interested in an estimate.

5. EXAMPLE: G_2

In this section we consider an example which illustrates the result of Section 4.

Let G be a connected, simply connected, complex algebraic group with $\mathfrak{g} \cong G_2$. Let K be a connected principal SL_2 -subgroup of G . As before, we set $\mathfrak{k} = Lie(K)$.

We order the fundamental weights of \mathfrak{g} so that $L(\omega_1) = 7$ and $\dim L(\omega_2) = 14$. For the rest of this section, we will refer to the representation $L(a\omega_1 + b\omega_2)$ (for $a, b \in \mathbb{N}$) as $[a, b]$. In Table 1, the entry in row i column j is $\dim[i, j]^K$:

Table 1 was generated by an implementation of the Weyl character formula (see Theorem 3.2.1) for the group G_2 using the computer algebra system MAPLE[®]. The characters were restricted to a maximal torus in K , thus allowing us to find the character of $[i, j]$ as a representation of $SL_2(\mathbb{C})$. This character was used to compute the dimension of the invariants for K .

Using the same implementation we can compute the values of $g_0(\lambda)$ for the pair $(\mathfrak{k}, \mathfrak{g})$. We display these data in the table below for $0 \leq i, j \leq 19$.

Of particular interest is the 7 in row 1 column 0. The irreducible G_2 -representation $[1, 0]$ is irreducible when restricted to a principal \mathfrak{sl}_2 subalgebra. And therefore, $g_0(\omega_1) = \dim[1, 0] = 7$. Note that most entries in Table 2 are 1. Following the proof in Section 4 we see that $b(\mathfrak{k}, \mathfrak{g}) = 7 + 1 = 8$. That is, every finite dimensional representation of G_2 contains an irreducible \mathfrak{k} -representation of dimension less than 8.

$i \setminus j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1	0	1	0	1	0	2	0	2	0	3	0	4	0	4	1	5	1	6	1
1	0	0	0	1	0	1	1	1	2	2	2	3	3	4	4	5	5	6	7	7
2	0	0	1	0	2	0	3	1	4	2	5	3	7	4	9	5	11	7	13	9
3	0	0	0	1	1	2	2	3	3	5	5	6	7	8	9	11	11	13	14	16
4	1	0	2	1	3	2	5	3	7	5	9	7	12	9	15	12	18	15	22	18
5	0	1	1	2	2	4	4	6	6	8	9	11	12	14	15	18	19	22	23	26
6	1	0	2	2	4	3	7	5	10	8	13	11	17	15	21	19	26	23	32	28
7	0	1	1	3	3	5	6	8	9	12	12	16	17	20	22	25	27	31	33	37
8	1	1	3	2	6	5	9	8	13	12	18	16	23	21	29	27	35	33	42	40
9	0	1	2	4	4	7	8	11	12	16	17	21	23	27	29	34	36	41	44	49
10	2	1	4	4	7	7	12	11	17	16	23	22	30	28	37	36	45	44	54	52
11	0	2	2	5	6	9	10	14	16	20	22	27	29	35	37	43	46	52	56	62
12	2	2	5	5	9	9	15	14	21	21	28	28	37	36	46	45	56	55	67	66
13	0	2	3	6	7	11	13	17	19	25	27	33	36	42	46	53	56	64	68	76
14	2	2	6	6	11	11	17	18	25	25	34	34	44	44	55	55	67	67	80	80
15	1	3	4	8	9	14	16	21	24	30	33	40	44	51	55	64	68	77	82	91
16	3	3	7	8	13	14	21	21	30	31	40	41	52	53	65	66	79	80	95	95
17	0	4	5	9	11	16	19	25	28	35	39	47	51	60	65	74	80	90	96	107
18	3	3	8	9	15	16	24	25	34	36	46	48	60	61	75	77	91	93	109	111
19	1	4	6	11	13	19	22	29	33	41	45	54	60	69	75	86	92	104	111	123

TABLE 1. $\dim[i, j]^K$ where K is a principle SL_2 -subgroup in G_2 .

Even more interesting is the fact that Table 2 suggests that there are only 26 ordered pairs (a, b) such that $g_0(a\omega_1 + b\omega_2) > 1$. This is indeed the case:

Theorem 5.0.12. *For all $a, b \in \mathbb{N}$, $[a, b]^K \neq (0)$ except for the following list of 26 exceptions:*

$$\begin{aligned}
& [0, 1], [0, 3], [0, 5], [0, 7], [0, 9], [0, 11], [0, 13], [1, 0], [1, 1], \\
& [1, 2], [1, 4], [2, 0], [2, 1], [2, 3], [2, 5], [3, 0], [3, 1], [3, 2], \\
& [4, 1], [5, 0], [6, 1], [7, 0], [9, 0], [11, 0], [13, 0], [17, 0].
\end{aligned}$$

Proof. By inspection of Table 1 (or Table 2), it suffices to show that for all $a, b \in \mathbb{N}$ such that $a > 24$ or $b > 24$ we have $\dim[a, b]^K > 0$. Let μ_1, \dots, μ_6 denote the highest weights of the representations $[0, 2], [0, 17], [4, 0], [15, 0], [5, 1]$, and $[1, 3]$. From the table we see that each of these representations has a K -invariant. Let $Z = \sum_{i=1}^6 \mathbb{N}\mu_i \subset P_+(\mathfrak{g})$. By Proposition 3.3.1, (for W trivial), each element of Z has a K -invariant. It is easy to see that $E = P_+(\mathfrak{g}) - Z$ is a finite set. In fact, computer calculations show that E consists of the following 194 elements:

$[0, 1], [0, 3], [0, 5], [0, 7], [0, 9], [0, 11], [0, 13], [0, 15], [1, 0], [1, 1], [1, 2], [1, 4], [1, 6], [1, 8], [1, 10], [1, 12], [1, 14], [1, 16], [1, 18], [2, 0], [2, 1], [2, 2], [2, 3], [2, 4], [0, 1], [0, 3], [0, 5], [0, 7], [0, 9], [0, 11], [0, 13], [0, 15], [1, 0], [1, 1], [1, 2], [1, 4], [2, 0], [2, 1], [2, 2], [2, 3], [2, 4], [2, 5], [2, 7], [3, 0], [3, 1], [3, 2], [3, 3], [3, 4], [3, 5], [3, 6], [3, 7], [3, 8], [3, 10], [4, 1], [4, 3], [4, 5], [4, 7], [4, 9], [4, 11], [4, 13], [5, 0], [5, 2], [5, 4], [6, 1], [6, 3], [6, 5], [7, 0], [7, 2], [7, 4], [8, 1], [8, 3], [8, 5], [9, 0], [9, 2], [9, 4], [10, 1], [10, 3], [10, 5], [11, 0], [11, 2], [11, 4], [12, 1], [12, 3], [12, 5], [13, 0], [13, 2], [13, 4], [14, 1], [14, 3], [14, 5], [16, 1], [17, 0], [17, 2], [17, 4], [18, 1], [18, 3], [18, 5], [2, 5], [2, 7], [2, 9], [2, 11], [2, 13], [2, 15], [2, 17], [2, 19], [2, 21], [3, 0], [3, 1], [3, 2], [3, 3], [3, 4], [3, 5], [3, 6], [3, 7], [3, 8], [3, 10], [3, 12], [3, 14], [3, 16], [3, 18], [3, 20], [3, 22], [3, 24], [4, 1], [4, 3], [4, 5], [4, 7], [4, 9], [4, 11], [4, 13], [4, 15], [5, 0], [5, 2], [5, 4], [5, 6], [5, 8], [5, 10], [5, 12], [5, 14], [5, 16], [6, 0], [6, 1], [6, 2], [6, 3], [6, 5], [6, 7], [6, 9], [6, 11], [6, 13], [6,$

$i \setminus j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	1	1	1	1	1
1	7	5	3	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	5	3	1	3	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	3	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
19	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

TABLE 2. Value of $g_0(i\omega_1 + j\omega_2)$ for a principal SL_2 -subgroup in G_2 .

15], [6, 17], [6, 19], [7, 0], [7, 1], [7, 2], [7, 3], [7, 4], [7, 5], [7, 6], [7, 8], [7, 10], [7, 12], [7, 14], [7, 16], [7, 18], [7, 20], [7, 22], [8, 1], [8, 3], [8, 5], [8, 7], [8, 9], [8, 11], [8, 13], [8, 15], [9, 0], [9, 2], [9, 4], [9, 6], [9, 8], [9, 10], [9, 12], [9, 14], [9, 16], [10, 0], [10, 1], [10, 3], [10, 5], [10, 7], [10, 9], [10, 11], [10, 13], [10, 15], [10, 17], [11, 0], [11, 1], [11, 2], [11, 3], [11, 4], [11, 6], [11, 8], [11, 10], [11, 12], [11, 14], [11, 16], [11, 18], [11, 20], [12, 1], [12, 3], [12, 5], [12, 7], [12, 9], [12, 11], [12, 13], [12, 15], [13, 0], [13, 2], [13, 4], [13, 6], [13, 8], [13, 10], [13, 12], [13, 14], [13, 16], [14, 0], [14, 1], [14, 3], [14, 5], [14, 7], [14, 9], [14, 11], [14, 13], [14, 15], [14, 17], [15, 1], [16, 1], [17, 0], [17, 2], [17, 4], [18, 0], [18, 1], [18, 3], [18, 5], [18, 7], [19, 1], [21, 0], [21, 2], [22, 0], [22, 1], [22, 3], [22, 5], [23, 1], [25, 0], [26, 0], [26, 1], [26, 3], [27, 1], [29, 0], [30, 1], [31, 1], [33, 0], [34, 1], [37, 0], [38, 1], [41, 0], [42, 1], [46, 1].

If one checks, we see that each of these has an invariant except for the 26 exceptional values in the statement of the theorem.

Unfortunately, our tables (in this paper) are not big enough to see all of these weights. For this reason, we can alternately consider the set $Z' = \sum_{i=1}^9 \mathbb{N}\mu'_i$, where μ'_1, \dots, μ'_9 are the highest weights of the representations $[0, 2]$, $[0, 17]$, $[4, 0]$, $[6, 0]$, $[15, 0]$, $[5, 1]$, $[1, 3]$, $[7, 1]$, and $[1, 6]$. Again, each of these has a K -invariant (from the tables) and therefore by Proposition 3.3.1, each element of Z' has a K -invariant. Set $E' = P_+(\mathfrak{g}) - Z'$. E' has 73 elements. They are: $[0, 1]$, $[0, 3]$, $[0, 5]$, $[0, 7]$, $[0, 9]$, $[0, 11]$, $[0, 13]$, $[0, 15]$, $[1, 0]$, $[1, 1]$, $[1, 2]$, $[1, 4]$, $[2, 0]$, $[2, 1]$, $[2, 2]$, $[2, 3]$, $[2, 4]$, $[2, 5]$, $[2, 7]$, $[3, 0]$, $[3, 1]$, $[3, 2]$, $[3, 3]$, $[3, 4]$, $[3, 5]$, $[3, 6]$, $[3, 7]$, $[3, 8]$, $[3, 10]$, $[4, 1]$, $[4, 3]$, $[4, 5]$, $[4, 7]$, $[4, 9]$, $[4, 11]$, $[4, 13]$, $[5, 0]$, $[5, 2]$, $[5, 4]$, $[6, 1]$, $[6, 3]$, $[6, 5]$, $[7, 0]$, $[7, 2]$, $[7, 4]$, $[8, 1]$, $[8, 3]$, $[8, 5]$, $[9, 0]$, $[9, 2]$, $[9, 4]$, $[10, 1]$, $[10, 3]$, $[10, 5]$, $[11, 0]$, $[11, 2]$, $[11, 4]$, $[12, 1]$, $[12, 3]$, $[12, 5]$, $[13, 0]$, $[13, 2]$, $[13, 4]$, $[14, 1]$, $[14, 3]$, $[14, 5]$, $[16, 1]$, $[17, 0]$, $[17, 2]$, $[17, 4]$, $[18, 1]$, $[18, 3]$, $[18, 5]$.

This time, each of these representations is on our tables. And so, the reader may (and should) check that, apart from the 26 exceptional values in the statement of the theorem, all of these have a positive dimension of K -invariants. \square

6. TABLES FOR MAXIMAL PARABOLIC SUBALGEBRAS.

In the proof of Corollary 3.5.2 we needed to see that if \mathfrak{g} was a simple Lie algebra not of type A2, then $\dim(\mathfrak{g}/[\mathfrak{p}, \mathfrak{p}]) > 3$ for a maximal parabolic subalgebra \mathfrak{p} . If $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$, $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}_{ss}$ then we can deduce that this inequality is equivalent to $\dim(\mathfrak{g}/\mathfrak{l}_{ss}) > 5$ (using the formula $\dim(\mathfrak{g}/[\mathfrak{p}, \mathfrak{p}]) = \frac{1}{2}(\dim \mathfrak{g}/\mathfrak{l}_{ss} + 1)$). It is possible to exhaust over all possible cases to establish this fact. The exceptional cases are in the following table:

Dimension of $\mathfrak{g}/\mathfrak{l}_{ss}$ for the exceptional groups.

$G \setminus \mathfrak{l}_{ss}$	1	2	3	4	5	6	7	8
G2	A1	A1						
$\dim \mathfrak{g}/\mathfrak{l}_{ss}$	11	11						
F4	C3	A2A1	A2A1	B3				
	31	41	41	31				
E6	D5	A5	A4A1	A2A2A1	A4A1	D5		
	33	43	51	59	51	33		
E7	D6	A6	A1A5	A1A2A3	A2A4	D5A1	E6	
	67	85	95	107	101	85	55	
E8	D7	A7	A1A6	A1A2A4	A4A3	D5A2	E6A1	E7
	157	185	197	213	209	195	167	115

For the classical cases, the situation reduces to the examination of several families of parabolic subalgebras. In each family, we can determine the dimension of $\mathfrak{g}/\mathfrak{l}_{ss}$ for any of parabolic $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$, from the formulas:

$$\begin{aligned} \dim A_n &= (n+1)^2 - 1 \\ \dim B_n &= n(2n+1) \\ \dim C_n &= n(2n+1) \\ \dim D_n &= n(2n-1) \end{aligned}$$

For example, if $\mathfrak{g} = A_n$ ($n \geq 2$) and $\mathfrak{l}_{ss} = A_p \oplus A_q$ for $p+q = n-1$ (set $\dim A_0 := 0$) we have $\dim(\mathfrak{g}/\mathfrak{l}_{ss}) = \dim A_n - \dim A_p - \dim A_q = n^2 - p^2 - q^2 + 2n + 2p + 2q - 2$. Upon examination of the possible values of this polynomial over the parameter space we see that the smallest value is 5. This case occurs only for $n = 2$. All other values of p and q give rise to larger values. In fact, for \mathfrak{g} classical and not of type A the dimension is also greater than 5. The low rank cases are summarized in the following tables:

Dimension of $\mathfrak{g}/\mathfrak{l}_{ss}$ for type A.

$G \setminus \mathfrak{l}_{ss}$	1	2	3	4	5
A2	A1	A1			
	5	5			
A3	A2	A1A1	A2		
	7	9	7		
A4	A3	A1A2	A1A2	A3	
	9	13	13	9	
A5	A4	A1A3	A2A2	A1A3	A4
	11	17	19	17	11

Dimension of $\mathfrak{g}/\mathfrak{l}_{ss}$ for type D.

$G \setminus \mathfrak{l}_{ss}$	1	2	3	4	5	6
D3	A1A1	A2	A2			
	9	7	7			
D4	A3	A1A1A1	A3	A3		
	13	19	13	13		
D5	D4	A1D3	A2A1A1	A4	A4	
	17	27	31	21	21	
D6	D5	A1D4	A2D3	A1A1A3	A5	A5
	21	35	43	45	31	31

Dimension of $\mathfrak{g}/\mathfrak{l}_{ss}$ for type B.

Group \ \mathfrak{l}_{ss}	1	2	3	4	5
B2	A1	A1			
	7	7			
B3	B2	A1A1	A2		
	11	15	13		
B4	B3	A1B2	A1A2	A3	
	15	23	25	21	
B5	B4	A1B3	A2B2	A1A3	A4
	19	31	37	37	31

Dimension of $\mathfrak{g}/\mathfrak{l}_{ss}$ for type C.

Group \ \mathfrak{l}_{ss}	1	2	3	4	5
C2	A1	A1			
	7	7			
C3	C2	A1A1	A2		
	11	15	13		
C4	C3	A1C2	A1A2	A3	
	15	23	25	21	
C5	C4	A1C3	A2C2	A1A3	A4
	19	31	37	37	31

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