Graded multiplicities of the nullcone for the algebraic symmetric pair of type G

by

Anthony Paul van Groningen

A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

at

The University of Wisconsin-Milwaukee
May 2007

Graded multiplicities of the nullcone for the algebraic symmetric pair of type G

by

Anthony Paul van Groningen

A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

at

The University of Wisconsin-Milwaukee
May 2007

Major Professor Date

Major Professor Date

Graduate School Approval Date

Abstract

Graded multiplicities of the nullcone for the algebraic symmetric pair of type G

by

Anthony Paul van Groningen

The University of Wisconsin-Milwaukee, 2007
Under the Supervision of Professor Allen D. Bell and
Professor Jeb W. Willenbring

The regular functions on the nullcone associated to the algebraic symmetric pair $(G_2, \mathfrak{so}(4,\mathbb{C}))$ of Type G is decomposed as a graded representation of $K = \mathrm{SO}(4,\mathbb{C})$. This is done in two complementary ways. First a formula for the graded multiplicities in terms of the principal branching rule for $\mathfrak{sp}(2,\mathbb{C})$ is derived using an application of Howe duality. Second, an explicit formula for the q-multiplicity of each K-type is given. As a consequence, a closed rational form for the Hilbert series of $\mathbb{C}[G_2]^K$ is determined.

Branching from a simple Lie algebra to a principal three-dimensional subalgebra is discussed in the context of Kostant's multiplicity formula. The support of the relevant partition function is described in terms of the Bruhat order. Using the methods of Heckman, asymptotic estimates for the principal branching multiplicities of a simple Lie algebra $\mathfrak{g}$ are shown to be related to the exponents $\mathfrak{g}$.

The case of $\mathfrak{g} = \mathfrak{sp}(2, \mathbb{C})$ is studied in further detail and these calculations are used to describe the asymptotic distribution of the graded multiplicities in Type G.

The symmetric pair $(G_2, \mathfrak{so}(4, \mathbb{C}))$ leads to the study of binary $(3, 1)$-forms for which a complete system of covariant generators is known. From this the Brion polytope for the nullcone is determined.

Major Professor Date

Major Professor Date

To Mum, Dad, and goddaughter Sophia.

In memory of Tony.

Table of Contents

List of Figures

List of Tables

Acknowledgements

This work would not of been possible without the continuing support of my advisors Allen Bell and Jeb Willenbring. I am incredibly grateful for their time, patience, and insight. In addition, I wish to thank committee members Craig Guilbault, Ian Musson, and Yi Ming Zou for their input into this work. Let me also assert my gratitude to Mark Teply who sadly passed away last year.

I have had so many wonderful professors at UWM who have influenced me both as a mathematician and as an educator—Ric Ancel, Eric Key, Ming Lee, James Arnold, Jay Beder, Hans Volkmer, Boris Okun, and the aforementioned members of my committee—thank you all. I am grateful to the departmental staff who have always been eager to assist in the day-to-day activities of a graduate student. Fellow students at UWM have helped make my time here a pleasure. In particular, I must thank Mirek Pryszczepko and Joseph Shomberg for their friendship these past years.

I am grateful to live in a time and place where intellectual pursuits are accessible to all and not merely the folly of the elite. This would be impossible without the efforts of those in my family who came before—be they butcher or bullockie, engineer or florist, soldier or pirate.

Thank you Mum, Dad, Monique, and Gerard for your support through this arduous process and thanks to my friends Jonathan, Jason, Brad, and Andy for your unwavering mateship.

While preparing this dissertation I was a Research Assistant for Dr. Willenbring with funding provided by the Graduate School Research Committee Award.

The whole of arithmetic now appeared
within the grasp of mechanism.

Charles Babbage

I rest not, 'tis best not, the world is a wide one–
And, caged for an hour, I pace to and fro;
I see things and dree things and plan while I'm sleeping,
I wander for ever and dream as I go.

I have stood by Table Mountain,
On the Lion at Capetown,
And I watched the sunset fading
From the roads that I marked down;
And I looked out with my brothers
From the heights behind Bombay,
Gazing north and west and eastward,
Over roads I'll tread some day.

For my ways are strange ways and new ways and old ways,
And deep ways and steep ways and high ways and low;
I'm at home and at ease on a track that I know not,
And restless and lost on a road that I know.

The Wander-light
Henry Lawson

Chapter 1

Introduction

The theory of algebraic symmetric pairs plays a crucial role both in the classification of real forms of simple complex Lie algebras and in the classification of compact symmetric spaces. The central object of study in this dissertation is the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is the exceptional complex Lie algebra of type G_2 and $\mathfrak{k} = \mathfrak{so}(4, \mathbb{C})$. This pair corresponds to the split real form of G_2. Let $K = \mathrm{SO}(4, \mathbb{C})$. There is a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is an irreducible representation of K. By a theorem of Kostant and Rallis, the algebra $\mathbb{C}[\mathfrak{p}]$ of regular functions on $\mathfrak{p}$ is a free module over the algebra $\mathbb{C}[\mathfrak{p}]^K$ of K-invariant functions on $\mathfrak{p}$. Precisely,

$$\mathbb{C}[\mathfrak{p}] \cong \mathbb{C}[\mathfrak{p}]^K \otimes \mathcal{H}$$

where $\mathcal{H}$ is the space of harmonic functions on $\mathfrak{p}$. Moreover, $\mathcal{H} = \oplus_{d \geq 0} \mathcal{H}_d$ inherits a gradation from $\mathbb{C}[\mathfrak{p}]$ with each homogeneous component $\mathcal{H}_d$ invariant under K. This leads to the question of how $\mathcal{H}_d$ decomposes as a representation of K.

This problem has a geometric interpretation. Let J be the ideal of $\mathbb{C}[\mathfrak{p}]$ generated by the $u \in \mathbb{C}[\mathfrak{p}]^K$ satisfying $u(0) = 0$. Then the vanishing set of J determines an affine variety $\mathcal{N} \subset \mathfrak{p}$ called the nullcone. Restricting the harmonic functions to the nullcone gives an isomorphism of graded K-representations $\mathcal{H} \cong \mathbb{C}[\mathcal{N}]$. Understanding the graded decomposition of $\mathcal{H}$ under the group K is equivalent to understanding the graded decomposition of $\mathbb{C}[\mathcal{N}]$.

In Chapter 3 these questions are settled. First we give a formula for the graded multiplicities in terms of a branching rule from $\mathfrak{sp}(2,\mathbb{C})$ to the so-called principal TDS. While elegant, this formula involves an alternating sum which at first glance obscures matters. A second approach describes the so-called q-multiplicity of each irreducible representation of K. For a given K-type λ, the q-multiplicity is a polynomial in q whose coefficient of q^d records the multiplicity of λ in $\mathcal{H}_d$. By computing a generating function for the q-multiplicities, various combinatorial information is deduced. In particular, a closed form for the Hilbert series of the algebra $\mathbb{C}[G_2]^{\mathrm{SO}(4,\mathbb{C})}$ is computed. The K-structure of $\mathcal{H}$ is closely related to a certain subgroup M of K. The lift of M to the simply-connected covering group $\mathrm{Spin}(4,\mathbb{C})$ of K is described.

In Chapter 2, principal branching rules analogous to that used in computing the graded multiplicities are discussed. Kostant's multiplicity formula is used to compute the branching multiplicities in terms of a partition function. The support of these partition functions are described in terms of the Bruhat order. Employing the methods of Heckman, the asymptotic behavior of the branching rule for a simple Lie algebra $\mathfrak{g}$ to its principal TDS is shown to be related to the exponents of $\mathfrak{g}$. Additional analysis is carried out in the case of $\mathfrak{sp}(2,\mathbb{C})$. This is put to use in Chapter 5 where an asymptotic description of the graded multiplicities for the pair $(G_2, \mathfrak{so}(4,\mathbb{C}))$ is provided. The distribution of graded multiplicities is shown to behave in a piecewise linear fashion. The Brion polytope for the nullcone is determined, thus providing an asymptotic picture of the support of the graded multiplicities.

In Chapter 4, the symmetric pair $(G_2, \mathfrak{so}(4,\mathbb{C}))$ is shown to be related to a problem in classical invariant theory. In 1946, J. A. Todd found a complete set of generators for the algebra of covariants of the double binary $(3,1)$-forms. Exploiting the above results and using a computer, polynomial identities among the covariant generators for the $(3,1)$-forms are determined.

1.1 Background

1.1.1 Conventions

Lie theory

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $l > 0$ and let G be its adjoint group. Thus, G is the connected complex Lie group with Lie algebra $\mathfrak{g}$. Each choice of Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ determines a root system $\Phi := \Phi(\mathfrak{h}) \subset \mathfrak{h}^*$. Choose a set of positive roots Φ^+ for Φ and let $\Delta \subset \Phi^+$ be a base of simple roots for Φ. Let $\beta \in \Phi^+$ and write $\beta = \sum_{\alpha\in\Delta} n_\alpha \, \alpha$, where the n_α are non-negative integers. The *height of* β is the number $\mathrm{ht}(\beta)$ defined by

$$\mathrm{ht}(\beta) = \sum_{\alpha\in\Delta} n_\alpha. \tag{1.1}$$

Let $\mathbb{E}$ be the real span of Φ in $\mathfrak{h}^*$ and let $(\, , \,)$ denote the restriction of the Killing form to $\mathbb{E}$. As $(\, , \,)$ is non-degenerate on $\mathfrak{h}^*$ we can associate to each $\lambda \in \mathbb{E}$ an element $H_\lambda \in \mathfrak{h}$ satisfying $(\lambda, \gamma) = \gamma(H_\lambda)$ for all $\gamma \in \mathfrak{h}^*$. Let

$$h_\lambda := \frac{2H_\lambda}{(\lambda, \lambda)}.$$

The *weight lattice of* $\mathfrak{g}$ is the set

$$P(\mathfrak{g}) = \{ \, \lambda \in \mathfrak{h}^* \mid (\lambda, h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Delta \, \}.$$

The *dominant integral weights* of $\mathfrak{g}$ is the set

$$P_+(\mathfrak{g}) = \{ \, \lambda \in P(\mathfrak{g}) \mid (\lambda, h_\alpha) \geq 0 \text{ for all } \alpha \in \Delta \, \}.$$

The set $P_+(\mathfrak{g})$ parametrizes the finite-dimensional regular representations of $\mathfrak{g}$ by associating to each $\lambda \in P_+(\mathfrak{g})$ the representation $L(\lambda)$ with highest weight λ.

Write $\Delta = \{ \alpha_1, \ldots, \alpha_l \}$. The *fundamental weights* for $\mathfrak{g}$ are the $\varpi_1, \ldots, \varpi_l \in \mathfrak{h}^*$ satisfying

$$\langle \varpi_i, h_{\alpha_j} \rangle = \delta_{ij}$$

for $1 \leq i, j \leq l$. Since $\mathfrak{g}$ is semisimple,

$$P_+(\mathfrak{g}) = \bigoplus_{i=1}^{l} \mathbb{Z}_{\geq 0}\, \varpi_i.$$

Let $T \subset G$ be a maximal torus in G. Then $G \cong (\mathbb{C}^\times)^l$ as an algebraic group where the number l is the rank of G (and of $\mathfrak{g}$). A *rational character of* G is a homomorphism of algebraic groups $\chi : T \to \mathbb{C}^\times$. Every rational character is determined by an l-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of integers. The corresponding character is denoted e^λ whose value at $(s_1, \ldots, s_l) \in T$ is

$$e^\lambda = e^\lambda(s_1, \ldots, s_l) = s_1^{\lambda_1} \cdots s_l^{\lambda_l}$$

where the character e^λ is identified with the corresponding rational expression.

For a finite-dimensional regular representation V of G, the *character of* V is the rational character $\operatorname{ch}(V) : T \to \mathbb{C}^\times$

$$\operatorname{ch}(V) = \sum_{\lambda \in P(\mathfrak{g})} \dim_{\mathbb{C}} V(\lambda)\, e^\lambda$$

where

$$V(\lambda) = \left\{ v \in V \mid s.v = e^\lambda(s)v \text{ for all } s \in T \right\}.$$

Suppose $W = \oplus_{d \geq 0} W_d$ is a graded vector space such that each W_d is G-invariant. By the *q-character of W* we mean the formal expression

$$\mathrm{ch}_q(W) = \sum_{d \geq 0} \mathrm{ch}(W_d)\; q^d$$

where q is an indeterminate. The *Hilbert series of W* is the formal expression

$$\mathrm{Hilb}(W) = \sum_{d \geq 0} (\dim_{\mathbb{C}} W_d)\; q^d.$$

Partitions

A *partition* λ is a finite decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$ of non-negative integers. We write $|\lambda| := \sum_i \lambda_i$ for the *size of* λ. If $N = |\lambda|$ then we say λ is a partition of N. The *depth of* λ is the number $\ell(\lambda) := \sup\{\, i \mid \lambda_i \neq 0 \,\}$. The *conjugate partition of* λ, denote $\lambda^\sharp$ is the partition whose i^{th} entry is

$$\left(\lambda^\sharp\right)_i = \#\{\, j \mid i \leq \lambda_j \,\}.$$

Define $\lambda!$ by $\lambda! = \prod_i^{\ell(\lambda)} \lambda_i!$ We need the following simple observation.

Lemma 1.1. *Let* $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$ *be a partition. Then*

$$\lambda^\sharp! = 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n}.$$

Representations of $\mathrm{SL}(2, \mathbb{C})$

Let $K = \mathrm{SL}(2, \mathbb{C})$. For $k \in \mathbb{Z}_{\geq 0}$, let $F^k = S^k\mathbb{C}^2$ denote the k^{th} symmetric power of the standard representation $\mathbb{C}^2$ of K. By convention, let $F^k = 0$ for $k < 0$. Write $\mathbb{C}^2 = \mathrm{span}_{\mathbb{C}}\{\, x_0,\, x_1 \,\}$. Then F^k may be taken as the space of homogeneous k-forms in the variables x_0 and x_1. A typical $f \in F^k$ can be written as

$$f = \sum_{i=0}^{k} \binom{k}{i} a_i \, x_0^{k-i} \, x_1^i. \tag{1.2}$$

For each k we have a representation $\rho_k : K \to \mathrm{GL}(F^k)$ given by linear change of coordinates. Explicitly, for $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in K$

$$\rho_k(g)(f) = \sum_{i=0}^{k} \binom{k}{i} (ax_0 + bx_1)^{k-i} \, (cx_0 + dx_1)^i \tag{1.3}$$

where f is given by (1.2).

Let T denote the maximal torus of K consisting of the diagonal matrices in K. The character of the representation F^k is denoted χ^k and its value at $\mathrm{diag}(s, s^{-1}) \in T$ is

$$\chi^k = \frac{s^{k+1} - s^{-k-1}}{s - s^{-1}}.$$

Let $\mathfrak{k} = \mathrm{Lie}(K)$. Then

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is a suitable basis for $\mathfrak{k}$. The triple (H, X, Y) satisfy the commutation relations

$$[\,H, X\,] = 2X, \quad [\,H, Y\,] = -2Y, \quad [\,X, Y\,] = H. \tag{1.4}$$

Every finite-dimensional irreducible representation of $\mathfrak{k}$ occurs as a differential $d\rho_k : \mathfrak{k} \to \mathfrak{gl}(F^k)$. Explicitly, $\mathfrak{k}$ acts on F^k by the differential operators

$$d\rho_k(X) = x_0 \frac{\partial}{\partial x_1}, \quad d\rho_k(H) = x_0 \frac{\partial}{\partial x_0} - x_1 \frac{\partial}{\partial x_1}, \quad d\rho_k(Y) = x_1 \frac{\partial}{\partial x_0}. \tag{1.5}$$

Theorem 1.2. *For each $k \geq 0$, there exists a unique (up to scalar) K-invariant non-degenerate bilinear form Γ_k on F^k. If k is even then Γ_k is symmetric; if k is odd then Γ_k is skew-symmetric.*

Proof. As K-representations, $F^k \otimes F^k \cong S^2F^k \oplus \wedge^2 F^k$ with

$$S^2F^k \cong \bigoplus_{i\geq 0} F^{2k-4i} \quad \text{and} \quad \wedge^2 F^k \cong \bigoplus_{j\geq 0} F^{2k-4j-2}. \tag{1.6}$$

The trivial representation $F^0 = \mathbb{C}$ occurs in the symmetric square when k is even and in the exterior square when k is odd. Projection onto the trivial term determines (up to scalar) a K-equivariant map $\Gamma_k : S^2F^k \to \mathbb{C}$ or $\Gamma_k : \wedge^2 F^k \to \mathbb{C}$ as the case may be. □

If $k \geq 0$ is even (resp. odd) set $G_{(k)} = \mathrm{SO}(F^k, \Gamma_k)$ (resp. $G_{(k)} = \mathrm{Sp}(F^k, \Gamma_k)$). Then Theorem 1.2 provides an embedding $\rho_k : K \hookrightarrow G_{(k)}$. The differential $d\rho_k : \mathfrak{k} \hookrightarrow \mathrm{Lie}(G_{(k)})$ determines an embedding of Lie algebras.

Lemma 1.3. *Let $\mathfrak{g} = \mathrm{Lie}(G_{(k)})$ and $l = \mathrm{rank}(\mathfrak{g})$. As a representation of $\mathfrak{k}$,*

$$\mathfrak{g} \cong \bigoplus_{i=1}^{l} F^{2(2i-1)}.$$

Proof. When k is even (resp. odd) the adjoint representation is $\wedge^2 F^k$ (resp. S^2F^k) and $l = k/2$ (resp. $l = (k+1)/2$). The K-decomposition of $\mathfrak{g}$ is then provided by (1.6). □

Representations of $\mathrm{GL}(2, \mathbb{C})$

Let $G = \mathrm{GL}(2, \mathbb{C})$. The finite dimensional *polynomial* representations of G are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq 0)$. We denote the representation corresponding to such a λ by F_2^λ. The reason for this notation is motivated by the following fact.

Proposition 1.4. [Ful97, Section 8.2] *The restriction of the* $\mathrm{GL}(2,\mathbb{C})$*-representation* F_2^λ *to* $\mathrm{SL}(2,\mathbb{C})$ *is* $F^{\lambda_1-\lambda_2}$.

Representations of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$

The group $K := \mathrm{SL}(2\,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ plays a crucial role in this dissertation. For each $k,l\geq 0$, let $F^{k,l} = S^k\mathbb{C}^2\widehat{\otimes}S^l\mathbb{C}^2$ and let

$$\rho_{k,l} = \rho_k\widehat{\otimes}\rho_l : \mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\to\mathrm{GL}(F^{k,l}).$$

We have written $\widehat{\otimes}$ to emphasize that this is an *outer* tensor product with each factor of $\mathrm{SL}(2,\mathbb{C})$ acting independently on either $S^k\mathbb{C}^2$ or $S^l\mathbb{C}^2$. The space $F^{k,l}$ may be viewed as the binary (k,l)-forms. A typical element $f\in F^{k,l}$ is of the form

$$f = \sum_{i=0}^{k}\sum_{j=0}^{l}\binom{k}{i}\binom{l}{j}\,a_{i,j}\,x_0^{k-i}x_1^i y_0^{l-j}y_1^j.$$

The action of K is given by the left (resp. right) factor of $\mathrm{SL}(2,\mathbb{C})$ changing coordinates in the $\{\,x_0,x_1\,\}$ (resp. $\{\,y_0,y_1\,\}$) as in (1.3). Let $\mathfrak{k} = \mathrm{Lie}(K)$. Then $\mathfrak{k} = \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\cong\mathfrak{so}(4,\mathbb{C})$. The finite-dimensional irreducible representations of $\mathfrak{k}$ are the differentials $d\rho_{k,l}:\mathfrak{k}\to\mathfrak{gl}(F^{k,l})$. The action $\mathfrak{k}$ is given by the differential operators (1.5) in both sets of variables.

Representations of $\mathrm{Sp}(l,\mathbb{C})$

Let $G=\mathrm{Sp}(l,\mathbb{C})$ and let $\mathfrak{g}=\mathfrak{sp}(l,\mathbb{C})$. Let $\varpi_1,\ldots,\varpi_l$ be the fundamental weights for $\mathfrak{g}$ corresponding to a choice of Cartan subalgebra of $\mathfrak{g}$ and compatible positive roots. Since $\mathfrak{g}$ is simple, each $\lambda\in P_+(\mathfrak{g})$ is of the form

$$\lambda = \sum_{i=1}^{l} a_i\,\varpi_i$$

where each $a_i \in \mathbb{Z}_{\geq 0}$. Alternatively, λ may be expressed as the partition of depth at most l

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 0)$$

where $\lambda_i - \lambda_{i+1} = a_i$, $i = 1, \ldots, l$. We denote the irreducible representation corresponding to λ by V_l^λ or just V^λ when the rank of G is clear.

1.1.2 The principal TDS

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank $l > 0$. The adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{g}$ is given by $\mathrm{ad}(X)(Y) = [\,X, Y\,]$, for all $X, Y \in \mathfrak{g}$. An element $X \in \mathfrak{g}$ is called *semisimple* (resp. *nilpotent*) if the operator $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is semisimple (resp. nilpotent). Let G be the adjoint group of $\mathfrak{g}$. Then G acts on $\mathfrak{g}$ via the adjoint action $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ which we refer to simply as *conjugation*.

Definition 1.5.

1. A triple (H, X, Y) of non-zero linear independent elements of $\mathfrak{g}$ is called a *standard triple of* $\mathfrak{g}$ if H, X, Y satisfy the commutation relations (1.4).
2. A Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ is called a *TDS (three-dimensional subalgebra) of* $\mathfrak{g}$ if $\mathfrak{u} \cong \mathfrak{sl}(2, \mathbb{C})$ as Lie algebras.

Clearly, any standard triple of $\mathfrak{g}$ spans a TDS of $\mathfrak{g}$. If $\mathfrak{u}$ is a TDS of $\mathfrak{g}$ then $\mathfrak{u}$ contains a standard triple (H, X, Y). In this case, H is necessarily a semisimple element of $\mathfrak{u}$ and X, Y are nilpotent in $\mathfrak{u}$. These properties also hold as elements of $\mathfrak{g}$ by the preservation of Jordan form [Hum78, Corollary 6.4].

Lemma 1.6. *Suppose (H, X, Y) is a standard triple in $\mathfrak{g}$. Then H is a semisimple element of $\mathfrak{g}$ and X, Y are nilpotent elements of $\mathfrak{g}$.*

If $X \in \mathfrak{g}$ and $\mathfrak{u}$ is any subset of $\mathfrak{g}$ then the *centralizer of X in* $\mathfrak{u}$ is defined as

$$\mathfrak{u}^X = \{\, Y \in \mathfrak{u} \mid [\,X, Y\,] = 0 \,\}\,.$$

Definition 1.7. 1. $X \in \mathfrak{g}$ is called *regular in* $\mathfrak{g}$ if $\dim_{\mathbb{C}} \mathfrak{g}^X \leq \dim_{\mathbb{C}} \mathfrak{g}^Y$ for all $Y \in \mathfrak{g}$.

2. A standard triple (H, X, Y) in $\mathfrak{g}$ is called a *principal standard triple of* $\mathfrak{g}$ if X is regular.

3. A TDS $\mathfrak{u}$ of $\mathfrak{g}$ is called a *principal TDS* $\mathfrak{g}$ if it contains a regular nilpotent element of $\mathfrak{g}$.

Clearly, a TDS $\mathfrak{u}$ of $\mathfrak{g}$ is a principal TDS if and only if it is spanned by a principal standard triple.

Let $\mathfrak{h}$ be a Cartan subalgebra for $\mathfrak{g}$ and let $\Phi := \Phi(\mathfrak{h})$. Choose a positive root system Φ^+ for Φ and let $\Delta \subset \Phi^+$ be a base of simple roots. Let $\mathbb{E}$ denote the real span of Φ and let $(\,,\,)$ denote the restriction of the Killing form to $\mathbb{E}$. Set $N = |\,\Phi^+\,|$, and $h = \frac{2N}{l}$. The number h is the *Coxeter number of* $\mathfrak{g}$.

Theorem 1.8. [CM93, Theorem 3.4.12,4.1.6] *Let H_0 be the unique element of $\mathfrak{h}$ satisfying $\alpha(H_0) = 2$ for all $\alpha \in \Delta$. Then there exists a principal standard triple (H_0, X, Y) of $\mathfrak{g}$ containing H_0. Let $\mathfrak{u}_{prin}$ be the principal TDS spanned by H_0, X, and Y. Any other principal TDS of $\mathfrak{g}$ is conjugate to $\mathfrak{u}_{prin}$.*

Remark 1.9. The algebra $\mathfrak{u}_{\mathrm{prin}}$ is unique only up to choice of Cartan subalgebra $\mathfrak{h}$ and choice of positive root system for $\Phi(\mathfrak{h})$. On the other hand, suppose $\mathfrak{u}$ is a principal TDS spanned by a principal standard triple (H, X, Y). Then there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $H \in \mathfrak{h}$ and the set $\Delta := \{\, \alpha \in \Phi(\mathfrak{h}) \mid \alpha(H) = 2 \,\}$ is a base for $\Phi(\mathfrak{h})$. Relative to this data, $\mathfrak{u}$ is the same principal TDS defined in 1.8.

Consider $\mathfrak{g}$ as a representation of $\mathfrak{u}_{\mathrm{prin}}$ via the restriction of the adjoint representation $\mathrm{ad} : \mathfrak{u}_{\mathrm{prin}} \to \mathfrak{gl}(\mathfrak{g})$. The decomposition of $\mathfrak{g}$ into irreducible representations of $\mathfrak{u}_{\mathrm{prin}}$ is a beautiful result connecting many important invariants of $\mathfrak{g}$.

Theorem 1.10. [Kos59, Theorem 5.2] *Let $\mathfrak{u}$ be a TDS of $\mathfrak{g}$. Then $\mathfrak{g}$ decomposes into l odd-dimensional irreducible representations of $\mathfrak{u}$ if and only if $\mathfrak{u}$ is a principal TDS.*

In this case,

$$\mathfrak{g} = \bigoplus_{i=1}^{l} F^{2e_i}$$

where $e_1, \ldots, e_l$ *are non-negative integers.*

By Lemma 1.3 we obtain the following description of the principal TDS for the simple groups $\mathfrak{sp}(l, \mathbb{C}) = \mathfrak{sp}_{2l}\mathbb{C}$ and $\mathfrak{so}(2l+1, \mathbb{C})$.

Corollary 1.11. *Let* $\mathfrak{u} = \mathfrak{sl}(2, \mathbb{C})$*,* $k \geq 1$*, and let* $\mathfrak{g} = \mathfrak{sp}_k\mathbb{C}$ *(when* k *is even) or* $\mathfrak{so}(k, \mathbb{C})$ *(when* k *is odd). Then the image* $d\rho_k(\mathfrak{u})$ *of the map* $d\rho_k : \mathfrak{u} \to \mathfrak{g}$ *is a principal TDS.*

The integers $e_1, \ldots, e_l$ of 1.10 are invariants called the *exponents of* $\mathfrak{g}$. In fact, the numbers $2e_i + 1$, $i = 1, \ldots, l$ are the Betti numbers of the complex manifold G [CM93, Section 4.4]. We need the following connection between the exponents of $\mathfrak{g}$ and root heights.

Theorem 1.12. [Hum90, Theorem 3.20] *Write the exponents of* $\mathfrak{g}$ *in decreasing order* $\lambda := (e_1 \geq e_2 \geq \ldots \geq e_l \geq 0)$ *as a partition. Then*

1. $|\,\lambda\,| = \sum_{i=1}^{l} e_i = |\,\Phi^+\,|$.

2. $e_1 = h - 1$ *and* $e_l = 1$*. In particular, the conjugate partition* $\lambda^\sharp$ *has depth* $h - 1$*.*

3. *Write* $\lambda^\sharp = (k_1 \geq k_2 \geq \ldots \geq k_{h-1} \geq 0)$*. Then*

$$k_i = \#\left\{\, \alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) = i \,\right\}.$$

   In particular, $k_1 = l$*,* $k_2 = l - 1$*, and* $k_{h-1} = 1$*.*

Kostant's branching formula for $\mathfrak{u}_{\mathbf{prin}}$

For $\lambda \in P_+(\mathfrak{g})$ and $k \geq 0$ define

$$B_{\mathfrak{g}}(k, \lambda) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}(2,\mathbb{C})}\left(F^k, L(\lambda)\right). \tag{1.7}$$

Since all principal three-dimensional subalgebras are conjugate, the multiplicities (1.7) are independent of the choice of Cartan subalgebra and positive root system used to define $\mathfrak{u}_{\text{prin}}$.

Suppose $\mathfrak{u}_{\text{prin}}$ is the span of the principal standard triple (H_0, X, Y) with X regular in $\mathfrak{g}$ and

$$\alpha(H_0) = 2, \quad \text{for all } \alpha \in \Delta. \tag{1.8}$$

Then $\mathfrak{h}_{\text{prin}} := \mathbb{C}\, H_0$ is a Cartan subalgebra for $\mathfrak{u}_{\text{prin}}$ contained in $\mathfrak{h}$. For $\gamma \in \mathfrak{h}^*$, write $\overline{\gamma} \in \mathfrak{h}^*_{\text{prin}}$ for the restriction of γ to $\mathfrak{h}_{\text{prin}}$. Let δ be the unique element of $\mathbb{E}$ satisfying $(\delta, \alpha) = 2$ for all $\alpha \in \Delta$. Then $\overline{\delta}(H_0) = 2$ and $\mathfrak{h}^*_{\text{prin}} = \mathbb{C}\,\overline{\delta}$. We may identify $\mathfrak{h}^*_{\text{prin}}$ with $\mathbb{C}$ by identifying $\overline{\delta}$ with $2 \in \mathbb{C}$. By (1.4), $\Phi_{\text{prin}} := \{\, 2, -2 \,\}$ is the root system of $\mathfrak{u}_{\text{prin}}$ determined by $\mathfrak{h}_{\text{prin}}$ and $\Phi^+_{\text{prin}} := \{\, 2 \,\}$ is a set of positive roots for Φ_{prin}.

Let $\lambda \in \mathbb{E}$. Since Δ is a basis for $\mathbb{E}$ we can write $\lambda = \sum_{\alpha \in \Delta} n_\alpha \, \alpha$. Then

$$\overline{\lambda} = \sum_{\alpha \in \Delta} n_\alpha \, \overline{\alpha} = 2 \sum_{\alpha \in \Delta} n_\alpha = (\delta, \lambda). \tag{1.9}$$

In particular, if $\beta \in \Phi^+$ is a positive root then by (1.1)

$$\overline{\beta} = 2 \, \mathrm{ht}(\beta). \tag{1.10}$$

Consider the partition function $\wp_{\mathfrak{g}} : \mathbb{R} \to \mathbb{R}$ defined by

$$\sum_{k \in \mathbb{Z}_{\geq 0}} \wp_{\mathfrak{g}}(k) \; q^k = \frac{1 - q^2}{\prod_{\beta \in \Phi^+} \left(1 - q^{2\, \mathrm{ht}(\beta)}\right)}. \tag{1.11}$$

As in 1.12, let k_i denote the number of positive roots of $\mathfrak{g}$ of height i, $i = 1, ..., h-1$. Then, by (1.10)

$$k_i = \# \left\{\, \alpha \in \Phi^+ \mid \overline{\alpha} = 2i \,\right\}.$$

By (1.11),

$$\sum_{k \in \mathbb{Z}_{\geq 0}} \wp_{\mathfrak{g}}(k)\, q^k = \frac{1-q^2}{\prod_{i=1}^{h-1}(1-q^{2i})^{k_i}}. \tag{1.12}$$

Let $W_{\mathfrak{g}}$ denote the Weyl group of $\mathfrak{g}$ and let

$$\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha.$$

For each $w \in W_{\mathfrak{g}}$, define $L_w : \mathbb{R} \times \mathbb{E} \to \mathbb{R}$ by

$$L_w(k,\lambda) = \overline{w(\lambda+\rho)-\rho} - k. \tag{1.13}$$

Theorem 1.13 (Principal branching formula)**.** *For $\lambda \in P_+(\mathfrak{g})$ and $k \geq 0$,*

$$B_{\mathfrak{g}}(k,\lambda) = \sum_{w \in W_{\mathfrak{g}}} \operatorname{sgn}(w)\, \wp_{\mathfrak{g}}(L_w(k,\lambda)).$$

Moreover, if $B_{\mathfrak{g}}(k,\lambda) \neq 0$ then $\overline{\lambda} - k \in 2\mathbb{Z}$.

Proof. Use [GW98, Theorem 8.2.1] or [Kna02, Theorem 9.20]. The partition function used in Kostant's formula is obtained by considering the multiset $\overline{\Phi^+} \setminus \Phi^+_{\text{prin}}$.

By (1.11), $\wp_{\mathfrak{g}}(n) = 0$ for n odd. Since all simple roots restrict to 2, all λ in the root lattice of $\mathfrak{g}$ restrict to even values. The parity of each $L_w(k,\lambda)$ agrees as $w \in W$ varies because the Weyl group preserves the root lattice of $\mathfrak{g}$. Thus the parity of each $L_w(k,\lambda)$ is determined by the parity of $L_1(k,\lambda) = \overline{\lambda} - k$. □

1.1.3 Algebraic symmetric pairs

In this section we review the necessary facts from the seminal paper [KR71] of Kostant and Rallis. Let $\mathfrak{g}$ be a semisimple complex Lie algebra and suppose $\theta : \mathfrak{g} \to \mathfrak{g}$ is an involutional automorphism of $\mathfrak{g}$. Thus θ is an automorphism of Lie algebras satisfying

$\theta^2 = 1$. Let

$$\mathfrak{k} = \{\, X \in \mathfrak{g} \mid \theta(X) = X \,\}$$

and

$$\mathfrak{p} = \{\, X \in \mathfrak{g} \mid \theta(X) = -X \,\}$$

denote the $+1$-eigenspace of θ and the -1-eigenspace of θ, respectively. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *Cartan decomposition* of $\mathfrak{g}$. Moreover, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a representation of $\mathfrak{k}$ via the adjoint action. The pair $(\mathfrak{g}, \mathfrak{k})$ is called an *algebraic symmetric pair.*

Let G be the adjoint group of $\mathfrak{g}$. We let K_θ be the subgroup of G consisting of those elements which commute with θ

$$K_\theta = \{\, g \in G \mid \mathrm{Ad}(g)\theta = \theta\,\mathrm{Ad}(g) \,\}\,.$$

Alternatively, since G is connected, θ determines an automorphism $\Theta : G \to G$ of Lie groups and

$$K_\theta = \{\, g \in G \mid \Theta(g) = g \,\}\,.$$

In fact, $\mathfrak{k}$ and $\mathfrak{p}$ are both invariant under K_θ. Moreover, $\mathfrak{k}$ is the Lie algebra of K_θ and $\mathfrak{p}$ is a representation of K_θ.

A Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ is called a *toral subalgebra* if it is abelian and each element of $\mathfrak{u}$ is semisimple in $\mathfrak{g}$. Any two maximal toral subalgebras of $\mathfrak{g}$ which contained in $\mathfrak{p}$ are conjugate under the action of K_θ. Thus, the dimension $r = \dim_{\mathbb{C}} \mathfrak{a}$ of any maximal toral subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is an invariant of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ and is called the *real rank.* The following definition generalizes the notion of regularity defined earlier for the a djoint case.

Definition 1.14. An element $X \in \mathfrak{p}$ is called *regular in* $\mathfrak{p}$ if $\dim_{\mathbb{C}} \mathfrak{k}^X \leq \dim_{\mathbb{C}} \mathfrak{k}^Y$, for all $Y \in \mathfrak{p}$.

Let $\mathcal{R}$ denote the set of regular elements of $\mathfrak{p}$, $\mathcal{S}$ the set of semisimple elements of $\mathfrak{p}$, and $\mathcal{N}$ the set of nilpotent elements $\mathfrak{p}$. Then $\mathcal{N}$ is called the *nullcone* of $\mathfrak{p}$.

Lemma 1.15. [KR71, Prop. 8] *Let $X \in \mathfrak{p}$. Then $X \in \mathcal{R}$ if and only if* $\dim_{\mathbb{C}} \mathfrak{p}^X = r$.

Maximal toral subalgebras arise as the centralizers of regular semisimple elements.

Lemma 1.16. [KR71, Lemma 20] *Let $X \in \mathfrak{p}$ and let $\mathfrak{a} := \mathfrak{p}^X$. Then $X \in \mathcal{R} \cap \mathcal{S}$ if and only if $\mathfrak{a}$ is a maximal toral subalgebra of $\mathfrak{p}$. In this case, the centralizer of $\mathfrak{a}$ in K_θ is equal to the stabilizer of X in K_θ*

$$\{ g \in K_\theta \mid \mathrm{Ad}\,(g)\ \mathfrak{a} = \mathfrak{a} \} = \{ g \in K_\theta \mid \mathrm{Ad}\,(g)\ X = X \}.$$

Functions on the nullcone

Let $\mathbb{C}[\mathfrak{p}]$ denote the space of regular functions on $\mathfrak{p}$. Then $\mathbb{C}[\mathfrak{p}]$ is a representation of K_θ according to the rule

$$(k.f)(X) = f(k^{-1}X),$$

for all $k \in K_\theta$, $f \in \mathbb{C}[\mathfrak{p}]$, and $X \in \mathfrak{p}$. Let $\mathbb{C}[\mathfrak{p}]^{K_\theta}$ be the subring of $\mathbb{C}[\mathfrak{p}]$ of K_θ-invariant functions. The Chevalley Restriction Theorem asserts that $\mathbb{C}[\mathfrak{p}]^{K_\theta} = \mathbb{C}[u_1, \ldots, u_r]$ is a polynomial ring minimally generated by invariants $u_1, \ldots, u_r$. Note that the number of generators is the real rank r.

Lemma 1.17. *The ideal of $\mathbb{C}[\mathfrak{p}]$ generated by the K_θ invariants $u_1, \ldots, u_r$ is a radical ideal and the variety it defines is the nullcone*

$$\mathcal{N} = \{ X \in \mathfrak{p} \mid u_i(X) = 0 \ \text{for}\ i = 1, \ldots, r \}. \tag{1.14}$$

Moreover, if K_θ is a connected group then $\mathcal{N}$ is irreducible as an affine variety.

Proof. The fact that $\mathcal{N}$ is given by (1.14) is [KR71, Prop. 11]. By [KR71], $\mathcal{R} \cap \mathcal{N}$ is a single K_θ-orbit which is dense in $\mathcal{N}$. Hence, if K_θ is connected $\mathcal{N}$ is irreducible as an affine algebraic variety. □

Let $S(\mathfrak{p})$ be the symmetric algebra on $\mathfrak{p}$ viewed as a representation of K_θ. Let $\partial : \mathfrak{p} \to \mathrm{End}_{\mathbb{C}}(\mathfrak{p})$ be the map associating to each $X \in \mathfrak{p}$ its directional derivative ∂_X

$$(\partial_X f)(Y) = \lim_{t \to 0} \frac{f(Y + tX) - f(Y)}{t}, \quad \text{for all } Y \in \mathfrak{p}.$$

This extends uniquely to a K_θ-map $\partial : S(\mathfrak{p}) \to \mathrm{End}_{\mathbb{C}}(\mathfrak{p})$. Moreover, $\mathbb{C}[\mathfrak{p}] \cong S(\mathfrak{p})$ as algebras and as graded K_θ-representations. We may then view $\partial : \mathbb{C}[\mathfrak{p}] \to \mathrm{End}_{\mathbb{C}}(\mathfrak{p})$.

Definition 1.18. A function $f \in \mathbb{C}[\mathfrak{p}]$ is called *harmonic* if $\partial(u)f = 0$ for all $u \in \mathbb{C}[\mathfrak{p}]^{K_\theta}$ satisfying $u(0) = 0$.

Let $\mathcal{H}(\mathfrak{p}) \subset \mathbb{C}[\mathfrak{p}]$ denote the space of harmonic functions on $\mathfrak{p}$. Then

$$\mathcal{H}(\mathfrak{p}) = \{ f \in \mathbb{C}[\mathfrak{p}] \mid \partial(u_i)f = 0, \ \text{ for } i = 1, \ldots, r \}.$$

For $d \geq 0$, define

$$\mathcal{H}_d(\mathfrak{p}) = \mathcal{H}(\mathfrak{p}) \cap \mathbb{C}[\mathfrak{p}]_d.$$

Then $\mathcal{H}(\mathfrak{p}) = \oplus_{d \geq 0} \mathcal{H}_d(\mathfrak{p})$ is a graded vector space and each $\mathcal{H}_d(\mathfrak{p})$ is K_θ-invariant.

Lemma 1.19. *Restriction of functions from $\mathfrak{p}$ to the nullcone $\mathcal{N}$ is an isomorphism of graded K_θ-representations $\mathcal{H}(\mathfrak{p}) \cong \mathbb{C}[\mathcal{N}]$.*

A central result in [KR71] is the following separation of variables for $\mathbb{C}[\mathfrak{p}]$.

Theorem 1.20. [GW98, Theorem 12.4.1] *Let $\mathfrak{a}$ be a maximal toral subalgebra of $\mathfrak{p}$ and let M_θ be the centralizer of $\mathfrak{a}$ in K_θ. Then multiplication of functions gives an* *isomorphism of K_θ-representations.*

$$\mathcal{H}(\mathfrak{p}) \otimes \mathbb{C}[\mathfrak{p}]^{K_\theta} \cong \mathbb{C}[\mathfrak{p}].$$

As a representation of K_θ,

$$\mathcal{H}(\mathfrak{p}) \cong \mathrm{Ind}_{M_\theta}^{K_\theta}(1) \tag{1.15}$$

is induced from the trivial representation of M_θ.

The key step to proving Theorem 1.20 is to observe that if $Z \in \mathfrak{a}$ is a regular semisimple element of $\mathfrak{p}$ then the restriction of functions from $\mathfrak{p}$ to the K_θ-orbit $\mathcal{O}_Z$ of Z is an isomorphism of K_θ-representations $\mathcal{H}(\mathfrak{p}) \cong \mathbb{C}[\mathcal{O}_Z]$. Now, $\mathcal{O}_Z \cong K_\theta/M_\theta$ by Lemma 1.16. Thus, $\mathbb{C}[\mathcal{O}_Z] \cong \mathbb{C}[K_\theta/M_\theta]$ proving that $\mathcal{H}(\mathfrak{p})$ is indeed induced from the trivial representation of M_θ.

Graded multiplicities

Suppose $\lambda \in P_+(\mathfrak{k})$ and $L(\lambda)$ is the corresponding finite-dimensional irreducible representation of K_θ with highest weight λ. By (1.15) and Frobenius reciprocity,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K_\theta}(L(\lambda), \mathcal{H}(\mathfrak{p})) = \dim_{\mathbb{C}} \left(L(\lambda)^{M_\theta}\right).$$

In particular, $L(\lambda)$ occurs in the harmonics with finite multiplicity. This fact, however, does not explain in which degrees $L(\lambda)$ appears. One would like to know the *graded multiplicity of λ in $\mathcal{H}(\mathfrak{p})$*, by which we mean the numbers

$$f_d(\lambda) := \dim_{\mathbb{C}} \mathrm{Hom}_{K_\theta}(L(\lambda), \mathcal{H}_d(\mathfrak{p})), \quad \text{for } \lambda \in P_+(\mathfrak{k}) \text{ and } d \in \mathbb{Z}_{\geq 0}. \tag{1.16}$$

It is convenient to arrange these statistics as the coefficients of a polynomial. For $\lambda \in P_+(\mathfrak{k})$, the *q-multiplicity of* λ is defined as

$$p_\lambda(q) := \sum_{d \geq 0} f_d(\lambda)\, q^d \tag{1.17}$$

where q is an indeterminate. In particular,

$$p_\lambda(1) = \dim_{\mathbb{C}} \operatorname{Hom}_{K_\theta}\left(L(\lambda), \mathcal{H}(\mathfrak{p})\right) = \dim_{\mathbb{C}} \left(L\,(\lambda)^{M_\theta} \right) < \infty.$$

Lemma 1.21. *Let d_i be the degree of the K_θ invariant u_i, $i = 1, \ldots, r$. Let* $\mathrm{wt}(\mathfrak{p}^*) \subset$ *$P(\mathfrak{k})$ be the set weights of $\mathfrak{p}^*$ as a K_θ-representation. Then*

$$\sum_{\lambda \in P_+(\mathfrak{k})} p_\lambda(q)\ \mathrm{ch}\left(L(\lambda)\right) = \frac{\prod_{i=1}^r (1 - q^{d_i})}{\prod_{\gamma \in \mathrm{wt}(\mathfrak{p}^*)} (1 - q e^\gamma)},$$

Proof. We have

$$\mathrm{ch}_q\ \mathbb{C}\,[\,\mathfrak{p}\,] = \frac{1}{\prod_{\gamma \in \mathrm{wt}(\mathfrak{p}^*)} (\,1 - q\, e^\gamma\,)}. \tag{1.18}$$

By (1.17),

$$\mathrm{ch}_q\ \mathcal{H}(\mathfrak{p}) = \sum_{\lambda \in P_+(\mathfrak{k})} p_\lambda(q)\ \mathrm{ch}\left(L(\lambda)\right).$$

By Theorem 1.20,

$$\begin{aligned} \mathrm{ch}_q\ \mathbb{C}\,[\,\mathfrak{p}\,] &= \mathrm{ch}_q\ \mathbb{C}\,[\,\mathfrak{p}\,]^{K_\theta} \cdot \mathrm{ch}_q\ \mathcal{H}(\mathfrak{p}) \\ &= \frac{1}{\prod_{i=1}^r (1 - q^{d_i})} \cdot \mathrm{ch}_q\ \mathcal{H}(\mathfrak{p}). \end{aligned}$$

Solving for $\mathrm{ch}_q\, \mathcal{H}(\mathfrak{p})$ and combining with (1.18) gives the desired result. □

For examples of symmetric pairs see Goodman and Wallach [GW98, Ch. 12.4.3] or Helgason [Hel01, Ch. X, Section 6]. Hesselink has found an alternating formula for the graded multiplicities in the adjoint case. Let $\mathfrak{k}$ be a semisimple complex Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$. Consider the pair $(\mathfrak{g}, \mathfrak{k})$ with $\mathfrak{k}$ diagonally embedded in $\mathfrak{g}$.

Theorem 1.22 ([Hes80]). *For $\lambda \in P_+(\mathfrak{k})$ and $d \geq 0$*

$$f_d(\lambda) = \sum_{w \in W} \operatorname{sgn}(w) \, \wp_d \left(w \left(\lambda + \rho \right) - \rho \right),$$

where W is the Weyl group of K and $\wp_d(\gamma)$ is the number of ways to write

$$\gamma = \sum_{\alpha \in \Phi_{\mathfrak{k}}^+} n_\alpha \, \alpha$$

with $n_\alpha \in \mathbb{Z}_{\geq 0}$ and $\sum n_\alpha = d$.

Other interesting cases where the graded multiplicity is understood are more sporadic. The pair $(\mathfrak{sl}(4, \mathbb{C}), \mathfrak{so}(4, \mathbb{C}))$ was treated in [WW00] while the pair $(F_4, \mathrm{Spin}(9))$ is multiplicity free [Joh76]. Wallach used an example of a symmetric pair to answer a question related to the space of 4-qubits [Wal05].

Chapter 2

Principal branching multiplicity

There are two fundamental difficulties in applying a branching formula such as that given by Theorem 1.13

$$B_{\mathfrak{g}}(k, \lambda) = \sum_{w \in W} \operatorname{sgn}(w) \, \wp_{\mathfrak{g}} \left(L_w(k, \lambda) \right).$$

Firstly, the alternation over the Weyl group W of $\mathfrak{g}$ will result in including many terms which contribute nothing to the total. When $L_w(k, \lambda) < 0$, $\wp((L_w(k, \lambda)) = 0$ and including such terms may be avoided. In Section 2.1 we make the simple observation that $w \in W$ is more likely to fall into this category as w gets larger in the sense of the Bruhat order.

Secondly, it is difficult to obtain useful expressions for evaluating the partition function $\wp_{\mathfrak{g}}$. In Section 2.2 we apply G. J. Heckman's asymptotic methods to show that the principal branching multiplicity is asymptotically related to the exponents of $\mathfrak{g}$. In Section 2.3 the principal branching rule for $\mathfrak{sp}(2, \mathbb{C})$ is investigated in further detail.

2.1 Bruhat order and weight restriction

Let $\mathbb{E}$ be a Euclidean space of dimension $l > 0$ with standard basis $\{ \varepsilon_1, \ldots, \varepsilon_l \}$ and inner-product $(\, , \,)$. Let $\Phi \subset \mathbb{E}$ be an irreducible crystallographic root system of rank l and make a choice $\Phi^+ \subset \Phi$ of positive roots. Let $\Delta \subset \Phi^+$ be a base of simple roots for Φ. Each root β determines a reflection $t_\beta \in \mathrm{GL}(\mathbb{E})$ according to the formula

$$t_\beta(\lambda) = \lambda - N_\beta(\lambda)\,\beta,$$

where $\lambda \in \mathbb{E}$ and

$$N_\beta(\lambda) := \frac{2(\lambda, \beta)}{(\beta, \beta)}.$$

It is a part of the definition of a root system that $N_\beta(\alpha) \in \mathbb{Z}$ whenever $\alpha, \beta \in \Phi$. If $\alpha \in \Delta$ then we write s_α in place of t_α and we let $S = \{ s_\alpha \mid \alpha \in \Delta \}$ be the corresponding set of simple reflections.

Let W be the Weyl group determined by the above data, i.e. W is the (finite) subgroup of $\mathrm{GL}(\mathbb{E})$ generated by S. In fact, W acts orthogonally on $\mathbb{E}$ which may be expressed as saying

$$(\lambda, w\gamma) = (w^{-1}\lambda, \gamma),$$

for all $w \in W$ and $\lambda, \gamma \in \mathbb{E}$. For each $w \in W$ the *length of* w is the number

$$\ell(w) = \# \left\{ \alpha \in \Phi^+ \mid w\alpha \notin \Phi^+ \right\}.$$

The *Bruhat order of* W is then the partial ordering $<$ on W defined as the transitive closure of the relations

$$w < w \cdot t_\beta \quad \Longleftrightarrow \quad \ell(w) < \ell(w \cdot t_\beta),$$

for $w \in W$ and $\beta \in \Phi^+$. Note that $<$ ultimately depends on the choice of simple roots Δ. We need the following fact which connects the Bruhat order with the action of W on $\mathbb{E}$.

Lemma 2.1. [BB05, Prop. 4.4.6] *For all $w \in W$ and $\beta \in \Phi^+$*

$$w(\beta) \in \Phi^- \quad \Longleftrightarrow \quad \ell(w \cdot t_\beta) < \ell(w).$$

Let

$$\mathcal{D} := \{\, \lambda \in \mathbb{E} \mid (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Delta \,\}.$$

We refer to elements of $\mathcal{D}$ as *dominant vectors*. Fix a dominant vector $\delta \in \mathcal{D}$ and let $J = \{\, \alpha \in \Delta \mid (\delta, \alpha) = 0 \,\}$. The stabilizer of δ in the Weyl group is the so-called parabolic subgroup $W_J := \langle\, s_\alpha \mid \alpha \in J \,\rangle$. Define

$${}^J W = \{\, w \in W \mid s_\alpha \cdot w > w \text{ for all } \alpha \in J \,\}.$$

Then ${}^J W$ forms the set of the unique minimal length *right* coset representatives for the quotient $W_J \backslash W$. Moreover, $({}^J W, <)$ is a partially ordered set under the restricted Bruhat order. [1]

Definition 2.2. For $\lambda \in \mathcal{D}$ and $k \in \mathbb{R}$, define

$$\mathcal{T}_\delta(k, \lambda) = \{\, w \in W \mid (\delta, w\lambda) \geq k \,\}.$$

Recall that a subset A of a partially ordered set $(P, <)$ is called an *order ideal of P* if for all $a, b \in P$

$$b < a \text{ and } a \in A \quad \Longrightarrow \quad b \in A.$$

Proposition 2.3. *Let $\lambda \in \mathcal{D}$, $k \in \mathbb{R}$. Set $A := \mathcal{T}_\delta(k, \lambda)$ and $B := A \cap {}^J W$. Then*

[1] See [BB05] or [Hum90] for details regarding parabolic subgroups and quotients. Both texts, however, focus on the *left* coset representatives of the quotient W/W_J.

1. *A is an order ideal in* W*;*

2. *B is an order ideal in* ${}^JW$*;*

3. $A = \bigcup {}^JW \cdot w$ *where the union is taken over the maximal elements of* B*.*

Proof. Let $w \in W$, $\beta \in \Phi^+$, and let $w' = w \cdot t_\beta$. For any $\lambda \in \mathbb{E}$,

$$(\delta, w'\lambda) = (w^{-1}\delta, t_\beta\lambda) = (w^{-1}\delta, \lambda) - N_\beta(\lambda)(w^{-1}\delta, \beta)$$

So,

$$(\delta, w\lambda) = (\delta, w'\lambda) + N_\beta(\lambda)(\delta, w\beta).$$

If $\lambda \in \mathcal{D}$ is dominant then $N_\beta(\lambda) \geq 0$. If, in addition, $w < w'$ then $\ell(w) < \ell(w')$ and $w\beta \in \Phi^+$ by Lemma 2.1. Since $\delta \in \mathcal{D}$ we also have $(\delta, w\beta) \geq 0$.

By the definition of Bruhat order we've shown for all $w, w' \in W$,

$$w < w' \quad \Longrightarrow \quad (\delta, w'\lambda) \leq (\delta, w\lambda) \text{ for all } \lambda \in \mathcal{D}.$$

In particular, if $w < w' \in W$ and $w' \in \mathcal{T}_\delta(k, \lambda)$ then $(\delta, w\lambda) \geq (\delta, w'\lambda) \geq k$. Hence, $w \in \mathcal{T}_\delta(k, \lambda)$ and $\mathcal{T}_\delta(k, \lambda)$ is an order ideal.

Suppose now that $w \in \mathcal{T}_\delta(k, \lambda)$ and $u \in W_J$ is in the stabilizer of δ. Then

$$(\delta, uw\lambda) = (u^{-1}\delta, w\lambda) = (\delta, w\lambda) \geq k.$$

This shows that if $w \in A$ then the right coset $W_J \cdot w \subset A$. The remaining claims follow from this fact. $\square$

Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Definition 2.4. For $\lambda \in \mathcal{D}$ and $k \in \mathbb{R}$, define

$$\mathcal{T}_\delta^*(k, \lambda) = \{ w \in W \mid (\delta, w(\lambda + \rho) - \rho) \geq k \}.$$

Lemma 2.5. *For any $\lambda \in \mathbb{E}$ and $k \in \mathbb{R}$*

$$\mathcal{T}_\delta^*(k, \lambda) = \mathcal{T}_\delta(k + (\delta, \rho), \lambda + \rho).$$

In particular, if $\lambda \in \mathcal{D}$ then $\mathcal{T}_\delta^*(k, \lambda)$ is an order ideal of W.

Proof. For $\lambda \in \mathbb{E}$,

$$\begin{aligned} w \in \mathcal{T}_\delta^*(k, \lambda) &\Longleftrightarrow (\delta, w(\lambda + \rho) - \rho) \geq k \\ &\Longleftrightarrow (\delta, w(\lambda + \rho)) - (\delta, \rho) \geq k \\ &\Longleftrightarrow (\delta, w(\lambda + \rho)) \geq k + (\delta, \rho) \\ &\Longleftrightarrow w \in \mathcal{T}_\delta(k + (\delta, \rho), \lambda + \rho). \end{aligned}$$

If λ is dominant then so is $\lambda + \rho$. By 2.3, $\mathcal{T}_\delta(k + (\delta, \rho), \lambda + \rho)$ is an order ideal. □

2.1.1 Bruhat order and principal branching

Suppose now that $\mathfrak{g}$ is a simple complex Lie algebra with rank $l \geq 2$ and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\Phi := \Phi(\mathfrak{h})$ and choose a set Φ^+ of positive roots for Φ. Let $\Delta \subset \Phi^+$ be a base of simple roots for Φ. Enumerate the simple roots $\Delta = \{ \alpha_1, \ldots, \alpha_l \}$. Let $\mathbb{E}$ be the real span of Φ and W the Weyl group of $\mathfrak{g}$ with Bruhat order $<$ determined by Δ. Let $\mathfrak{u}_{\text{prin}}$ be the principal TDS of $\mathfrak{g}$ determined by the above data as in Theorem 1.8. Let $\delta \in \mathbb{E}$ satisfying $(\delta, \alpha) = 2$ for all $\alpha \in \Delta$. The next proposition refines Theorem 1.13.

Proposition 2.6. *For $k \geq 0$, $\lambda \in P_+(\mathfrak{g})$, $\overline{\lambda} - k \in 2\mathbb{Z}$*

$$B_{\mathfrak{g}}(k,\lambda) = \sum_{w \in S} \operatorname{sgn}(w)\, \wp_{\mathfrak{g}}\left(L_w(k,\lambda)\right)$$

where S may be taken to be the order ideal $\mathcal{T}_\delta^*(k,\lambda)$ of $(W,<)$. In particular, if $S = \emptyset$, then $B_{\mathfrak{g}}$ is identically zero. If $\overline{\lambda} - k < 0$ then $B_{\mathfrak{g}}(k,\lambda) = 0$.

Proof. By Section 1.1.2,

$$L_w(k,\lambda) = \overline{w(\lambda+\rho)-\rho} - k = (\delta, w(\lambda+\rho)-\rho) - k.$$

By Definition 2.4, $L_w(k,\lambda) \geq 0$ if and only if $w \in \mathcal{T}_\delta^*(k,\lambda)$.

Suppose $\overline{\lambda} - k < 0$. Then $(\delta,\lambda) < k$ and it must be that the identity element of W is not in $\mathcal{T}_\delta^*(k,\lambda)$. Since $\mathcal{T}_\delta^*(k,\lambda)$ is an order ideal, $\mathcal{T}_\delta^*(k,\lambda) = \emptyset$. Therefore, $\wp_{\mathfrak{g}}\left(L_w(k,\lambda)\right) = 0$ for all $w \in W$ and the branching multiplicity vanishes. □

2.2 μ-Partition functions

In this section we apply G. J. Heckman's method [Hec82] for establishing asymptotic estimates of partition functions for a special case relevant to the partition function (1.11). We then use this to establish an asymptotic estimate for the principal branching multiplicities $B_{\mathfrak{g}}(k,\lambda)$.

Fix an integer $M > 0$ and choose a partition $\mu = (\mu_1 \geq \mu_2 \geq \ldots \mu_n \geq 0)$ of M. We associate to μ a partition function $\wp_\mu : \mathbb{R} \to \mathbb{R}$ given by the generating function

$$\sum_{k \geq 0} \wp_\mu(k)\, q^k = \frac{1}{\prod_{i=1}^n (1-q^i)^{\mu_i}}. \tag{2.1}$$

Thus, for a non-negative integer k, $\wp_\mu(k)$ is the number of ways to write

$$k = \sum_{i=1}^{n} i \left(\sum_{j=1}^{\mu_i} a_{i,j} \right),$$

where each $a_{i,j}$ must be a non-negative integer. In particular, $\wp_\mu(0) = 1$. Note that since each $\mu_i \geq 0$, $\wp_\mu(k)$ is finite. If $k \in \mathbb{R}$ is not a non-negative integer then $\wp_\mu(k) = 0$.

Let $\mathbb{E} = \mathbb{R}^M$ be a Euclidean space of dimension M. The partition μ determines an indexing of the standard basis

$$\mathbb{E} = \mathrm{span}\{ e(i,j) \mid 1 \leq i \leq n,\, 1 \leq j \leq \mu_i \}.$$

Define $L_\mu : \mathbb{E} \rightarrow \mathbb{R}$ to be the linear function determined by

$$L_\mu \left(e\left(i,j\right)\right) = i, \quad \text{for } 1 \leq i \leq n,\, 1 \leq j \leq \mu_i.$$

Set

$$\mathbb{E}_{\geq 0} := \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \mu_i}} \mathbb{R}_{\geq 0}\, e(i,j).$$

Then

$$\wp_\mu(k) = \#\text{ of integral points in } L_\mu^{-1}(k) \cap \mathbb{E}_{\geq 0}.$$

For example, Figure 2.1 shows the set $L_\mu^{-1}(10) \cap \mathbb{E}_{\geq 0} \subset \mathbb{R}^3$ for $\mu = (2,1)$.

Heckman's approach to estimating $\wp_\mu(k)$ is to compute the volume of $L_\mu^{-1}(k) \cap \mathbb{E}_{\geq 0}$. normalized by the volume of a fundamental block formed by the integral points.

The integral solutions to the equation

$$\sum_{i=1}^{n} i \left(\sum_{j=1}^{\mu_i} a_{i,j}\, e(i,j) \right) = 0 \tag{2.2}$$

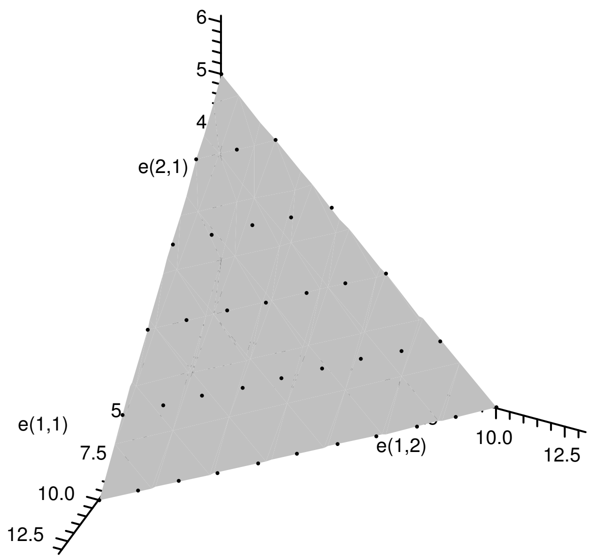

Figure 2.1: $L_\mu^{-1}(10) \cap \mathbb{E}_{\geq 0}$ with $\mu = (2, 1)$.

in the unknowns $\{ a_{i,j} \}$ form a lattice K_μ of rank $M - 1$ in $\ker L_\mu$. Define vectors

$$v_{i,j} = -i \cdot e(1,1) + 1 \cdot e(i,j), \quad \text{for } 1 \leq i \leq n, \quad 1 \leq j \leq \mu_i, \quad (i,j) \neq (1,1). \tag{2.3}$$

These $M - 1$ non-zero vectors are independent and determine a set of generators for K_μ as an abelian group

$$K_\mu = \bigoplus_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \mu_i \\ (i,j) \neq (1,1)}} \mathbb{Z}\, v_{i,j}.$$

Let N_μ denote the volume of a fundamental domain in K_μ.

Lemma 2.7.

$$N_\mu = \sqrt{\sum_{i=1}^{n} \mu_i\, i^2}. \tag{2.4}$$

Proof. We must compute the volume of the $(M-1)$-parallelepiped $\mathcal{P}$ determined by the $M-1$ vectors $v_{i,j}$

$$\mathcal{P} := \left\{ \sum_{\substack{1\le i\le n \\ 1\le j\le \mu_i \\ (i,j)\ne(1,1)}} t_{i,j}\, v_{i,j} \mid 0 \le t_{i,j} \le 1 \text{ for all } i,j \right\}.$$

Then $N_\mu = \sqrt{\det\left(AA^t\right)}$, were A is the matrix whose rows are the $v_{i,j}$'s. Making the natural choice of ordering of the $v_{i,j}$'s we may take

$$A = \left[\begin{array}{c|c|c|c|c|c}
-1 & & & & & \\
\vdots & I_{\mu_1-1} & & & & \\
-1 & & & & & \\ \hline
-2 & & & & & \\
\vdots & & I_{\mu_2} & & & \\
-2 & & & & & \\ \hline
-3 & & & & & \\
\vdots & & & I_{\mu_3} & & \\
-3 & & & & & \\ \hline
\vdots & & & & \ddots & \\ \hline
-n & & & & & \\
\vdots & & & & & I_{\mu_n} \\
-n & & & & &
\end{array}\right]_{(M-1)\times M}$$

where I_k denotes the $k\times k$ identity matrix. By [Blo79, 5.7.4],

$$\det\left(AA^t\right) = \sum_{k=1}^{M} \left(\det B_k\right)^2$$

where B_k is the $(M-1)\times(M-1)$-submatrix of A obtained by deleting the k^{th} column from A. By performing minor expansion along the k^{th}-row of B_k the determinant is easily computed: by deleting any of the first μ_1 columns of A the resulting submatrix has squared determinant 1; by deleting any of the next μ_2 columns of A the resulting submatrix has squared determinant 2; and so on. Hence,

$$\det\left(AA^t\right) = \sum_{i=1}^{n} \mu_i i^2.$$

□

Recall that $\mu^\sharp$ is the conjugate partition of μ and $\mu! = \prod_{i=1}^{n} \mu_i!$.

Lemma 2.8. *For $k \geq 0$,*

$$\operatorname{vol}\left(L_\mu^{-1}(k) \cap \mathbb{E}_{\geq 0}\right) = \frac{N_\mu}{(M-1)!\,\mu^\sharp!}\,k^{M-1}.$$

Proof. Let $u_{i,j} = \frac{k}{i} \cdot e(i,j)$, for $1 \leq i \leq n$, $1 \leq j \leq \mu_i$. Then $L_\mu^{-1}(k) \cap \mathbb{E}_{\geq 0}$ is the convex hull in $\mathbb{E}$ of the set $\{\, u_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq \mu_i \,\}$. This is an $(M-1)$-simplex in $\mathbb{R}^M$ whose volume is given by

$$\frac{1}{(M-1)!}\sqrt{\det\left(AA^t\right)}$$

where A is the matrix with rows $w_{i,j} := u_{i,j} - u_{1,1}$, $1 \leq i \leq n$, $1 \leq j \leq \mu_i$, $(i,j) \neq (1,1)$. With the appropriate ordering of the $w_{i,j}$'s, we may take

$$A = \left[\begin{array}{c|c|c|c|c|c}
-k & & & & & \\
\vdots & k \cdot I_{\mu_1 - 1} & & & & \\
-k & & & & & \\
\hline
-k & & & & & \\
\vdots & & \frac{k}{2} \cdot I_{\mu_2} & & & \\
-k & & & & & \\
\hline
-k & & & & & \\
\vdots & & & \frac{k}{3} \cdot I_{\mu_3} & & \\
-k & & & & & \\
\hline
\vdots & & & & \ddots & \\
\hline
-k & & & & & \\
\vdots & & & & & \frac{k}{n} \cdot I_{\mu_n} \\
-k & & & & &
\end{array}\right]_{(M-1)\times M} .$$

In a similar fashion as Lemma 2.7 we delete columns to obtain the determinant of AA^t

$$\begin{aligned}
\det\left(AA^t\right) &= \sum_{i=1}^{n} \mu_i \left[\frac{k^{\mu_1} \left(\frac{k}{2}\right)^{\mu_2} \cdots \left(\frac{k}{n}\right)^{\mu_n}}{\frac{k}{i}} \right]^2 \\
&= \sum_{i-1}^{n} \mu_i \, i^2 \left[\frac{k^{M-1}}{1^{\mu_1} 2^{\mu_2} \cdots n^{\mu_n}} \right]^2
\end{aligned}$$

which by Lemma 1.1 is

$$\begin{aligned}
&= \sum_{i-1}^{n} \mu_i \, i^2 \left[\frac{k^{M-1}}{\mu^{\sharp}!} \right]^2 \\
&= N_\mu^2 \left[\frac{k^{M-1}}{\mu^{\sharp}!} \right]^2 .
\end{aligned}$$

Taking the square root and dividing by $(M-1)!$ obtains the result. □

The volume of $L_\mu^{-1}(k) \cap \mathbb{E}_{\geq 0}$ normalized by N_μ provides the following first order estimate.

Proposition 2.9. *Let μ be a partition and set $M = |\,\mu\,|$.*

1. *There exists a constant $C > 0$ such that for all non-negative integers k,*

$$\left| \wp_\mu(k) - \frac{k^{M-1}}{(M-1)!\,\mu^\sharp!} \right| \leq C\,(1+k)^{M-2}.$$

2. *Assume that $\ell(\mu) \geq 2$, i.e. $\mu_1 \geq \mu_2 > 0$. Fix $\xi \in \mathbb{Z}$. There exists a constant $C_\xi > 0$ such that for all $k \in \mathbb{Z}$,*

$$\left| \wp_\mu(k+\xi) - \frac{k^{M-1}}{(M-1)!\,\mu^\sharp!} \right| \leq C_\xi\,(1+k)^{M-2}.$$

Proof. In the notation of [Hec82, Section 2] we have $V = \mathbb{R}$, $L = \mathbb{Z}$, $A = \{\, i \mid \mu_i > 0 \,\}$, $m_i = \mu_i$. Then $\sum_{i \in A} m_i = M$ and the $\mathbb{Z}$-span of A is the rank one lattice $\mathbb{Z}$ in V. Apply [Hec82, Lemma 2.3] for part (1).

For part (2), if $i \in A$ then $A \setminus \{\, i \,\}$ still determines a rank one lattice by assumption on the depth of μ. We can then apply [Hec82, Lemma 2.4]. □

Example 2.10. Let $\mu = (1,1,1)$. The generating function for $\wp_\mu$ is

$$\frac{1}{(1-q)(1-q^2)(1-q^3)}.$$

We have $M = 3$, $\mu^\sharp = (3)$, and $\mu^\sharp! = 3! = 6$. The estimate given by 2.9 is $k^2/12$. In fact, for integral $k \geq 0$

$$\wp_\mu(k) = \left[\frac{(k+3)^2}{12} \right]$$

where $[n]$ denotes the nearest integer to n [Slo].

Example 2.11. Let $\mu = (2,1)$. The generating function for $\wp_\mu$ is

$$\frac{1}{(1-q)^2(1-q^2)}.$$

We have $M = 3$, $\mu^\sharp = (2,1)$, $\mu^\sharp! = 2$. The estimate in this example is $k^2/4$. In fact, for integral $k \geq 0$

$$\wp_\mu(k) = \left\lfloor \frac{(k+2)^2}{4} \right\rfloor$$

where $\lfloor n \rfloor$ denotes the floor of n [Slo].

2.2.1 Asymptotics for principal branching

Let $\mathfrak{g}$ be a simple complex Lie algebra with rank $l \geq 2$ and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\Phi := \Phi(\mathfrak{h})$ and choose a set of positive roots Φ^+ for Φ. Let $\Delta \subset \Phi^+$ be a base of simple roots for Φ. Let $\mathbb{E}$ denote the real span of Φ and let W be the Weyl group of $\mathfrak{g}$. Let $N = |\,\Phi^+\,|$ be the number of positive roots and let $h = \frac{2N}{l}$ be the Coxeter number of $\mathfrak{g}$. For $i = 1, \ldots, h-1$, let k_i be the number of positive roots with height i. Let

$$\mu = (k_1 - 1 \geq k_2 \geq k_3 \geq \ldots \geq k_{h-1}).$$

Then μ is indeed a partition by Theorem 1.12 (3) and $|\,\mu\,| = N - 1$. Let

$$\lambda = (e_1 \geq e_2 \geq \ldots \geq e_l)$$

be the exponents of $\mathfrak{g}$ written in decreasing order. Note that $\mu^\sharp \neq \lambda$ due to the subtraction in the first entry of μ.

Lemma 2.12. $\mu^\sharp! = \lambda!$

Proof. Use the fact that $e_l = 1$. □

By (2.1),

$$\sum_{k \geq 0} \wp_\mu(k)\, q^k = \frac{1}{(1-q)^{k_1-1} \prod_{i=2}^{n} (1-q^i)^{k_i}} = \frac{1-q}{\prod_{i=1}^{h-1} (1-q^i)^{k_i}}$$

Comparing with the partition function $\wp_{\mathfrak{g}}$ defined in (1.11),

$$\sum_{k\geq 0} \wp_{\mathfrak{g}}(k)\, q^k = \frac{1-q^2}{\prod_{i=1}^{h-1}(1-q^{2i})^{k_i}} = \sum_{k\geq 0} \wp_{\mu}(k)\, q^{2k}.$$

Thus,

$$\wp_{\mathfrak{g}} = \begin{cases} \wp_{\mu}(k/2), & \text{if } k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

In this context, Proposition 2.9 is restated as follows.

Lemma 2.13. *1. There exists a constant $C > 0$ such that for all non-negative $k \in 2\mathbb{Z}$,*

$$\left| \wp_{\mathfrak{g}}(k) - \frac{(k/2)^{N-2}}{(N-2)!\, e_1! e_2! \cdots e_l!} \right| \leq C\,(1+k)^{N-3}.$$

2. Fix $\xi \in \mathbb{Z}$. There exists a constant $C_\xi > 0$ such that for all $k \in \mathbb{Z}$ satisfying $k - \xi \in 2\mathbb{Z}$,

$$\left| \wp_{\mathfrak{g}}(k+\xi) - \frac{(k/2)^{N-2}}{(N-2)!\, e_1! e_2! \cdots e_l!} \right| \leq C_\xi\,(1+k)^{N-3}.$$

Proof. $|\,\mu\,| = (\sum k_i) - 1 = |\,\Phi^+\,| - 1 = N-1$. Use Lemma 2.12 and Proposition 2.9 (1) with $M = N-1$.

For (2), note that $\mu_1 = k_1 - 1 = k_2 = l - 1 \geq 0$ by Theorem 1.12 (3) and the assumption on the rank of $\mathfrak{g}$. Thus, $\ell(\mu) \geq 2$ and we can apply Proposition 2.9 (2). □

Let

$$P_{\mathfrak{g}}(k) = \begin{cases} \frac{(k/2)^{N-2}}{(N-2)!\, e_1! e_2! \cdots e_l!}, & \text{if } k \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $P_{\mathfrak{g}}(nk) = n^{N-2}\, P_{\mathfrak{g}}(k)$ for all $n > 0$. Heckman defines the *asymptotic branching function* by replacing $\wp_{\mathfrak{g}}$ with its asymptotic counterpart $P_{\mathfrak{g}}$. Define $b_{\mathfrak{g}}$:

$\mathbb{R} \times \mathbb{E} \to \mathbb{R}$ by

$$b_{\mathfrak{g}}(k, \lambda) = \sum_{w \in W_{\mathfrak{g}}} \operatorname{sgn}(w)\, P_{\mathfrak{g}}\left(\overline{w\lambda} - k\right). \tag{2.5}$$

Then

$$b_{\mathfrak{g}}(nk, n\lambda) = n^{N-2} b_{\mathfrak{g}}(k, \lambda).$$

Recall that the principal branching multiplicity is

$$B_{\mathfrak{g}}(k, \lambda) = \sum_{w \in W_{\mathfrak{g}}} \operatorname{sgn}(w)\, \wp_{\mathfrak{g}}\left(L_w\left(k, \lambda\right)\right) \tag{2.6}$$

with $L_w(k, \lambda)$ defined in (1.13). The next theorem specializes [Hec82, Lemma 3.6] to the current situation.

Theorem 2.14. *Let $\mathfrak{g}$ be a simple complex Lie algebra of rank $l \geq 2$. Let N be the number of positive roots of $\mathfrak{g}$ and let $e_1, \ldots, e_l$ be the exponents of $\mathfrak{g}$. There exists a constant $C > 0$ such that for all $\lambda \in P_+(\mathfrak{g})$ and $k \geq 0$ with $\overline{\lambda} - k \in 2\mathbb{Z}$ we have*

$$|\, B_{\mathfrak{g}}(k, \lambda) - b_{\mathfrak{g}}(k, \lambda)\,| \leq C \left(1 + \overline{\lambda}\right)^{N-3}.$$

Proof. By (2.6) and (2.5),

$$\begin{aligned} |\, B_{\mathfrak{g}}(k, \lambda) - b_{\mathfrak{g}}(k, \lambda)\,| &\leq \sum_{w \in W} \left|\, \wp_{\mathfrak{g}}(\overline{w(\lambda + \rho) - \rho} - k) - P_{\mathfrak{g}}(\overline{w\lambda} - k) \,\right| \\ &= \sum_{w \in W} \left|\, \wp_{\mathfrak{g}}(\overline{w\lambda} - k + \overline{w(\rho) - \rho}) - P_{\mathfrak{g}}(\overline{w\lambda} - k) \,\right|. \end{aligned}$$

For each $w \in W$, apply Lemma 2.13 with $\xi = \overline{w(\rho) - \rho}$ to find constants $C_w > 0$ such that

$$| B_{\mathfrak{g}}(k,\lambda) - b_{\mathfrak{g}}(k,\lambda) | \leq \sum_{w\in W} C_w(1 + |\overline{w\lambda} - k|)^{N-3}$$
$$\leq C_0(1 + \overline{\lambda} + k)^{N-3}.$$

where $C_0 > 0$ is constant.

If $\overline{\lambda} - k \geq 0$, then

$$| B_{\mathfrak{g}}(k,\lambda) - b_{\mathfrak{g}}(k,\lambda) | \leq C_0(1 + 2\overline{\lambda})^{N-3} \leq C(1 + \overline{\lambda})^{N-3},$$

for some constant $C > 0$.

If $\overline{\lambda} - k < 0$, $B_{\mathfrak{g}}(k,\lambda) = 0$ so that for $n \geq 0$,

$$| b_{\mathfrak{g}}(k,\lambda) | = \frac{1}{n^{N-2}} | b_{\mathfrak{g}}(nk, n\lambda) | \leq \frac{C'}{n^{N-2}}(1 + n\overline{\lambda} + nk)^{N-3}.$$

As $n \to \infty$, $| b_{\mathfrak{g}}(k,\lambda) | \to 0$. Thus, $b_{\mathfrak{g}}(k,\lambda) = 0$ when $\overline{\lambda} - k < 0$. □

2.3 Principal branching for $\mathfrak{sp}(2,\mathbb{C})$

In this section we describe in detail the principal branching rule in the rank two symplectic case. In particular, we obtain a quadratic estimate of the branching multiplicities which will be useful in Section 5.2.

Let $K = \mathrm{SL}(2,\mathbb{C})$ and let $V = F^3$ be the four-dimensional irreducible representation of K with ordered basis $\{ x_0^3,\, 3\,x_0^2x_1,\, 3\,x_0x_1^2,\, x_1^3 \}$. By 1.2, V has a skew-symmetric non-degenerate K-invariant form Γ, which in terms of the given basis may be taken to be

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

(See [GW98, 5.1.22]). Then $G_\Gamma = \mathrm{Sp}(F^3, \Gamma)$ is the rank two symplectic group and the embedding $\rho_3 : K \hookrightarrow G_\Gamma$ is explicitly given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^3 & 3\,a^2c & 3\,ac^2 & c^3 \\ a^2b & a + 3\,abc & 3\,bc^2 + 2\,c & c^2d \\ ab^2 & 3\,b^2c + 2\,b & 3\,bcd + d & cd^2 \\ b^3 & 3\,b^2d & 3\,bd^2 & d^3 \end{bmatrix},$$

where $ad - bc = 1$. By definition,

$$G_\Gamma = \left\{\, g \in \mathrm{GL}(F^3) \mid \rho_3(g)^t\, J\, \rho_3(g) = J \,\right\}.$$

Let $\mathfrak{g} = \mathrm{Lie}(G_\Gamma) = \mathfrak{sp}(2, \mathbb{C})$ and $\mathfrak{k} = \mathrm{Lie}(K)$. By Corollary 1.11 the image of the differential $d\rho_3 : \mathfrak{k} \longrightarrow \mathfrak{g}$ is a principal TDS of $\mathfrak{g}$.

The exponents of $\mathfrak{g}$ are $\lambda := (3 \geq 1)$ and the conjugate partition is then $\lambda^\sharp = (2, 1, 1)$. Indeed, $\mathfrak{g}$ has 2 simple roots of height 1; 1 positive root with height 2; and the unique highest root has height 3. By (1.11), the partition function $\wp_{\mathfrak{g}}$ for the principal branching multiplicities is determined by the generating function

$$\sum_{k \geq 0} \wp_{\mathfrak{g}}(k)\, q^k = \frac{1 - q^2}{(1 - q^2)^2(1 - q^4)(1 - q^6)} = \frac{1}{(1 - q^2)(1 - q^4)(1 - q^6)}.$$

While we could apply the asymptotic methods of this chapter, the rank is sufficiently small to find an exact formula for $\wp_{\mathfrak{g}}(k)$. In fact,

$$\wp_{\mathfrak{g}}(n) = \begin{cases} \left[\frac{(n+6)^2}{48}\right], & \text{if } n \geq 0 \text{ is even;} \\ 0, & \text{otherwise} \end{cases} \tag{2.7}$$

where $[n]$ denotes the nearest integer to n [Slo].

Let $\mathfrak{h}$ be a Cartan subalgebra for $\mathfrak{g}$, $\Phi := \Phi(\mathfrak{h})$, and let $\mathbb{E}$ be the real span of Φ. There exists a basis $\{\varepsilon_1, \varepsilon_2\}$ for $\mathbb{E}$ such that

$$\Delta := \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \quad \alpha_2 := 2\varepsilon_2\} \tag{2.8}$$

is a base of simple roots for Φ and the vectors

$$\Phi^+ := \{\alpha_1, \alpha_2, \alpha_3 := \alpha_1 + \alpha_2, \quad \alpha_4 := 2\alpha_1 + \alpha_2\}$$

form a set of positive roots for Φ. The Killing form restricted to $\mathbb{E}$ is then the usual inner product $(\ ,\ )$ relative to the basis $\{\varepsilon_1, \varepsilon_2\}$. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = 2\varepsilon_1 + \varepsilon_2$. The dominant weight lattice is given by

$$P_+(\mathfrak{g}) = \mathbb{Z}_{\geq 0}\,\varpi_1 \oplus \mathbb{Z}_{\geq 0}\,\varpi_2,$$

where

$$\varpi_1 = \varepsilon_1 \quad \text{and} \quad \varpi_2 = \varepsilon_1 + \varepsilon_2 \tag{2.9}$$

are the fundamental weights. The finite-dimensional irreducible representations $V_2^\lambda = V^\lambda$ of $\mathfrak{g}$ are parametrized by $\lambda = l\,\varpi_1 + m\,\varpi_2$, $l, m \in \mathbb{Z}_{\geq 0}$.

The Weyl group $W := W_{\mathfrak{g}}$ has order 8 and is generated by the simple reflections

$s_i \in \mathrm{GL}(\mathbb{E})$ corresponding to the simple roots α_i, $i = 1, 2$. Precisely,

$$s_1(a\,\varepsilon_1 + b\,\varepsilon_2) = b\,\varepsilon_1 + a\,\varepsilon_2$$
$$s_2(a\,\varepsilon_1 + b\,\varepsilon_2) = a\,\varepsilon_1 - b\,\varepsilon_2$$

Index the elements of W by the symbols w_i, $i = 1, \ldots, 8$ as follows

$$\begin{array}{lll} w_1 = 1, & w_2 = s_1, & w_3 = s_2, \\ w_4 = s_1 s_2, & w_5 = s_2 s_1, & w_6 = s_1 s_2 s_1, \\ w_7 = s_2 s_1 s_2, & w_8 = s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1. & \end{array}$$

The Bruhat order $<$ on W determined by the simple reflections s_1 and s_2 is depicted in Figure 2.2 by giving the Hasse diagram of the poset $(W, <)$.

Let $\delta \in \mathbb{E}$ such that $(\delta, \alpha_1) = 2 = (\delta, \alpha_2)$. By (2.8)

$$\delta = 3\varepsilon_1 + \varepsilon_2. \tag{2.10}$$

Recall that for $w \in W$, $L_w : \mathbb{R} \times \mathbb{E} \to \mathbb{R}$ is defined by

$$L_w(k, \lambda) = \overline{w(\lambda + \rho) - \rho} - k,$$

where $k \in \mathbb{R}$, $\lambda \in \mathbb{E}$, and $\overline{\lambda} = (\delta, \lambda)$. We prefer to work in the coordinates of the fundamental weights ϖ_1 and ϖ_2. By (2.9) and (2.10),

$$(\delta, \varpi_1) = (\delta, \varepsilon_1) \qquad = (3\varepsilon_1 + \varepsilon_2, \varepsilon_1) = 3$$

and

$$(\delta, \varpi_2) = (\delta, \varepsilon_2) \qquad = (3\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2) = 4.$$

Thus,

$$\overline{l\,\varpi_1 + m\,\varpi_2} = 3\,l + 4\,m. \tag{2.11}$$

To simplify notation, write

$$L_w(k, l, m) := L_w(k, \lambda), \quad \text{where } \lambda = l\,\varpi_1 + m\,\varpi_2.$$

For $i = 1, \ldots, 8$ we abuse notation and write L_i for L_{w_i} when the context requires. For each $w \in W$, the value of $L_w(k, l, m)$ appears next to the corresponding node for w in Figure 2.2.

Definition 2.15. For $k, l, m \geq 0$ define

$$A(k, l, m) = \{\, w \in W \mid L_w(k, l, m) \geq 0 \,\}\,.$$

We set $A(k, l, m) = \emptyset$ when any of k, l, m are negative. A subset $S \subset W$ is called *admissible* if $S = A(k, l, m)$ for some $k, l, m \geq 0$.

Lemma 2.16. *The non-empty admissible subsets of W are in the list*

$$\{\,1\,\}, \{\,1, 2\,\}, \{\,1, 3\,\}, \{\,1, 2, 3\,\}, \{\,1, 2, 3, 4\,\}, \{\,1, 2, 3, 5\,\}, \tag{2.12}$$

where we have identified $w_i \in W$ with its index i.

Proof. $A(k, l, m) = \mathcal{T}_\delta^*(k, \lambda)$ where $\lambda = l\,\varpi_1 + m\,\varpi_2$. Thus an admissible subset is an order ideal of W. By Figure 2.2, we observe that $L_8(k, l, m), L_7(k, l, m)$, and $L_6(k, l, m)$ are strictly negative when $k, l, m \geq 0$ from which we conclude that

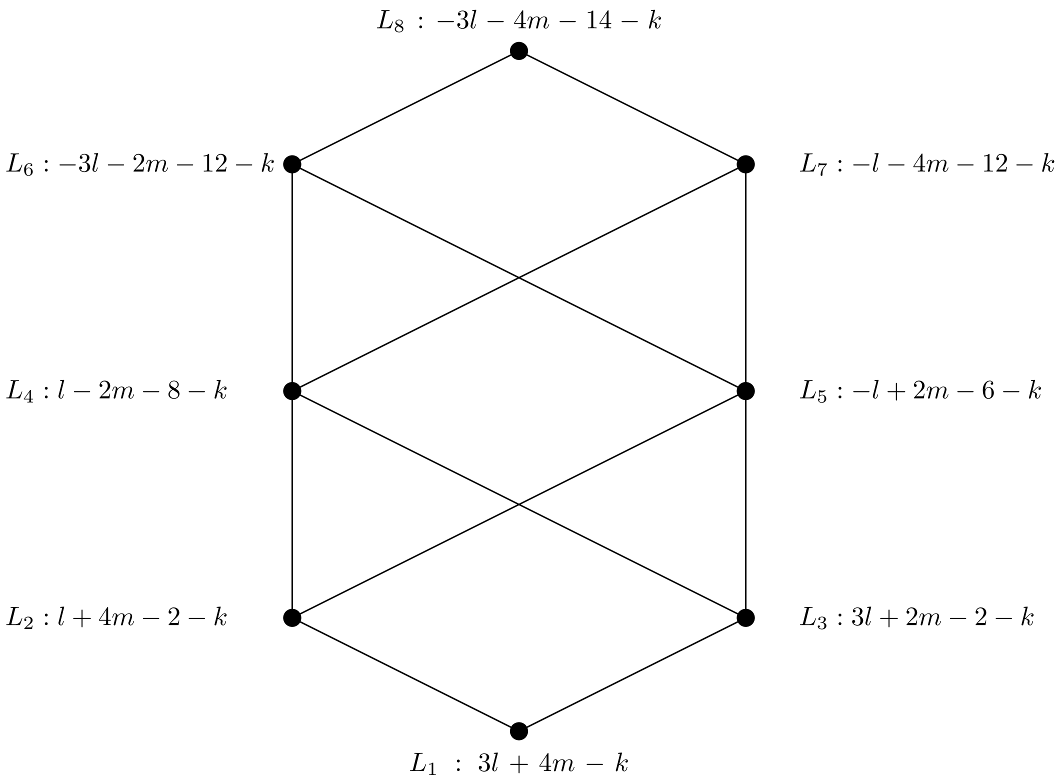

Figure 2.2: Hasse diagram for the Weyl group of $\mathfrak{sp}(2, \mathbb{C})$ with nodes labeled by $L_w(k, l, m)$

w_8, w_7, w_6 never appear in an admissible subset.

It follows by direct calculation that

$$L_w(k, l, m) + L_{w'}(k, l, m) = -2k - 14 < 0$$

whenever $w' = w_8 w$. In particular, w_4 and w_5 never appear in an admissible subset together as $w_4 = w_8 w_5$. Therefore, any admissible subset must be contained in either $M_1 := \{1, 2, 3, 4\}$ or $M_2 := \{1, 2, 3, 5\}$. It is possible to demonstrate $k, l, m \geq 0$ with $A(k, l, m)$ appearing in (2.12) and these are exactly the order ideals contained in M_1 or M_2. $\square$

Denote the principal branching multiplicity of Proposition 1.13 by

$$B(k, l, m) := B_{\mathfrak{g}}(k, (l + m, m)). \tag{2.13}$$

Proposition 2.17. *For integers $k, l, m \geq 0$, $B(k,l,m) = 0$ whenever $k - l \notin 2\mathbb{Z}$;*

Proof. Let $\lambda = l\varpi_1 + m\varpi_2$. By 1.13, $B_{\mathfrak{g}}(k,\lambda) = 0$ when $\overline{\lambda} - k$ is odd. By (2.11), $\overline{\lambda} - k = 3l + 4m - k$ which is even if and only if l and k have matching parity. □

To obtain a strong estimate of the branching multiplicities, we replace $\wp_{\mathfrak{g}}(n)$ with the quadratic $(n+6)^2/48$. For this to be effective, particular attention must be paid to which elements of the Weyl group are relevant in the computation.

Definition 2.18. For $S \subset W$, let

$$Q_S(k,l,m) = \sum_{w \in S} \operatorname{sgn}(w) \, \frac{1}{48} \left(L_w(k,l,m) + 6 \right)^2 . \tag{2.14}$$

Lemma 2.19. *There exists a constant $C > 0$ such that for all integers $k, l, m \geq 0$, $k - l \in 2\mathbb{Z}$,*

$$|\, B(k,l,m) - Q_S(k,l,m) \,| \leq C, \tag{2.15}$$

where $S = A(k,l,m)$.

Proof. Recall that $[\, n \,]$ denotes the nearest integer to n. Then

$$\begin{aligned} |\, B(k,l,m) - Q_S(k,l,m) \,| &\leq \sum_{w \in S} \left| \frac{(L_w(k,l,m)+6)^2}{48} - \left[\frac{(L_w(k,l,m)+6)^2}{48} \right] \right| \\ &\leq \sum_{w \in S} 1 = |\, S \,| \leq |\, W \,| = 8. \end{aligned}$$

□

Remark 2.20. Empirically, we have observed that the error in (2.15) has the following tight bound

$$|\, B(k,l,m) - Q_S(k,l,m) \,| \leq 11/12,$$

for $k, l, m \geq 0$, $k - l \in 2\mathbb{Z}$.

Remark 2.21. Let $U \subset G_\Gamma$ and $N \subset K$ be the naturally chosen maximal unipotent subgroups. The algebra

$$A := \mathbb{C}\,[\,N\backslash G_\Gamma/U\,] := \{\, f \in \mathbb{C}\,[\,G_\Gamma\,] \mid f(ngu) = f(g) \text{ for all } n \in N, g \in G_\Gamma, u \in U \,\}$$

is the appropriate model for understanding the branching rule and was studied extensively by Papageorgiou in [Pap98]. In particular, a minimal system of generators is determined for A is determined. Moreover, the author gives a formula for the branching rule albeit a recursive one.

Chapter 3

Graded multiplicities in Type G

3.1 The symmetric pair $(G_2, \mathfrak{so}_4)$

In this section we review the construction of the Cartan involution which gives rise to the algebraic symmetric pair $(G_2, \mathfrak{so}_4)$. The primary reference is [Kna02, Chapter VI]. Let $\mathfrak{g}$ denote the complex simple Lie algebra of type G_2 and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Set $\Phi := \Phi(\mathfrak{h})$. Then the real span of Φ is a 2-dimensional Euclidean space $\mathbb{E} \subset \mathfrak{h}^*$. Consider $\mathbb{R}^3$ with standard basis $\{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \}$. Then $\mathbb{E}$ may be identified with the plane

$$\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 = 0 \}.$$

Define the following vectors in $\mathbb{E}$:

$$\begin{aligned} \alpha_1 &= \varepsilon_1 - \varepsilon_2, & \alpha_2 &= -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ \alpha_3 &= \alpha_1 + \alpha_2, & \alpha_4 &= 2\alpha_1 + \alpha_2, \\ \alpha_5 &= 3\alpha_1 + \alpha_2, & \alpha_6 &= 3\alpha_1 + 2\alpha_2. \end{aligned} \tag{3.1}$$

Then Φ may be taken as $\Phi^+ \cup (-\Phi^+)$ where $\Phi^+ = \{ \alpha_i \mid 1 \leq i \leq 6 \}$ [Kna02, C.2]. The set $\Delta = \{ \alpha_1, \alpha_2 \}$ is then a base of simple roots for Φ.

The root space decomposition for $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h} \}.$$

Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denote the Killing form of $\mathfrak{g}$. Since κ is non-degenerate on $\mathfrak{h}$ we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$. In particular, for each root $\alpha \in \mathfrak{h}^*$ let $H_\alpha \in \mathfrak{h}$ satisfying

$$\kappa(H, H_\alpha) = \alpha(H).$$

The Killing form determines a non-degenerate bilinear form on $\mathfrak{h}^*$ according to

$$(\alpha, \beta) = \kappa(H_\alpha, H_\beta), \quad \text{for } \alpha, \beta \in \mathfrak{h}^*.$$

The realization (3.1) of the root system is constructed so that $(\, , \,)$ agrees with the usual inner product on $\mathbb{E} \subset \mathbb{R}^3$ relative to the basis $\{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \}$.

For each root $\alpha \in \Phi$, let

$$h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}.$$

To simplify notation write α_{-i} for the negative root $-\alpha_i$, where $1 \leq i \leq 6$. Similarly, set $h_i = h_{\alpha_i}$ and $h_{-i} = h_{-\alpha_i} = -h_i$.

For $i = 1, \ldots, 6$ there exist non-zero root vectors $X_i \in \mathfrak{g}_{\alpha_i}$ and $X_{-i} \in \mathfrak{g}_{-\alpha_i}$ satisfying

(I) $[h_i, h_j] = 0$, for $1 \leq i, j \leq 6$;

(II) $[h_i, X_j] = \alpha_j(h_i) \, X_j$, for $1 \leq i \leq 6$, $1 \leq | j | \leq 6$;

(III) $[X_i, X_{-i}] = h_i$, for $1 \leq i \leq 6$;

(IV) $[\,X_i, X_j\,] = 0$, when $\alpha_i + \alpha_j \notin \Phi$, $j \neq -i$;

(V) $[\,X_i, X_j\,] = N(i,j)\,X_k$, when $\alpha_i + \alpha_j = \alpha_k \in \Phi$.

The structure constants $N(i,j)$ satisfy

(i) $N(i,j) = -N(-i,-j)$;

(ii) $N(i,j) = \epsilon(i,j)(1+p)$ where $\epsilon(i,j) = \pm 1$ and $p \geq 0$ is the maximal integer satisfying $\alpha_j - p\alpha_i \in \Phi$, $i \neq j$.

The choice of signs $\epsilon(i,j)$ is not uniquely determined. By [Sam90, Section 2.8, Prop. A],

(a) $\epsilon(i,j) = -\epsilon(j,i)$, $1 \leq |\,i\,|, |\,j\,| \leq 6$;

(b) $\epsilon(i,j) = \epsilon(j,k) = \epsilon(k,i)$, for pairwise independent roots $\alpha_i, \alpha_j, \alpha_k \in \Phi$ satisfying $\alpha_i + \alpha_j + \alpha_k = 0$

$N(i,j)$	1	2	3	4	5	6
1		1	2	3		
2	−1				1	
3	−2			−3		
4	−3		3			
5		−1				
6						

$N(i,-j)$	1	2	3	4	5	6
1			3	2	1	
2			−1			1
3	−3	1		−2		−1
4	−2		2		−1	1
5	−1			1		−1
6		−1	1	−1	1	

Table 3.1: Structure constants $N(i,j)$ for G_2.

In Table 3.1 the constants $N(i,j)$, $1 \leq i,j \leq 6$ are provided with the same choice of signs as [Đok98]. This determines $N(-i,-j)$, $1 \leq i,j \leq 6$ by (i). To determine the mixed case $N(i,-j)$, $1 \leq i,j \leq 6$ one uses (b). The remaining structure of $\mathfrak{g}$ is encoded by the constants $\alpha_i(h_j)$ which appear in Table 3.2.

The Cartan involutions of $\mathfrak{g}$ are classified by its Vogan diagram. For Type G the diagram appears in Figure 3.1 [Kna02, VI.10] and it determines a unique involution

$\alpha_j(h_i)$	α_1	α_2	α_3	α_4	α_5	α_6
$h_1 = H_L$	2	−3	−1	1	3	0
h_2	−1	2	1	0	−1	1
h_3	−1	3	2	1	0	3
h_4	1	0	1	2	3	3
h_5	1	−1	0	1	2	1
$h_6 = H_R$	0	1	1	1	1	2

Table 3.2: Structure constants $\alpha_j(h_i)$ for G_2.

$\theta : \mathfrak{g} \to \mathfrak{g}$. The "unpainted" node of Figure 3.1 corresponds to the short simple root α_1 while the "painted" node corresponds to the long simple root α_2.

By the proof of [Kna02, Theorem 6.88], $\theta|_{\mathfrak{h}}$ is the identity; $\theta(X_1) = X_1$; and $\theta(X_2) = -X_2$. For $\alpha_i \in \Phi$, $\theta(X_i) = a_i X_i$, for some constants $a_i = \pm 1$. The a_i satisfy $a_{i+j} = a_i a_j$ whenever α_i, α_j and $\alpha_i + \alpha_j$ are roots. By (3.1),

$$\begin{aligned} \theta(X_1) &= X_1, & \theta(X_2) &= -X_2, \\ \theta(X_3) &= -X_3, & \theta(X_4) &= -X_4, \\ \theta(X_5) &= -X_5, & \theta(X_6) &= X_6, \end{aligned} \tag{3.2}$$

and $\theta(X_{-i})$ is similarly determined.

Let $\mathfrak{k}$ (resp. $\mathfrak{p}$) denote the $+1$-eigenspace (resp. -1-eigenspace) of θ.

$$\begin{aligned} \mathfrak{k} &= \{ X \in \mathfrak{g} \mid \theta(X) = X \}, \\ \mathfrak{p} &= \{ X \in \mathfrak{g} \mid \theta(X) = -X \}. \end{aligned} \tag{3.3}$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{p}$ is a representation of $\mathfrak{k}$.

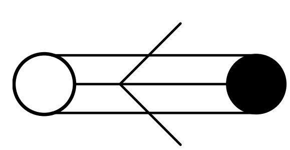

Figure 3.1: Vogan diagram for G_2.

We have

$$\mathfrak{k} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{-\alpha_6} = \mathfrak{k}_L \oplus \mathfrak{k}_R$$

where

$$\mathfrak{k}_L := \mathfrak{g}_{\alpha_1} \oplus \mathbb{C}h_1 \oplus \mathfrak{g}_{-\alpha_1} \cong \mathfrak{sl}(2, \mathbb{C}),$$
$$\mathfrak{k}_R := \mathfrak{g}_{\alpha_6} \oplus \mathbb{C}h_6 \oplus \mathfrak{g}_{-\alpha_6} \cong \mathfrak{sl}(2, \mathbb{C}).$$

Indeed, the roots α_1 and α_6 are orthogonal so that the algebras $\mathfrak{k}_L$ and $\mathfrak{k}_R$ commute and $\mathfrak{k} \cong \mathfrak{k}_L \oplus \mathfrak{k}_R \cong \mathfrak{so}(4, \mathbb{C})$ as Lie algebras. Set

$$\begin{aligned} X_L &= X_1, & X_R &= X_6, \\ H_L &= h_1, & H_R &= h_6, \\ Y_L &= X_{-1}, & Y_R &= X_{-6}. \end{aligned} \tag{3.4}$$

Then (H_L, X_L, Y_L) and (H_R, X_R, Y_R) are standard triples in $\mathfrak{g}$ spanning $\mathfrak{k}_L$ and $\mathfrak{k}_R$, respectively.

By (3.2)

$$\mathfrak{p} = \sum_{i=2}^{5} \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i}. \tag{3.5}$$

The action of $\mathfrak{k}$ on $\mathfrak{p}$ can be organized as a 4×2 matrix with the arrows showing which root space the corresponding operator moves a root vector.

$$\begin{array}{c|cc|c} 0 & \Leftarrow & X_R & 0 \\ \hline & \mathfrak{g}_{\alpha_5} & \mathfrak{g}_{-\alpha_2} & \\ \Uparrow & \mathfrak{g}_{\alpha_4} & \mathfrak{g}_{-\alpha_3} & Y_L \\ X_L & \mathfrak{g}_{\alpha_3} & \mathfrak{g}_{-\alpha_4} & \Downarrow \\ & \mathfrak{g}_{\alpha_2} & \mathfrak{g}_{-\alpha_5} & \\ \hline 0 & Y_R & \Rightarrow & 0 \end{array}$$

The $\mathfrak{k}$-representation $\mathfrak{p}$ can be identified with the representation $d\rho_{3,1} : \mathfrak{k} \to \mathrm{GL}(F^{3,1})$.

The map $\phi : \mathfrak{p} \to F^{3,1}$ sending

$$\begin{aligned} X_5 &\mapsto x_0^3 y_0, & X_{-2} &\mapsto x_0^3 y_1, \\ X_4 &\mapsto 3\,x_0^2 x_1 y_0, & X_{-3} &\mapsto 3\,x_0^2 x_1 y_1, \\ X_3 &\mapsto 3\,x_0 x_1^2 y_0, & X_{-4} &\mapsto 3\,x_0 x_1^2 y_1, \\ X_2 &\mapsto x_1^3 y_0 & X_{-5} &\mapsto x_1^3 y_1 \end{aligned} \tag{3.6}$$

provides the isomorphism of $\mathfrak{k}$-representations. To see this, one compares the action of the operators $d\rho_{3,1}(\mathfrak{k})$ with the structure constants of $\mathfrak{g}$ in Tables 3.1 and 3.2. To illustrate, compare the action of X_L on the root vectors $X_i \in \mathfrak{p}$ with the action of $d\rho_{3,1}(X_L)$ on $\phi(X_i)$.

$$\begin{aligned} [\,X_L, X_5\,] &= 0, & d\rho_{3,1}(X_L)(x_0^3 y_0) &= 0, \\ [\,X_L, X_4\,] &= 3\,X_3, & d\rho_{3,1}(X_L)(3\,x_0^2 x_1 y_0) &= 3\,x_0^3 y_0 = 3\,\phi(X_3), \\ [\,X_L, X_3\,] &= 2\,X_4, & d\rho_{3,1}(X_L)(3\,x_0 x_1^2 y_0) &= 6\,x_0^2 x_1 y_0 = 2\,\phi(X_4), \\ [\,X_L, X_2\,] &= X_3, & d\rho_{3,1}(X_L)(x_1^3 y_0) &= 3\,x_0 x_1^2 y_0 = \phi(X_3). \end{aligned}$$

Let G be the adjoint group of $\mathfrak{g}$ and let K_θ be the subgroup of G whose elements commute with θ

$$K_\theta = \{\, g \in G \mid \mathrm{Ad}(g).\theta(X) = \theta(\mathrm{Ad}(g).X), \text{ for all } X \in \mathfrak{g} \,\}.$$

Then K_θ is the connected Lie group $\mathrm{SO}(4,\mathbb{C})$ [GW98, Section 12.4.3]. Both $\mathfrak{k}$ and $\mathfrak{p}$ are K_θ-invariant. It will be convenient to work with the simply connected cover $K := \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}(4,\mathbb{C})$ of K_θ. We have

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow K \xrightarrow{\pi} K_\theta \longrightarrow 0,$$

where $\mathbb{Z}_2 \cong \langle (\pm I, \pm I) \rangle$.

The next task is to determine a maximal toral subalgebra of $\mathfrak{p}$. By 1.16, these arise as the centralizers of regular semisimple elements of $\mathfrak{p}$. It is known that the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ has real rank equal to two and as such any maximal toral subalgebra is two-dimensional [GW98, 12.4.3].

Lemma 3.1. *Let $Z = X_5 + X_{-5} + \sqrt{-1}\ (X_3 + X_{-3})$. Then Z is a regular semisimple element of $\mathfrak{p}$ and the centralizer $\mathfrak{p}^Z$ is the maximal toral subalgebra of $\mathfrak{p}$ spanned by $X_3 + X_{-3}$ and $X_5 + X_{-5}$.*

Proof. Both $Z_5 := X_5 + X_{-5}$ and $Z_3 := X_3 + X_{-3}$ are semisimple by 1.6. Since the roots α_5 and α_3 are orthogonal $[\,Z_5, Z_3\,] = 0$. In particular, $\mathrm{ad}(Z_3)$ and $\mathrm{ad}(Z_5)$ are simultaneously diagonalizable. Hence, $Z = Z_5 + \sqrt{-1}\, Z_3$ is semisimple in $\mathfrak{g}$. Direct computation in G_2 shows $\mathfrak{p}^Z$ is two-dimensional and spanned by $\{\, Z_5, Z_3 \,\}$. By Lemma 1.15, Z is regular in $\mathfrak{p}$. □

Let $\mathfrak{a} = \mathfrak{p}^Z$ be the maximal toral subalgebra of $\mathfrak{p}$ found in Lemma 3.1. Let M_θ be the centralizer of $\mathfrak{a}$ in K_θ

$$M_\theta = \{\, g \in K_\theta \mid \mathrm{Ad}(g)(X) = X, \text{ for all } X \in \mathfrak{a} \,\}.$$

The next theorem describes the lift of M_θ to $K = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$.

Theorem 3.2. *Let $M = \pi^{-1}(M_\theta)$. Then M is isomorphic to the eight element non-abelian group of basic unit quaternions*

$$\mathbb{H} := \langle \pm 1, \pm i, \pm j, \pm k \,\rangle / \langle\, i^2 = j^2 = k^2 = ijk = -1 \,\rangle.$$

Explicitly, the embedding $\mathbb{H} \hookrightarrow \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ *is determined by*

$$\pm 1 \mapsto \pm \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$\pm i \mapsto \pm \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right)$$

$$\pm j \mapsto \pm \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$

$$\pm k \mapsto \pm \left(\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right)$$

Proof. By Lemma 1.16, M_θ is the stabilizer in K_θ of Z. Hence, M is the stabilizer of Z in K or equivalently the stabilizer of $\phi(Z) \in F^{3,1}$. By (3.6),

$$f := \phi(Z) = x_0^3 y_0 + x_1^3 y_1 + 3\sqrt{-1}\left(x_0 x_1^2 y_0 + x_0^2 x_1 y_1 \right).$$

Let $(g,h) \in K$, $\det g = \det h = 1$. The equation $\rho_{3,1}((g,h))(f) - f = 0$ is solved for the entries in the matrices g and h resulting in the given embedding. □

The algebra $\mathbb{C}[\mathfrak{p}]^K$ of K-invariant polynomial functions on $\mathfrak{p}$ is well-known to be a polynomial algebra generated by two invariants u_2 and u_6 of degree 2 and 6, respectively [GW98, Section 12.4.3]. This fact, however, will be observed in the sequel. By Theorem 1.20 we have

$$\mathbb{C}[\mathfrak{p}] = \mathcal{H} \otimes \mathbb{C}[u_2, u_6], \tag{3.7}$$

where $\mathcal{H}$ is the space of harmonic functions

$$\mathcal{H} = \{\, f \in \mathbb{C}[\,\mathfrak{p}\,] \mid \partial(u_2) f = 0 = \partial(u_6) f \,\} . \tag{3.8}$$

Let $\mathcal{H}_d = \mathcal{H} \cap \mathbb{C}[\,\mathfrak{p}\,]_d$ denote the harmonic functions of degree d. We view $\mathcal{H}_d$ as a representation of K and denote the graded multiplicities defined in (1.16) by

$$f_d(k,l) := \dim_{\mathbb{C}} \operatorname{Hom}_K \left(F^{k,l}, \mathcal{H}_d\right) = \dim_{\mathbb{C}} \operatorname{Hom}_{K_\theta} \left(F^{k,l}, \mathcal{H}_d\right) , \tag{3.9}$$

for integers $d, k, l \geq 0$. In Section 3.2 we give a formula for the graded multiplicities in terms of the branching rule studied in Section 2.3.

The q-multiplicities defined in (1.17) are denoted

$$p_{k,l}(q) := \sum_{d \geq 0} f_d(k,l)\, q^d , \tag{3.10}$$

where q is an indeterminate. In Section 3.3, we give explicit formulae for the q-multiplicities as rational expressions in q.

Since $\mathcal{H}_d$ is first and foremost a representation of $K_\theta \cong \mathrm{SO}(4, \mathbb{C})$ we observe the following fact.

Lemma 3.3. *If $k - l \notin 2\mathbb{Z}$ then $f_d(k,l) = 0$ for all $d \geq 0$. In particular, $p_{k,l}(q) = 0$ when the integers k and l have different parity.*

Proof. The representations of $\mathrm{SO}(4, \mathbb{C})$ which lift to $\mathrm{Spin}(4, \mathbb{C})$ are precisely the $F^{k,l}$ with $k - l \in 2\mathbb{Z}$. □

Let T denote the maximal torus of K consisting of the diagonal matrices

$$T = \left\{ \left(\begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}, \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \right) \mid \text{for } s, t \in \mathbb{C}^\times \right\} .$$

For a representation V of T, the character of V is the rational function $\mathrm{ch}(V) : T \to \mathbb{C}^\times$

$$\mathrm{ch}(V)(s,t) = \sum_{k,l \in \mathbb{Z}} [\dim V(k,l)]\; s^k t^l,$$

where $V(k,l)$ is the weight space

$$V(k,l) = \left\{ v \in V \mid \left(\mathrm{diag}(s,s^{-1}), \mathrm{diag}(t,t^{-1})\right).v = s^k t^l v \right\}.$$

The character of an irreducible K-representation $F^{k,l}$ is denoted $\chi^{k,l} = \mathrm{ch}(F^{k,l})$ and is given by

$$\chi^{k,l} = \chi^{k,l}(s,t) = \frac{(s^{k+1} - s^{-k-1})(t^{l+1} - t^{-l-1})}{(s - s^{-1})(t - t^{-1})}. \tag{3.11}$$

For example, the character of $\mathfrak{p} = F^{3,1}$ is

$$\chi^{3,1} = s^3 t + \frac{s^3}{t} + st + \frac{s}{t} + \frac{t}{s} + \frac{1}{st} + \frac{t}{s^3} + \frac{1}{s^3 t}. \tag{3.12}$$

By (3.12), the non-zero weight spaces of $\mathfrak{p}$ are all one dimensional with characters $s^k t^l$, where $k \in \{-3,-1,1,3\}$ and $l \in \{-1,1\}$. The following is Lemma 1.21 for the pair $(G_2, \mathfrak{so}(4,\mathbb{C}))$.

Lemma 3.4.

$$\sum_{k,l \geq 0} p_{k,l}(q)\, \chi^{k,l} = \frac{(1-q^2)(1-q^6)}{\prod_{\substack{i \in \{-3,-1,1,3\} \\ j \in \{-1,1\}}} (1 - q s^i t^j)}. \tag{3.13}$$

3.2 An alternating formula for graded multiplicities

In this section we use an application of Howe duality to derive an alternating formula for the graded multiplicities in Type G. For convenience write $K = K_L \times K_R$ where K_L and K_R are both isomorphic to $\mathrm{SL}(2,\mathbb{C})$. Fix $n > 0$ and let V be the irreducible representation $F^{2n-1}\widehat{\otimes}F^1$ of K. By 1.2, F^{2n-1} possesses a non-degenerate K_L-invariant skew-symmetric form $\Gamma := \Gamma_{2n-1}$ which determines an embedding of K_L into a symplectic group. We have

$$K = K_L \times K_R \hookrightarrow \mathrm{Sp}_{2n} \times \mathrm{GL}_2, \tag{3.14}$$

where $\mathrm{Sp}_{2n} := \mathrm{Sp}(F^{2n-1}, \Gamma)$ is the rank n symplectic group preserving Γ and $\mathrm{GL}_2 = \mathrm{GL}(2, \mathbb{C})$. Viewed as a representation of $\mathrm{Sp}_{2n} \times \mathrm{GL}_2$ we have $V \cong \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \cong M_{2n\times 2}$ with the groups acting via the matrix multiplication:

$$(g, h).X = gXh \quad \text{for all } g \in \mathrm{Sp}_{2n},\ h \in \mathrm{GL}_2 \text{ and } X \in M_{2n\times 2}.$$

Consider the algebra $\mathbb{C}[\,V\,]$ of polynomial functions on V. By [How95, Theorem 3.8.5.3],

$$\mathbb{C}[\,V\,] \cong \check{\mathcal{H}} \otimes \mathbb{C}[\,V\,]^{\mathrm{Sp}_{2n}}$$

where $\check{\mathcal{H}}$ is the space of *symplectic harmonic functions*, i.e. the polynomials annihilated by constant coefficient differential operators which commute with the action of Sp_{2n}

$$\check{\mathcal{H}} := \left\{ f \in \mathbb{C}[\,V\,] \mid \partial(\theta) f = 0 \text{ for all } \theta \in \mathbb{C}[\,V\,]^{\mathrm{Sp}_{2n}} \right\}. \tag{3.15}$$

The First Fundamental Theorem of Invariant Theory for Sp_{2n} describes the invariants. By [GW98, Theorem 4.2.2],

$$\mathbb{C}[\,V\,]^{\mathrm{Sp}_{2n}} = \mathbb{C}\left[\,\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}\,\right]^{\mathrm{Sp}_{2n}} = \mathbb{C}[\,\xi_2\,]$$

where ξ_2 is (up to scalar) the quadratic invariant determined by the form Γ. Specifically, if $X \in M_{2n\times 2}$, then $\xi_2(X)$ is given by contracting the columns of X via the symplectic form Γ. Under the action of GL_2, ξ_2 transforms according to the determinant. Thus,

$$\check{\mathcal{H}} = \{ f \in \mathbb{C}[V] \mid \partial(\xi_2) f = 0 \}.$$

The space $\check{\mathcal{H}}$ is graded by degree

$$\check{\mathcal{H}} = \bigoplus_{d \geq 0} \check{\mathcal{H}}_d,$$

where $\check{\mathcal{H}}_d := \check{\mathcal{H}} \cap \mathbb{C}[V]_d$. By [How95, 3.8.5.3], each graded component of $\check{\mathcal{H}}$ is multiplicity-free under the action of $\mathrm{Sp}_{2n} \times \mathrm{GL}_2$:

$$\check{\mathcal{H}}_d = \bigoplus_{\substack{\lambda=(\lambda_1 \geq \lambda_2 \geq 0) \\ \lambda_1+\lambda_2=d}} V_n^\lambda \widehat{\otimes} F_2^\lambda. \tag{3.16}$$

Notice, that only the representations λ with $\ell(\lambda) \leq 2$ appear in (3.16).

Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. By restricting to $K = K_L \times K_R \subset \mathrm{Sp}_{2n} \times \mathrm{GL}_2$ we obtain the following lemma.

Lemma 3.5. *The K-representation $F^{k,l}$ occurs in $\check{\mathcal{H}}_d$ with multiplicity $B_{\mathfrak{g}}(k, (l+m, m))$ where $d = l + 2m$*

$$\check{\mathcal{H}}_d = \bigoplus_{\substack{k,l,m \geq 0 \\ d=l+2m}} B_{\mathfrak{g}}(k, (l+m, m))\ F^{k,l}.$$

Proof. Write $\lambda = (\lambda_1 \geq \lambda_2) = (l+m, m)$ with $l, m \geq 0$. If the weight λ occurs in (3.16) then $d = \lambda_1 + \lambda_2 = l + 2m$. Restricting to $K_L \times \mathrm{GL}_2$ we have

$$V_n^\lambda \widehat{\otimes} F_2^l = \bigoplus_{k,l,m \geq 0} B_{\mathfrak{g}}(k, (l+m, m))\ F^k \widehat{\otimes} F_2^\lambda. \tag{3.17}$$

The GL_2-representation F_2^λ restricts to the SL_2-representation with highest weight $\lambda_1 - \lambda_2 = l$. Hence, (3.17) becomes

$$V_n^\lambda \widehat{\otimes} F_2^l = \bigoplus_{k,l,m \geq 0} B_{\mathfrak{g}}(k, (l+m, m))\ F^k \widehat{\otimes} F^l. \tag{3.18}$$

□

To obtain a formula for the graded multiplicities (3.9) we specialize Lemma 3.5 to the case $n = 2$. Then $\mathfrak{g} = \mathfrak{sp}(2, \mathbb{C})$ and $V = M_{4\times 2}$ is in fact the K-representation $\mathfrak{p} \cong F^{3,1}$ in the G_2 Cartan decomposition (3.3). Moreover, the $Sp_4 \times \mathrm{GL}_2$-invariant ξ_2 is the K-invariant u_2. By direct computation, $B_{\mathfrak{g}}(0, (3, 3)) = 1$. Thus, when $l = 0$ and $m = 3$ there is a trivial K_L-representation in degree $d = 2l + m = 6$. This proves the existence of a K-invariant function $u_6 \in \check{\mathcal{H}}_6$.

Theorem 3.6. *Let* $B(k, l, m) := B_{\mathfrak{sp}(2,\mathbb{C})}\left(k, (l + m, m)\right)$*. For integers* $k, l, m \geq 0$

$$f_d(k, l) = B(k, l, m) - B(k, l, m - 3), \tag{3.19}$$

where $d = l + 2m$*.*

Proof. The harmonic polynomials (3.8) are contained in the symplectic harmonics (3.15)

$$\begin{aligned} \mathcal{H} &= \{\, f \in \mathbb{C}\,[\,\mathfrak{p}\,] \mid \partial(u_6)f = 0 = \partial(u_2)f \,\} \\ &= \{\, f \in \check{\mathcal{H}} \mid \partial(u_6)f = 0 \,\}. \end{aligned} \tag{3.20}$$

View $u_6 : \check{\mathcal{H}}_{d-6} \to \check{\mathcal{H}}_d$ and $\partial(u_6) : \check{\mathcal{H}}_d \to \check{\mathcal{H}}_{d-6}$ as linear operators. We obtain an orthogonal decomposition[1]

$$\check{\mathcal{H}}_d = \ker \partial(u_6) \oplus \operatorname{img} u_6.$$

By (3.20), $\ker \partial(u_6) = \mathcal{H}_d$. Therefore,

$$\check{\mathcal{H}}_d = \mathcal{H}_d \oplus \left(u_6 \cdot \check{\mathcal{H}}_{d-6}\right). \tag{3.21}$$

[1] $\langle f, g\rangle := \partial(\overline{g})f$ is a Hermitian inner product on each $\mathcal{H}_d$; the bar denotes complex conjugation. The operators u_6 and $\partial(u_6)$ are adjoint with respect to this form.

By Lemma 3.5, the multiplicity of $F^{k,l}$ in $\check{\mathcal{H}}$ is $B(k,l,m)$ where $d = l + 2m$. Since u_6 is K-invariant, the multiplicity of $F^{k,l}$ in $u_6 \cdot \check{\mathcal{H}}_{d-6}$ is $B(k,l,m')$ where $d-6 = l+2m'$. Thus, $m' = m - 3$. By (3.21),

$$f_d(k,l) = B(k,l,m) - B(k,l,m') = B(k,l,m) - B(k,l,m-3).$$

yielding (3.19). □

3.3 *a priori* formulae for the q-multiplicities

In this section we give explicit formulae for the q-multiplicities in Type G. Recall that these are the polynomials (3.10)

$$p_{k,l}(q) = \sum_{d \geq 0} f_d(k,l)\, q^d, \quad k,l \geq 0,$$

where

$$f_d(k,l) = \dim_{\mathbb{C}} \operatorname{Hom}_K \left(F^{k,l}, \mathcal{H}_d\right).$$

Surprisingly, we do not make use of Theorem 3.6 to achieve this result. In fact, we will do not make any attempt at deriving these formulae, but rather give the answer *a priori* and verify it is correct by comparing the appropriate generating functions. Each q-multiplicity is presented as a rational expression; the general form is summarized by the following theorem.

Theorem 3.7. *For* $k,l \geq 0$,

$$p_{k,l}(q) = \frac{\tilde{p}_{k,l}(q)}{(1-q^2)(1-q^4)} \tag{3.22}$$

where $\tilde{p}_{k,l}(q)$ *is a polynomial divisible by* $(1-q^2)(1-q^4)$. *If* $p_{k,l}(q) \neq 0$ *then* $k-l \in 2\mathbb{Z}$.

Note the parity condition has already been observed. Before describing the numerators $\tilde{p}_{k,l}(q)$ in (3.22), we need to define some auxiliary expressions.

Definition 3.8. Let $\lfloor n \rfloor$ denote the floor of n. For $k, l \in \mathbb{Z}_{\geq 0}$ define

$$A_k(q) = \begin{cases} 1 + q^2 + q^4 + q^6 & \text{if } k \text{ is even} \\ q + 2q^3 + q^5 & \text{if } k \text{ is odd} \end{cases} \tag{3.23}$$

$$B_{k,l}(q) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 - q^2 & \text{if } k \text{ and } l \text{ are both even} \\ q^6(1 - q^2) & \text{if } k \text{ is even and } l \text{ is odd} \end{cases} \tag{3.24}$$

$$L_0(l) = \left\lfloor \frac{l}{6} \right\rfloor + \begin{cases} 0 & \text{if } l = 1 \bmod 6 \\ 1 & \text{otherwise} \end{cases} \tag{3.25}$$

$$L_2(l) = \left\lfloor \frac{l}{6} \right\rfloor + \begin{cases} 0 & \text{if } l = 0 \bmod 6 \\ 1 & \text{if } l = 2, 3 \bmod 6 \\ 2 & \text{otherwise} \end{cases} \tag{3.26}$$

$$L_6(l) = \left\lfloor \frac{l}{6} \right\rfloor + \begin{cases} -1 & \text{if } l = 0, 3 \bmod 6 \\ 1 & \text{if } l = 4 \bmod 6 \\ 0 & \text{otherwise} \end{cases} \tag{3.27}$$

$$L_8(l) = \left\lfloor \frac{l}{6} \right\rfloor + \begin{cases} 1 & \text{if } l = 3, 5 \bmod 6 \\ 0 & \text{otherwise} \end{cases} \tag{3.28}$$

$$K_1(k) = L_8(k) \tag{3.29}$$

$$K_2(k) = \left\lfloor \frac{k}{6} \right\rfloor + \begin{cases} 0 & \text{if } k = 0, 3 \bmod 6 \\ 1 & \text{if } k = 1, 2, 5 \bmod 6 \\ 2 & \text{if } k = 4 \bmod 6 \end{cases} \tag{3.30}$$

Proposition 3.9. *In the notation of Theorem 3.7,*

$$\tilde{p}_{k,k+2l}(q) = q^{k+2l}\left[K_1\,(1-q^8) + K_2\,q^2(1-q^4) + B_{k,l}(q) - q^{2l+4}(1-q^{2k+2})\right] \tag{3.31}$$

$$\tilde{p}_{3k+l,k+l}(q) = q^{k+l}\left[L_0 + L_2\,q^2 + q^4 - L_6\,q^6 - L_8\,q^8 + q^{4k+2l+6} - q^{k+1}A_{k+1}(q)\right] \tag{3.32}$$

$$\tilde{p}_{2k+3l,l}(q) = q^{k+l}\left[A_k(q) - q^{l+1}A_{k+l+1}(q) - q^{k+2}(1-q^{4l+4})\right] \tag{3.33}$$

where $K_i := K_i(k)$ *and* $L_i := L_i(l)$*.*

The proof of Proposition 3.9 and hence Theorem 3.7 is straightforward, but requires a considerable amount of computation which is carried out in the following two sections. The use of a computer algebra system to verify the correctness of the calculation was indispensable. Appendix A contains `Maple` code which parallels the following calculations.

3.3.1 Step 1. The generating function

The purpose of this section is to compute a closed form for the expression

$$\mathcal{W}(X,Y) := \sum_{k,l\geq 0} \frac{\tilde{p}_{k,l}(q)}{(1-q^2)(1-q^4)}\,X^kY^l. \tag{3.34}$$

First, the function $n \mapsto \left\lfloor \frac{n}{6} \right\rfloor$ is determined by the generating function

$$\mathcal{F}(Z) := \sum_{n\geq 0} \left\lfloor \frac{n}{6} \right\rfloor Z^n = \frac{Z^6}{(1-Z)(1-Z^6)}. \tag{3.35}$$

It is now straightforward to determine generating functions for the auxiliary expressions (3.25)-(3.30).

$$\begin{aligned}\mathcal{L}_0(Y) &:= \sum_{l\geq 0} L_0(l)\,Y^l \\ &= \mathcal{F}(Y) + \frac{1+Y^2+Y^3+Y^4+Y^5}{1-Y^6} \qquad (3.36)\\ \mathcal{L}_2(Y) &:= \sum_{l\geq 0} L_2(l)\,Y^l \\ &= \mathcal{F}(Y) + \frac{2Y+Y^2+Y^3+2Y^4+2Y^5}{1-Y^6} \qquad (3.37)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_6(Y) &:= \sum_{l\geq 0} L_6(l)\,Y^l \\ &= \mathcal{F}(Y) + \frac{-1-Y^3+Y^4}{1-Y^6} \qquad (3.38)\\ \mathcal{L}_8(Y) &:= \sum_{l\geq 0} L_8(l)\,Y^l \\ &= \mathcal{F}(Y) + \frac{Y^3+Y^5}{1-Y^6} \qquad (3.39)\end{aligned}$$

$$\begin{aligned}\mathcal{K}_1(X) &:= \sum_{k\geq 0} K_1(k)\,X^k \\ &= \mathcal{L}_8(X) \qquad (3.40)\\ \mathcal{K}_2(X) &:= \sum_{k\geq 0} K_2(k)\,X^k \\ &= \mathcal{F}(X) + \frac{X+X^2+2X^4+X^5}{1-X^6} \qquad (3.41)\end{aligned}$$

Next we determine generating functions for the various forms of $A_k(q)$ and $B_{k,l}(q)$ that appear in Proposition 3.9. Since they all heavily depend on various parity conditions it is useful to define the following intermediate expression to aid in programming

$$\mathcal{D}(a,\, b,\, c,\, d\,;\, X, Y) := \frac{a + bY + cX + dXY}{(1 - X^2)(1 - Y^2)}. \tag{3.42}$$

The coefficient of X^kY^l in $\mathcal{D}(a, b, c, d; X, Y)$ can be viewed as the (k, l)-entry in the table below where the uppermost left corner is the $(0, 0)$-entry.

a	c	a	c	a
b	d	b	d	b
a	c	a	c	a
b	d	b	d	b

In particular,

$$\mathcal{D}(a,\, b,\, 0,\, 0;\, X, 0) = \sum_{\substack{k \geq 0 \\ k \text{ even}}} a\, X^k + \sum_{\substack{k \geq 0 \\ k \text{ odd}}} b\, X^k.$$

In terms of $\mathcal{D}$, we have

$$\begin{aligned} \mathcal{A}_1(X) &:= \sum_{k \geq 0} A_{k+1}(q)\, X^k \\ &= \mathcal{D}(q + 2q^3 + q^5,\, 1 + q^2 + q^4 + q^6,\, 0,\, 0;\, X, 0) \end{aligned} \tag{3.43}$$

$$\begin{aligned} \mathcal{A}_2(X) &:= \sum_{k \geq 0} A_k(q)\, X^k \\ &= \mathcal{D}(1 + q^2 + q^4 + q^6,\, q + 2q^3 + q^5,\, 0,\, 0;\, X, 0) \end{aligned} \tag{3.44}$$

$$\begin{aligned} \mathcal{A}_3(X) &:= \sum_{k,l \geq 0} A_{k+l+1}(q)\, X^kY^l \\ &= \mathcal{D}(q + 2q^3 + q^5,\, 1 + q^2 + q^4 + q^6,\, 1 + q^2 + q^4 + q^6,\, q + 2q^3 + q^5;\, X, Y) \end{aligned} \tag{3.45}$$

$$\begin{aligned} \mathcal{B}(X, Y) &:= \sum_{k,l \geq 0} B_{k,l}(q)\, X^kY^l \\ &= \mathcal{D}(1 - q^2,\, 0,\, q^6(1 - q^2),\, 0;\, X, Y). \end{aligned} \tag{3.46}$$

We can now find generating functions for the three cases defined in Proposition 3.9. Define

$$\mathcal{P}_1(X,Y) := \sum_{k,l\geq 0} \tilde{p}_{k,k+2l}(q)X^kY^l$$
$$\mathcal{P}_2(X,Y) := \sum_{k,l\geq 0} \tilde{p}_{3k+l,k+l}(q)X^kY^l$$
$$\mathcal{P}_3(X,Y) := \sum_{k,l\geq 0} \tilde{p}_{2k+3l,l}(q)X^kY^l.$$

To compute $\mathcal{P}_1$ we first expand (3.31) and isolate any factors that are dependent on k and l:

$$\begin{aligned}\tilde{p}_{k,k+2l}(q) &= (1-q^8)q^{k+2l}K_1(k)\\ &+ q^2(1-q^4)q^{k+2l}K_2(k)\\ &+ q^{k+2l}B_{k,l}(q)\\ &- q^4q^{k+4l}\\ &+ q^6q^{3k+4l}.\end{aligned}$$

Next, sum these coefficients for X^kY^l, $k,l\geq 0$.

$$\begin{aligned}\mathcal{P}_1(X,Y) &= (1-q^8)\frac{\mathcal{K}_1(qX)}{1-q^2Y}\\ &+ q^2(1-q^4)\frac{\mathcal{K}_2(qX)}{1-q^2Y}\\ &+ \mathcal{B}(qX,q^2Y)\\ &- q^4\frac{1}{(1-qX)(1-q^4Y)}\\ &+ q^6\frac{1}{(1-q^3X)(1-q^4Y)}.\end{aligned} \tag{3.47}$$

Similarly, from (3.32) and (3.33)

$$\begin{aligned}\mathcal{P}_2(X,Y) = \frac{\mathcal{L}_0(qY)}{1-qX} + q^2\frac{\mathcal{L}_2(qY)}{1-qX} + \frac{q^4}{(1-qY)(1-qX)} - q^6\frac{\mathcal{L}_6(qY)}{1-qX} \\ - q^8\frac{\mathcal{L}_8(qY)}{1-qX} + \frac{q^6}{(1-q^5X)(1-q^3Y)} - \frac{q\mathcal{A}_1(q^2X)}{1-qY},\end{aligned} \tag{3.48}$$

and

$$\mathcal{P}_3(X,Y) = \frac{\mathcal{A}_2(qX)}{1-qY} - q\,\mathcal{A}_3(qX,q^2Y) - \frac{q^2}{(1-q^2X)(1-qY)} + \frac{q^6}{(1-q^2X)}(1-q^5Y). \tag{3.49}$$

The equations (3.31)-(3.33) defining the $\tilde{p}_{k,l}$ overlap on the two "diagonals" (k,k) and $(3k,k)$.

Lemma 3.10 (Diagonals). *1. Equation* (3.31) *with* $l=0$ *agrees with* (3.32) *with* $k=0$.

2. Equation (3.32) *with* $l=0$ *agrees with* (3.33) *with* $k=0$.

Proof. It suffices to check this at the level of generating functions, i.e. show that $\mathcal{P}_1(Z,0) = \mathcal{P}_2(0,Z)$ and $\mathcal{P}_2(Z,0) = \mathcal{P}_3(0,Z)$. This is performed in Appendix A. □

Define

$$\Delta_{1,1}(Z) = \sum_{k\geq 0} \tilde{p}_{k,k}(q)\,Z^k = \mathcal{P}_2(0,Z) = \mathcal{P}_1(Z,0), \tag{3.50}$$

and

$$\Delta_{3,1}(Z) = \sum_{k\geq 0} \tilde{p}_{3k,k}(q)\,Z^k = \mathcal{P}_3(0,Z) = \mathcal{P}_2(Z,0). \tag{3.51}$$

By Lemma 3.10, both expressions are well-defined. We can now close the expression in (3.34) by making the appropriate substitutions into equations (3.47)-(3.51) and taking into account the Inclusion-Exclusion Principle for the overlapping diagonals.

Lemma 3.11. *The closed form for* (3.34) *is*

$$(1-q^2)(1-q^4)\mathcal{W}(X,Y) = \mathcal{P}_1(XY,Y^2) + \mathcal{P}_2(X^3Y,XY) + \mathcal{P}_3(X^2,X^3Y) \\ - \Delta_{1,1}(XY) - \Delta_{3,1}(X^3Y). \tag{3.52}$$

3.3.2 Step 2: Averaging over the Weyl group

To complete the proof of Proposition 3.9, we need to compare (3.52) and (3.13) after the appropriate algebraic modifications. Recall that

$$\sum_{k,l\geq 0} p_{k,l}(q)\,\chi^{k,l} = \frac{(1-q^2)(1-q^6)}{\prod_{\substack{i\in\{-3,-1,1,3\}\\ j\in\{-1,1\}}}(1-qs^it^j)}.$$

By (3.11),

$$\sum_{w_1,w_2\in\{\pm 1\}}(-1)^{w_1w_2}\sum_{k,l\geq 0} p_{k,l}(q)\,s^{w_1(k+1)}t^{w_2(l+1)} = \frac{(s-s^{-1})(t-t^{-1})\,(1-q^2)(1-q^6)}{\prod_{\substack{i\in\{-3,-1,1,3\}\\ j\in\{-1,1\}}}(1-qs^it^j)}.$$

We need to average $\mathcal{W}(X,Y)$ over the Weyl group $\mathbb{Z}_2\times\mathbb{Z}_2 \cong \{(\pm 1,\pm 1)\}$ of K in a similar fashion and make the needed substitutions. The following lemma concludes the proof of Proposition 3.9 and Theorem 3.7.

Lemma 3.12.

$$\sum_{w_1,w_2\in\{\pm 1\}}(-s)^{w_1}(-t)^{w_2}\,\mathcal{W}(s^{w_1},t^{w_2}) = \frac{(s-s^{-1})(t-t^{-1})\,(1-q^2)(1-q^6)}{\prod_{\substack{i\in\{-3,-1,1,3\}\\ j\in\{-1,1\}}}(1-qs^it^j)}. \tag{3.53}$$

Proof. The left-hand side is computed using Lemma 3.11. This is shown to agree with the right-hand side in Appendix A. □

3.4 Hilbert series for $\mathbb{C}\left[\, G_2 \,\right]^{SO(4,\mathbb{C})}$

We now have obtained a generating function for the q-multiplicities

$$\mathcal{W}(X,Y) = \sum_{k,l\geq 0} p_{k,l}(q) X^k Y^l \tag{3.54}$$

which can be closed to a rational expression. We derive some important combinatorial data from (3.54).

The first observation will be useful in Chapter 4. Let U be the subgroup of upper triangular unipotent matrices in K

$$U = \left\{ \left(\begin{bmatrix} 1 & u_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & u_2 \\ 0 & 1 \end{bmatrix} \right) \mid u_1, u_2 \in \mathbb{C} \right\}. \tag{3.55}$$

Proposition 3.13.

$$\mathrm{ch}_q \left(\mathbb{C}\left[\, \mathfrak{p} \,\right]^U \right)(s,t) = \frac{\mathcal{W}(s,t)}{(1-q^2)(1-q^6)}. \tag{3.56}$$

Proof. By highest-weight theory $\left(F^{k,l}\right)^U$ is one dimensional with character $s^k t^l$. Combine (3.54) with separation of variables (3.7). □

To obtain the Hilbert series of $\mathbb{C}\left[\, G_2 \,\right]^{SO(4,\mathbb{C})}$ we first need to appeal to the classical situation of spherical harmonics.

Theorem 3.14 (Classical spherical harmonics)**.** *Let* $K_0 = \mathrm{SL}(2,\mathbb{C})$ *and let* $\mathfrak{k}_0 = \mathrm{Lie}(K_0)$. *Then*

$$\mathbb{C}\left[\, \mathfrak{k}_0 \,\right] = \mathbb{C}\left[\, \mathfrak{k}_0 \,\right]^{K_0} \otimes \mathcal{H}^0,$$

where $\mathcal{H}^0 = \oplus_{d\geq 0} \mathcal{H}^0_d$ *is a graded* K_0*-representation. Moreover,* $\mathbb{C}\left[\, \mathfrak{k}_0 \,\right]^{K_0}$ *is a polynomial ring generated by a quadratic invariant and the q-character of* $\mathcal{H}^0$ *as a represen-*

tation of K_0 is given by

$$\mathrm{ch}_q\, \mathcal{H}^0 = \sum_{d\geq 0} q^d\, \chi^{2d}.$$

Proof. Note that K_0 is locally isomorphic to $\mathrm{SO}(3,\mathbb{C})$ and $\mathfrak{k}_0 \cong \mathfrak{so}(3,\mathbb{C})$. The adjoint representation of $\mathfrak{k}_0$ is then the standard representation $V = \mathbb{C}^3$ of $\mathrm{SO}(3,\mathbb{C})$. Let $\{ x_1, x_2, x_3 \}$ be a basis for V^*. Then $r^2 := x_1^2 + x_2^2 + x_3^2 \in \mathbb{C}[V]$ is $\mathrm{SO}(3,\mathbb{C})$-invariant. The space of spherical harmonic functions is defined as

$$\mathcal{H}^0 = \{ f \in \mathbb{C}[V] \mid \Delta(f) = 0 \},$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the usual Laplacian. By [Vin89, Section 9.4],

$$\mathbb{C}[V] \cong \mathbb{C}\left[r^2\right] \otimes \mathcal{H}^0,$$

where $\mathcal{H}^0_d := \mathcal{H}^0 \cap \mathbb{C}[V]_d$ is the irreducible $\mathrm{SO}(3,\mathbb{C})$-representation of dimension $2d+1$. Therefore, in terms of K_0-representations, $\mathrm{ch}\, \mathcal{H}^0_d = \chi^{2d}$. □

Corollary 3.15.

$$C[\mathfrak{k}] \cong C[\mathfrak{k}]^K \otimes \mathcal{H}(\mathfrak{k})$$

where $\mathcal{H}(\mathfrak{k})$ is a graded K-representation. Moreover, $\mathbb{C}[\mathfrak{k}]^K$ is a polynomial ring minimally generated by two quadratic invariants and

$$\mathrm{ch}_q\, \mathcal{H}(\mathfrak{k}) = \sum_{k,l\geq 0} q^{k+l}\, \chi^{2k,2l}. \tag{3.57}$$

Proof. We have $\mathfrak{k} = \mathfrak{k}_L \oplus \mathfrak{k}_R$ and apply 3.14 to each factor of $\mathfrak{sl}(2,\mathbb{C})$. The corresponding q-characters can then be multiplied since $\chi^{k,0} \cdot \chi^{0,l} = \chi^{k,l}$, for all $k, l \geq 0$. □

Lemma 3.16.

$$\sum_{k,l\geq 0} q^{k+l}\, p_{2k,2l}(q) = \frac{\mathcal{W}\left(\sqrt{q},\sqrt{q}\right)+\mathcal{W}\left(-\sqrt{q},\sqrt{q}\right)}{2} \tag{3.58}$$

Proof. By 3.7, $p_{2k,2l+1}(q)=0$ for all $k,l\geq 0$. Hence,

$$\begin{aligned}\mathcal{W}\left(\sqrt{q},\sqrt{q}\right)+\mathcal{W}\left(-\sqrt{q},\sqrt{q}\right) &= 2\sum_{k,l\geq 0} p_{2k,l}(q)\left(\sqrt{q}\right)^{2k}\left(\sqrt{q}\right)^{l}\\ &= 2\sum_{k,l\geq 0} p_{2k,2l}(q)\left(\sqrt{q}\right)^{2k+2l}\\ &= 2\sum_{k,l\geq 0} p_{2k,2l}(q)\, q^{k+l}.\end{aligned}$$

□

Theorem 3.17. *Let $\mathfrak{g}$ be the complex Lie algebra G_2. Then*

$$\mathrm{Hilb}\left(\mathbb{C}\left[\,\mathfrak{g}\,\right]^{\mathrm{SO}(4,\mathbb{C})}\right)=\frac{h(q)}{\left(1-q^2\right)^2\left(1-q^4\right)^2\left(1-q^6\right)^2\left(1-q^9\right)\left(1-q^{10}\right)}, \tag{3.59}$$

where $h(q)$ is the palindromic polynomial

$$\begin{aligned}h(q) &= q^{29}+q^{27}+q^{25}+4q^{23}+q^{22}+5q^{21}+5q^{20}+5q^{19}+7q^{18}+9q^{17}+8q^{16}\\ &+10q^{15}+10q^{14}+8q^{13}+9q^{12}+7q^{11}+5q^{10}+5q^{9}+5q^{8}+q^{7}+4q^{6}+q^{4}+q^{2}+1.\end{aligned}$$

In particular, $\mathbb{C}\left[\,\mathfrak{g}\,\right]^{\mathrm{SO}(4,\mathbb{C})}$ is Gorenstein.

Proof. We decompose under K. By Theorem 1.20 and Corollary 3.15,

$$\mathbb{C}\left[\,\mathfrak{g}\,\right]^{K}=\mathbb{C}\left[\,\mathfrak{k}\oplus\mathfrak{p}\,\right]^{K}=\mathbb{C}\left[\,\mathfrak{k}\,\right]^{K}\otimes\mathbb{C}\left[\,\mathfrak{p}\,\right]^{K}\otimes\left(\mathcal{H}\left(\mathfrak{k}\right)\otimes\mathcal{H}\left(\mathfrak{p}\right)\right)^{K}. \tag{3.60}$$

By 3.10 and 3.57,

$$\begin{aligned}
\operatorname{ch}_q \mathcal{H}(\mathfrak{k}) \otimes \mathcal{H}(\mathfrak{p}) &= \operatorname{ch}_q \mathcal{H}(\mathfrak{k}) \cdot \operatorname{ch}_q \mathcal{H}(\mathfrak{p}) \\
&= \sum_{m,p\geq 0} q^{m+p}\, \chi^{2m,2p} \cdot \sum_{k,l\geq 0} p_{k,l}(q)\, \chi^{k,l} \\
&= \sum_{m,p,k,l\geq 0} q^{m+p} p_{k,l}(q) \sum_{\substack{0\leq s\leq \min\{2m,k\} \\ 0\leq t\leq \min\{2p,l\}}} \chi^{2m+k-2s,2p+l-2t}
\end{aligned}$$

where the last equality is the Clebsch-Gordan rule for tensor products of K-representations. In particular, the character $\chi^{0,0}$ appears if and only if $\min\{2m,k\} = (2m+k)/2$ and $\min\{2p,l\} = (2p+l/2)$; if and only if $k = 2m$ and $l = 2p$. Thus,

$$\operatorname{Hilb}\left(\left(\mathcal{H}(\mathfrak{k}) \otimes \mathcal{H}(\mathfrak{p})\right)^K \right) = \sum_{k,l\geq 0} q^{k+l}\, p_{2k,2l}(q).$$

Combining Lemma 3.16 with the q-characters for the remaining terms in (3.60) we obtain

$$\operatorname{Hilb}\left(\mathbb{C}[\mathfrak{g}]^K \right) = \frac{1}{(1-q^2)^2} \cdot \frac{1}{(1-q^2)(1-q^6)} \cdot \frac{\mathcal{W}\left(\sqrt{q}, \sqrt{q}\right) + \mathcal{W}\left(-\sqrt{q}, \sqrt{q}\right)}{2}.$$

After some minor algebraic manipulation the resulting expression is equal to (3.59).

By a theorem of Hochster and Roberts, $\mathbb{C}[\mathfrak{g}]^{\mathrm{SO}(4,\mathbb{C})}$ is Cohen-Macaulay [Eis95, p.467]. Since $h(q)$ is palindromic, $\mathbb{C}[\mathfrak{g}]^{\mathrm{SO}(4,\mathbb{C})}$ is Gorenstein by a result of Stanley [Eis95, Exercise 21.19]. □

Chapter 4

Invariant theory

4.1 Covariants of double binary forms

In this section we review the theory of covariants for double binary forms. See [Olv99, Ch. 10] for a classical treatment although our treatment is inspired by [Dol03, Ch. 5]. For a vector space V, let $\mathcal{P}(V)$ be the space of polynomial functions on V. Then $\mathcal{P}(V) = \oplus_{d\geq 0}\mathcal{P}_d(V)$ is graded by degree.

Let $K = \mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$. The irreducible K-representation $F^{m,p} = S^m\mathbb{C}^2\widehat{\otimes}S^p\mathbb{C}^2$ is viewed as the space of binary (m,p)-forms of degree m in the variables x_0, x_1 and degree p in the variables y_0, and y_1. A typical $f \in F^{m,p}$ is of the form

$$f = f(x_0, x_1, y_0, y_1) = \sum_{i=0}^{m}\sum_{j=0}^{p}\binom{m}{i}\binom{p}{j}a_{i,j}\, x_0^{m-i}x_1^i\, y_0^{p-j}y_1^j \tag{4.1}$$

with $a_{i,j} \in \mathbb{C}$.

Definition 4.1. A *covariant of $F^{m,p}$ of degree d and order (k,l)* is a K-equivariant map $J : F^{m,p} \to F^{k,l}$ given by homogeneous polynomial coordinates of degree d. In terms of coordinates,

$$J(f) = \sum_{s=0}^{k}\sum_{t=0}^{l}\binom{k}{s}\binom{l}{t}b_{s,t}\, x_0^{k-s}x_1^s\, y_0^{l-t}y_1^t \tag{4.2}$$

where f is as in (4.1) and each $b_{s,t}$ is a homogeneous polynomial of degree d in the coordinates $a_{i,j}$ of f.

Equivalently, a covariant $J : F^{m,p} \to F^{k,l}$ of degree d and order (k,l) is an element of the space

$$\mathrm{Cov}(F^{m,p})(d;k,l) := \left[\mathcal{P}_d\left(F^{m,p}\right) \otimes F^{k,l}\right]^K \tag{4.3}$$

where $f \otimes w \in \mathcal{P}_d\left(F^{m,p}\right) \otimes F^{k,l}$ is identified with $J : v \mapsto f(v) \cdot w$, for $v \in F^{m,p}$. A covariant of order $(0,0)$ is an *invariant* of $F^{m,p}$.

Let $W = (F^{1,0} \oplus F^{0,1})^*$. The polynomial functions on W provides a model for all finite-dimensional irreducible representations of K. For $d \geq 0$,

$$\begin{aligned}
\mathcal{P}_d(W) &\cong S^d(F^{1,0} \oplus F^{0,1}) \\
&\cong \bigoplus_{k+l=d} S^k(\mathbb{C}^2) \mathbin{\widehat{\otimes}} S^l(\mathbb{C}^2) \\
&\cong \bigoplus_{k+l=d} F^{k,l}.
\end{aligned}$$

Thus $\mathcal{P}(W) \cong \oplus_{k,l\geq 0} F^{k,l}$.

Definition 4.2. The *algebra of covariants of* $F^{m,p}$ is the algebra

$$\mathrm{Cov}(F^{m,p}) := \mathcal{P}\left(F^{m,p} \times W\right)^K \cong \left[\mathcal{P}\left(F^{m,p}\right) \otimes \mathcal{P}\left(W\right)\right]^K. \tag{4.4}$$

Consider $D \geq 0$,

$$\begin{aligned}
\mathcal{P}_D(F^{m,p} \times W)^K &\cong \bigoplus_{d+e=D} \bigoplus_{e=k+l} \left[\mathcal{P}_d(F^{m,p}) \otimes F^{k,l}\right]^K \\
&\cong \bigoplus_{d+e=D} \bigoplus_{e=k+l} \mathrm{Cov}(F^{m,p})(d;k,l),
\end{aligned}$$

so that $\mathrm{Cov}(F^{m,p})$ is a triply-graded algebra whose homogeneous components are $\mathrm{Cov}(F^{k,l})(d;k,l)$ for $d,k,l \geq 0$.

Let U be the subgroup of K of upper triangular unipotent matrices (3.55).

Definition 4.3. The algebra $\mathcal{P}(F^{m,p})^U$ is called the algebra of *semiinvariants of* $F^{m,p}$.

Lemma 4.4. *There is an isomorphism of triply graded algebras*

$$\mathrm{Cov}(F^{m,p}) \cong \mathcal{P}(F^{m,p})^U. \tag{4.5}$$

Proof. Fix a non-zero U-invariant vector $w_0 \in W$. Consider the map

$$\Psi : \mathcal{P}(F^{m,p}) \otimes \mathcal{P}(W) \to \mathcal{P}(F^{m,p})$$

sending $f \otimes g \mapsto g(w_0)f$, where $f \in \mathcal{P}(F^{m,p})$ and $g \in \mathcal{P}(W)$. Restriction to $\mathrm{Cov}(F^{m,p})$ gives the isomorphism. □

We emphasize that a covariant is a certain kind of function between representations of K. As such, it should be evaluated at a binary form which we typically denote by f as in (4.1). We make a slight abuse of notation in the case of a degree 1 covariant of $F^{m,p}$ and order (m,p). Such a covariant is unique up to scalar being a multiple of the identity $F^{m,p} \to F^{m,p}$ which we denote by the generic form f itself.

It is possible to combine covariants via *transvection*. Define the differential operators

$$\Omega_L = \frac{\partial^2}{\partial u_1 \partial x_0} - \frac{\partial^2}{\partial x_1 \partial u_0}$$

and

$$\Omega_R = \frac{\partial^2}{\partial v_1 \partial y_0} - \frac{\partial^2}{\partial y_1 \partial v_0}.$$

Let $J_1 \in \mathrm{Cov}(F^{m,p})(d_1; k_1, l_1)$ and $J_2 \in \mathrm{Cov}(F^{m,p})(d_2; k_2, l_2)$. The (s,t)-*transvectant* *of* J_1 *and* J_2 is the covariant $(J_1, J_2)^{s,t}$ given by

$$(J_1, J_2)^{s,t}(f) = \Omega_L^s \, \Omega_R^t \, [J_1(f(x_0, x_1, y_0, y_1)) \cdot J_2(f(u_0, u_1, v_0, v_1))]\Big|_{\substack{u_0=x_0, u_1=x_1 \\ v_0=y_0, v_1=y_1}}$$

Then $(J_1, J_2)^{s,t}$ is a covariant of degree $d_1 + d_2$ and order $(k_1 + k_2 - 2s, l_1 + l_2 - 2t)$.

A famous result of Gordan says that $\mathrm{Cov}(F^{m,p})$ is generated as an algebra by a finite number of iterated transvectants of the base form f. In particular, $\mathrm{Cov}(F^{m,p})$ is finitely generated as an algebra by *homogeneous* elements. A generating set has been determined in the cases $\mathrm{Cov}(F^{1,1})$, $\mathrm{Cov}(F^{2,1})$, $\mathrm{Cov}(F^{2,2})$ by Peano [Pea82],[Tur22]. The cases $\mathrm{Cov}(F^{3,1})$ and $\mathrm{Cov}(F^{4,1})$ were treated by J.A. Todd [Tod46a],[Tod46b].

The generators for the $(3, 1)$ case are given in Table 4.1. Each covariant is denoted by the symbol $J_{d,k,l}$ identifying it as a covariant of degree d and order (k, l). Luckily, the order and degree of each generator is a unique triple (d, k, l) and no confusion can arise. The base form is also denoted $f := J_{1,3,1} : F^{3,1} \to F^{3,1}$. In some cases we have introduced a constant coefficient for computational purposes.

Degree				
1	$J_{1,3,1} := f$			
2	$J_{2,4,0} := \frac{1}{18}\left(f, f\right)^{1,1}$	$J_{2,2,2} := \frac{1}{72}\left(f, f\right)^{2,0}$	$J_{2,0,0} := \frac{1}{72}\left(f, f\right)^{3,1}$	
3	$J_{3,5,1} := \frac{1}{6}\left(f, J_{2,4,0}\right)^{1,0}$	$J_{3,3,3} := \frac{1}{3}\left(f, J_{2,2,2}\right)^{1,0}$	$J_{3,1,1} := \frac{1}{216}\left(f, J_{2,4,0}\right)^{3,0}$	
4	$J_{4,4,0} := \left(f, J_{3,1,1}\right)^{0,1}$	$J_{4,2,2} := \left(f, J_{3,1,1}\right)^{1,0}$	$J_{4,0,4} := \frac{1}{2}\left(J_{2,2,2}, J_{2,2,2}\right)^{2,0}$	
5	$J_{5,3,1} := \left(J_{2,2,2}, J_{3,1,1}\right)^{0,1}$	$J_{5,1,3} := \left(J_{2,2,2}, J_{3,1,1}\right)^{1,0}$		
6	$J_{6,6,0} := \left(J_{3,5,1}, J_{3,1,1}\right)^{0,1}$	$J_{6,4,2} := \left(J_{3,3,3}, J_{3,1,1}\right)^{0,1}$	$J_{6,2,4} := \left(J_{3,3,3}, J_{3,1,1}\right)^{1,0}$	$J_{6,0,0} := \frac{1}{2}\left(J_{3,1,1}, J_{3,1,1}\right)^{1,1}$
7	$J_{7,1,3} := \left(f, {J_{3,1,1}}^2\right)^{2,0}$			
8	$J_{8,0,4} := \left(J_{2,2,2}, {J_{3,1,1}}^2\right)^{2,0}$			
9	$J_{9,1,5} := \left(J_{3,3,3}, {J_{3,1,1}}^2\right)^{2,0}$			
12	$J_{12,0,6} := \left(J_{3,3,3}, {J_{3,1,1}}^3\right)^{3,0}$			

Table 4.1: Todd's generators for $\mathrm{Cov}(F^{3,1})$.

4.2 Syzygies for the binary $(3, 1)$-forms

Fix $m, p \geq 0$. Suppose we know a finite generating set

$$\mathcal{G} = \{\, J_i \in \mathrm{Cov}(F^{m,p})(d_i; k_i, l_i) \mid i = 1, \ldots, r \,\}$$

of homogeneous covariants for $\mathrm{Cov}(F^{m,p})$. Let

$$S_{\mathcal{G}} = \mathbb{C}\,[\, X_1, \ldots, X_r \,]$$

be the triply-graded polynomial algebra with the degree[1] of X_i equal to (d_i, k_i, l_i), $i = 1, \ldots, r$. Write $S_{\mathcal{G}}(d, k, l)$ for the homogeneous elements of $S_{\mathcal{G}}$ of degree (d, k, l). There exists a presentation for $\mathrm{Cov}(F^{m,p})$

$$0 \to I_{\mathcal{G}} \to S_{\mathcal{G}} \xrightarrow{\pi} \mathrm{Cov}(F^{m,p}) \to 0, \tag{4.6}$$

such that $\pi(X_i) = J_i$ for $i = 1, \ldots, r$ and $I_{\mathcal{G}}$ is a finitely generated homogeneous ideal of $S_{\mathcal{G}}$. Write $I_{\mathcal{G}}(d, k, l) = S_{\mathcal{G}}(d, k, l) \cap I_{\mathcal{G}}$. We call an element of $I_{\mathcal{G}}$ a *syzygy*[2] of $F^{m,p}$.

For a triply graded vector space $W = \oplus_{d,k,l \geq 0} W_{d,k,l}$ we denote the multivariate Hilbert series of W by

$$H(W)(q, s, t) = \sum_{d,k,l \geq 0} [\, \dim_{\mathbb{C}} W_{d,k,l} \,]\; q^d s^k t^l.$$

By (4.6),

$$H(\mathrm{Cov}(F^{m,p}))(q, s, t) = H(S_{\mathcal{G}})(q, s, t) - H(I_{\mathcal{G}})(q, s, t).$$

Suppose $\mathcal{F}$ is a finite subset of triples from $(\mathbb{Z}_{\geq 0})^3$. For each $(d, k, l) \in \mathcal{F}$ it is a simple linear algebra problem to compute a basis for the space $I_{\mathcal{G}}(d, k, l)$ and hence,

[1] In this section the entire triple (d, k, l) is referred to simply as *degree*.

[2] A syzygy is then a polynomial identity in the covariants. In homological terms, the ideal $I_{\mathcal{G}}$ is called the *first* syzygy.

determine a set of generators for the ideal

$$I_{\mathcal{F}} := \langle\, I_{\mathcal{G}}(d,k,l) \mid (d,k,l) \in \mathcal{F} \,\rangle \subset I_{\mathcal{G}}.$$

Unfortunately, it is not clear *a priori* when $I_{\mathcal{F}} = I_{\mathcal{G}}$. If, however, the Hilbert series for $\mathrm{Cov}(F^{m,p})$ is known in advance then a dimension count gives

$$I_{\mathcal{F}} = I_{\mathcal{G}} \quad \Longleftrightarrow \quad H(S_{\mathcal{G}}/I_{\mathcal{F}}) = H(S_{\mathcal{G}}/I_{\mathcal{G}}).$$

We now apply this to the case of the binary $(3,1)$-forms. Recall from Chapter 3, that $F^{3,1}$ is the -1-eigenspace $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the symmetric algebraic pair $(G_2, \mathfrak{so}(4,\mathbb{C}))$. By Lemma 4.4, $\mathrm{Cov}(F^{3,1}) \cong \mathbb{C}\,[\,\mathfrak{p}\,]^U$ and by Proposition 3.13

$$H(\mathrm{Cov}(F^{3,1})) = \frac{\mathcal{W}(s,t)}{(1-q^2)(1-q^6)}. \tag{4.7}$$

We proceed as follows:

Step 1. Let $\mathcal{G}$ be the twenty covariant generators $J_i := J_{d_i,k_i,l_i}$ given in Table 4.1. Then $S_{\mathcal{G}}$ is a triply-graded polynomial algebra in the generators $X_i := X_{d_i,k_i,l_i}$.

Step 2. Let $\mathcal{M}(d,k,l)$ be the set of monomials of degree (d,k,l) in the X_i. Computationally, this may be obtained by finding the coefficient $C_{d,k,l}$ of $q^d s^k t^k$ in the series

$$\frac{1}{\prod_{i=1}^{20}(1 - q^{d_i} s^{k_i} t^{l_i} X_i)}.$$

Then $\mathcal{M}(d,k,l)$ consists of the additive terms in $C_{d,k,l}$.

Step 3. Choose a finite subset $\mathcal{F}$ of $(\mathbb{Z}_{\geq 0})^3$. For each $(d,k,l) \in \mathcal{F}$ compute a basis for $I_{\mathcal{G}}(d,k,l)$ as follows:

(a) Form the linear combination

$$P(X_1, \ldots, X_{20}) = \sum_{M \in \mathcal{M}(d,k,l)} A_M \, M$$

in unknowns A_M. Then

$$\pi(P(X_1, \ldots, X_{20})) = P(J_1, \ldots, J_{20}) : F^{3,1} \to F^{k,l} \tag{4.8}$$

is a covariant of $F^{3,1}$ of degree d and order (k, l).

(b) Evaluate (4.8) at a sufficiently generic form (to be explained momentarily) $f \in F^{3,1}$ and set $P(J_1, \ldots, J_{20})(f) = P(J_1(f), \ldots, J_{20}(f)) = 0$.

(c) Collect like terms in the x_0, x_1, y_0, y_1. The coefficients of each term is a linear equation in the A_M which we solve simultaneously to determine a basis for the space $I_{\mathcal{G}}(d, k, l)$.

When done we have a generating set for the ideal $I_{\mathcal{F}}$.

Step 4. Compute the Hilbert series for $S_{\mathcal{G}}/I_{\mathcal{F}}$ and compare with (4.7). If they agree then $I_{\mathcal{F}} = I_{\mathcal{G}}$ and we have a complete set of syzygies for $F^{3,1}$.

The next lemma makes precise what constitutes a sufficiently generic form.

Lemma 4.5. *A covariant $J : F^{3,1} \to F^{m,p}$ is identically zero if and only if it vanishes on a maximal toral subalgebra $\mathfrak{a} \subset \mathfrak{p} = F^{3,1}$. In particular, $J \in \mathrm{Cov}(F^{3,1})$ is identically zero if and only if*

$$J(a(x_0^3\, y_0 + x_1^3\, y_1) + b(x_0\, x_1^2\, y_0 + x_0^2\, x_1\, y_1)) = 0$$

for all $a, b \in \mathbb{C}$.

Proof. Since a covariant is K-equivariant, it vanishes at a particular $f \in F^{3,1}$ if and only if it vanishes on the orbit $K \cdot f$. Every semisimple element of $\mathfrak{p} = F^{3,1}$

is conjugate to an element of any fixed maximal toral subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Since the semisimple elements are dense in $\mathfrak{p}$ it is enough to check vanishing on $\mathfrak{a}$. Lemma 3.1 gives one such choice for $\mathfrak{a}$. □

Theorem 4.6. *Let*

$$\begin{aligned}\mathcal{F} = \{\, &(5,7,3),(6,6,2),(6,4,4),(6,6,6),(6,8,4),(6,10,2),(7,5,5),(7,7,3),\\ &(7,5,3),(7,9,1),(8,6,4),(8,4,4),(8,4,6),(8,8,2),(8,6,2),(9,3,5),\\ &(9,5,3),(9,7,1),(9,5,7),(9,7,5),(9,9,3),(9,11,1),(10,4,6),(10,6,4),\\ &(10,8,2),(10,4,4),(10,2,6),(10,6,2),(11,3,7),(11,5,5),(11,3,5),(11,5,3),\\ &(11,7,3),(11,9,1),(12,2,6),(12,4,4),(12,4,8),(12,6,6),(12,8,4),(12,10,2),\\ &(12,12,0),(13,3,7),(13,5,5),(13,3,5),(13,7,3),(14,2,8),(14,2,6),(14,4,6),\\ &(14,6,4),(15,1,7),(15,3,9),(15,5,7),(15,7,5),(16,2,8),(16,4,6),(17,1,9),\\ &(17,3,7),(18,2,10),(18,4,8),(18,6,6),(19,1,9),(21,1,11),(24,0,12)\,\}.\end{aligned}$$

Then $I_{\mathcal{F}} = I_{\mathcal{G}}$ and this ideal is minimally generated by the 104 syzygies that appear in Appendix B.2.

The proof of Theorem 4.6 requires a significant amount of computer computation. Section B.1 has source code for a C++ program called `syzygies` that implements Step 3. It accepts as input (d, k, l) and returns a basis for $I_{\mathcal{G}}(d, k, l)$. The commutative algebra software `Macaulay2` [Gra] calls this external program for each (d, k, l) appearing in $\mathcal{F}$. The results are combined to form the ideal $I_{\mathcal{F}}$ and a minimal set of generators for this ideal are computed. The 104 resulting generators appear in Section B.2. Next, the Hilbert series of $S_{\mathcal{G}}/I_{\mathcal{F}}$ is computed. This takes several hours to complete on a modern computer platform. The result is compared to the known Hilbert series (4.7) for $\mathrm{Cov}(F^{3,1})$. By observing that they are equal, the theorem is proved. Section B.2 demonstrates the necessary input to `Macaulay2`.

Chapter 5

Asymptotic methods for graded multiplicities

5.1 The Brion polytope

Let K be a connected semisimple algebraic Lie group with maximal torus T. Let $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{h} = \mathrm{Lie}(T)$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi := \Phi(\mathfrak{h})$ and choose a set of positive roots Φ^+ for Φ. Let $\mathbb{E}$ denote the real span of Φ. For $\lambda \in P_+(\mathfrak{k})$, let $L(\lambda)$ denote the finite-dimensional irreducible representation of K with highest weight λ. We write $\lambda^* \in P_+(\mathfrak{k})$ for the highest weight of the contragredient representation $L(\lambda)^*$.

Fix a finite-dimensional irreducible representation V of K and suppose $X \subset V$ is a K-invariant affine cone-variety. Then K acts on the algebra $\mathbb{C}[X]$ of regular functions on X and the space of functions $\mathbb{C}[X]_d$ is K-stable. Let $f_d(\lambda)$ denote the multiplicity of $\lambda \in P_+(\mathfrak{k})$ in $\mathbb{C}[X]_d$

$$f_d(\lambda) = \dim_{\mathbb{C}} \mathrm{Hom}_K\left(L(\lambda), \mathbb{C}[X]_d\right).$$

Example 5.1. If $(\mathfrak{g}, \mathfrak{k})$ is an algebraic symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ then the nullcone $\mathcal{N} \subset \mathfrak{p}$ is an affine cone-variety. By Lemma 1.19, restriction provides an isomorphism of $\mathbb{C}[\mathcal{N}]$ with the harmonic functions $\mathcal{H} \subset \mathbb{C}[\mathfrak{p}]$. Thus, $f_d(\lambda)$ is precisely the graded multiplicity (1.16).

Definition 5.2. The *Brion polytope* of X is

$$\mathrm{Bri}(X) = \left\{ \frac{\lambda}{d} \in P_+(\mathfrak{k}) \otimes_{\mathbb{Z}} \mathbb{Q} \mid \exists\, \lambda \in P_+(\mathfrak{k}) \text{ and } d > 0 \text{ such that } f_d(\lambda^*) \neq 0 \right\}.$$

Naming $\mathrm{Bri}(X)$ after M. Brion is adopted from [Smi04]. Brion proved that the closure of $\mathrm{Bri}(X)$ in $\mathbb{E}$ is a convex polytope whenever X is irreducible [Bri87]. We provide our own proof of this fact in the following context. For each $m > 0$, let $\mathcal{G}^m = (\mathbb{Z}_{\geq 0})^m$ viewed as an additive semigroup

Theorem 5.3. *Suppose*

$$A = \bigoplus_{(d,\lambda) \in \mathbb{Z} \times \mathcal{G}^m} A(d, \lambda)$$

is a $\mathbb{Z} \times \mathcal{G}^m$-graded integral domain over $\mathbb{C}$ which is finitely generated by homogeneous elements $X_i \in A(d_i, \lambda_i)$ where $d_i > 0$, $\lambda_i \in \mathcal{G}^m$, for $i = 1, \ldots, r$. Let

$$\mathcal{C} = \left\{ \frac{\lambda}{d} \in \mathbb{Q}^m \mid \exists\, \lambda \in \mathcal{G} \text{ and } d > 0 \text{ such that } A(d, \lambda) \neq 0 \right\}$$

Then the closure of $\mathcal{C}$ in $\mathbb{R}^m$ is the convex hull of the points $\xi_i := \lambda_i / d_i \in \mathbb{Q}^m$, $i = 1, \ldots, r$.

Proof. Let $\mathcal{S} = \mathrm{Hull}(\xi_1, \ldots, \xi_r) \cap \mathbb{Q}^m$. Suppose $0 \neq A(d, \lambda)$ for some $d > 0$. Choose a monomial $X_1^{n_1} \cdots X_r^{n_r} \in A$ such that $d = \sum_{i=1}^r n_i d_i$ and $\lambda = \sum_{i=1}^r n_i \lambda_i$. Let $c_j = \frac{n_j d_j}{d}$, for $j = 1, \ldots, r$. Then $\sum_{j=1}^r c_j = 1$ and

$$\frac{\lambda}{d} = \sum_{i=1}^r \frac{n_i}{d}\, \lambda_i = \sum_{j=1}^r c_j\, \xi_j \in \mathcal{S}.$$

This proves $\mathcal{C} \subset \mathcal{S}$.

Conversely, suppose $\xi = \sum_{i=1}^{r} c_i\,\xi_i \in \mathcal{S}$ where $\sum_{i=1}^{r} c_i = 1$ and each $c_i = m_i/l_i$ is rational with $m_i \in \mathbb{Z}_{\geq 0}$ and $l_i \in \mathbb{N}$. Let $d = \operatorname{lcm}(l_1 d_1, \ldots, l_r d_r)$ and let

$$n_i = \frac{c_i d}{d_i} = \frac{m_i d}{l_i d_i} \in \mathbb{Z}_{\geq 0}, \quad \text{for } i = 1, \ldots, r. \tag{5.1}$$

Since $c_i = n_i d_i/d$,

$$d = d\sum_{i=1}^{r} c_i = \sum_{i=1}^{r} n_i\, d_i.$$

Since A is a domain, $0 \neq X_1^{n_1} \cdots X_r^{n_r} \in A(d, \lambda)$ where $\lambda = \sum_{i=1}^{r} n_i\,\lambda_i$. By (5.1),

$$\frac{\lambda}{d} = \sum_{i=1}^{r} \frac{n_i\,\lambda_i}{d} = \sum_{i=1}^{r} \frac{c_i\,\lambda_i}{d_i} = \sum_{i=1}^{r} c_i\,\xi_i = \xi$$

which proves $\mathcal{S} \subset \mathcal{C}$. □

5.1.1 The Brion polytope for Type G

In this section we return to the symmetric pair $(\mathfrak{g}, \mathfrak{k}) := (G_2, \mathfrak{so}_4)$. In the notation of the previous section, we take $K = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$; $V = \mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition; and $X = \mathcal{N}$ is the nullcone of $\mathfrak{p}$. The Brion polytope can then be viewed as the set

$$\mathrm{Bri}(\mathcal{N}) = \{\, (k/d, l/d) \in \mathbb{Q}^2 \mid \exists\, k, l \geq 0 \text{ and } d > 0 \text{ with } f_d(k, l) \neq 0 \,\},$$

where $f_d(k, l)$ is the graded multiplicity (3.9).

Proposition 5.4. *The nullcone*

$$\mathcal{N} = \{\, X \in \mathfrak{p} \mid u_2(X) = 0 = u_6(X) \,\}$$

for the pair $(G_2, \mathfrak{so}_4)$ is irreducible. Moreover, the closure of $\mathrm{Bri}(\mathcal{N})$ *in* $\mathbb{R}^2$ *is a convex polytope.*

Proof. As $K_\theta = \mathrm{SO}(4,\mathbb{C})$ is connected, irreducibility follows from Lemma 1.17. The algebra of semiinvariants $A = \mathbb{C}[\mathcal{N}]^U$ is then a triply graded integral domain. The homogeneous component $A(d,k,l)$ is non-zero if and only if $f_d(k,l) \neq 0$, in which case $A(d,k,l)$ is one dimensional. Convexity follows from 5.3 applied to A. □

The boundary of the Brion polytope can be completely determined from Todd's generators for the covariants of the binary $(3,1)$-forms. This is because

$$\mathbb{C}[\mathcal{N}]^U \cong (\mathbb{C}[\mathfrak{p}]/\langle u_2, u_6\rangle)^U \cong \mathbb{C}[\mathfrak{p}]^U/\langle u_2, u_6\rangle \cong \mathrm{Cov}\left(F^{3,1}\right)/\langle J_{2,0,0}, J_{6,0,0}\rangle.$$

Thus, for all of the *non-invariant* covariant generators $J_{d,k,l}$ that appear in Table 4.1 we plot the point $(x,y) = (k/d, l/d)$ and take the convex hull. Observe that distinct covariant generators may give rise to the same point in $\mathrm{Bri}(\mathcal{N})$. The resulting polytope appears in Figure 5.1 and is bounded by the lines

$$x = 0, \quad y = 0, \quad y = 1, \quad x + 2y = 1, \quad x - y = 2. \tag{5.2}$$

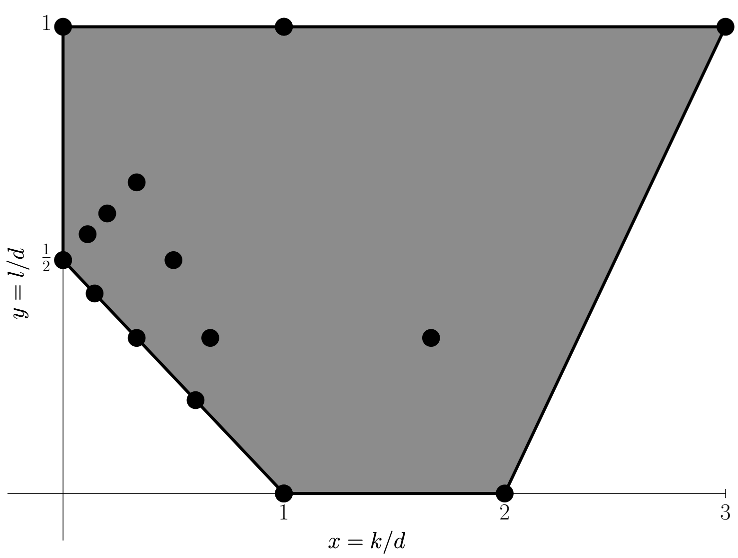

Figure 5.1: The Brion polytope for the nullcone of $(G_2, \mathfrak{so}(4,\mathbb{C}))$.

5.2 Density of the graded multiplicities in Type G

Let Ω denote the interior of the closure of $\mathrm{Bri}(\mathcal{N})$ in $\mathbb{R}^2$. The goal of this section is to construct a piecewise-linear function $g : \Omega \to \mathbb{R}$ which describes the asymptotic density of the graded multiplicities $f_d(k,l)$ for the pair $(G_2, \mathfrak{so}(4,\mathbb{C}))$. In order to achieve this we take advantage of two earlier results:

1. the quadratic estimate (2.14) for the principal branching multiplicities $B(k,l,m)$ for $\mathfrak{sp}(2,\mathbb{C})$;

2. the formula (3.19) for the graded multiplicities

$$f_d(k,l) = B(k,l,m) - B(k,l,m-3), \quad \text{where } d = l + 2m.$$

We briefly recall the relevant notation from Section 2.3. The dominant integral weights for $\mathfrak{sp}(2,\mathbb{C})$ are of the form $\lambda = l\,\varpi_1 + m\,\varpi_2$, $l,m \geq 0$ where ϖ_1 and ϖ_2 are the fundamental weights. The Weyl group of $\mathfrak{sp}(2,\mathbb{C})$ is denoted by W. For $w \in W$, $k,l,m \in \mathbb{R}$

$$L_w(k,l,m) = \overline{w(\lambda+\rho)-\rho} - k, \quad \text{where } \lambda = l\,\varpi_1 + m\,\varpi_2 \tag{5.3}$$

and explicit formulae for each L_w are provided in Figure 2.2. For real numbers $k,l,m \geq 0$,

$$A(k,l,m) = \{\, w \in W \mid L_w(k,l,m) \geq 0 \,\}.$$

If any of k,l,m are negative then we set $A(k,l,m) = \emptyset$. For $S \subset W$ and $k,l,m \in \mathbb{R}$,

$$Q_S(k,l,m) = \sum_{w \in S} \mathrm{sgn}(w) \frac{1}{48} \left(L_w(k,l,m) + 6\right)^2.$$

The following convention will be used for the remainder of this section.

Convention 1. Whenever the symbols d, l, m appear together it should be understood that

$$\boxed{d = l + 2m.} \tag{5.4}$$

With this in mind, define

$$g_d(k,l) = Q_S(k,l,m) - Q_S(k,l,m-3),$$

where $S = A(k,l,m)$. Thus,

$$g_d(k,l) = \sum_{w \in A(k,l,m)} \frac{1}{48} \operatorname{sgn}(w) \left(\left(L_w\left(k,l,m\right)+6\right)^2 - \left(L_w\left(k,l,m-3\right)+6\right)^2 \right). \tag{5.5}$$

Set $S = A(k,l,m)$ and $S' = A(k,l,m-3)$. When $S = S'$, Lemma 2.19 immediately yields

$$|\, f_d(k,l) - g_d(k,l) \,| \le C$$

for some constant C. Things are more delicate when $S \neq S'$ as we are unable to appeal to 2.19 directly. Fortunately, the terms in (5.5) that may or may not appear due to differences between S and S' contribute little to the overall calculation.

Lemma 5.5. *For each $w \in W$ there exists a constant $C_w > 0$ such that for all k, l, m, $S = A(k,l,m), S' = A(k,l,m-3)$ we have*

1. *If $w \in S \backslash S'$ then*

$$-C_w \le L_w(k,l,m-3) < 0.$$

2. *If $w' \in S' \backslash S$ then*

$$0 \le L_{w'}(k,l,m-3) < C_{w'}.$$

Proof. By (5.3),

$$L_w(k,l,m-3) = L_w(k,l,m) - 3\overline{w(\varpi_2)} = L_w(k,l,m) - \epsilon_w C_w,$$

where $C_w := 3\left|\overline{w(\varpi_2)}\right|$ and $\epsilon_w = \pm 1$ is chosen accordingly. If $w \in S \backslash S'$, then $L_w(k,l,m) \geq 0$, but $L_w(k,l,m-3) < 0$. Thus,

$$0 > L_w(k,l,m-3) = L_w(k,l,m) - \epsilon_w C_w \geq 0 - \epsilon_w C_w \geq -\epsilon_w C_w.$$

The second case is similar. □

Proposition 5.6. *There exists a constant $C > 0$, such that for all integers $k, l, d \geq 0$, $k - l \in 2\mathbb{Z}$,*

$$|\, f_d(k,l) - g_d(k,l)\,| \leq C.$$

Proof. Let $S = A(k,l,m)$ and $S' = A(k,l,m-3)$. Using 2.19, there is a constant $C' > 0$ such that

$$\begin{aligned} |\, f_d(k,l) - g_d(k,l)\,| &\leq |\, B(k,l,m) - Q_S(k,l,m)\,| + |\, B(k,l,m-3) - Q_S(k,l,m-3)\,| \\ &\leq C' + |\, B(k,l,m-3) - Q_S(k,l,m-3)\,| \\ &= C' + |\, B(k,l,m-3) - Q_{S'}(k,l,m-3) + Q_{S'}(k,l,m-3) - Q_S(k,l,m-3)\,| \\ &\leq C' + |\, B(k,l,m-3) - Q_{S'}(k,l,m-3)\,| + |\, Q_S(k,l,m-3) - Q_{S'}(k,l,m-3)\,| \\ &\leq 2C' + |\, Q_S(k,l,m-3) - Q_{S'}(k,l,m-3)\,|. \end{aligned}$$

Estimate the second term by applying the Inclusion-Exclusion Principle:

$$S \cup S' = (S \cap S') \cup (S \backslash S') \cup (S' \backslash S).$$

Since terms from the intersection cancel,

$$\begin{aligned}
\left| Q_S(k,l,m-3) - Q_{S'}(k,l,m-3) \right| &= \left| Q_{S\setminus S'}(k,l,m-3) - Q_{S'\setminus S}(k,l,m-3) \right| \\
&\leq \sum_{w\in S\setminus S'} \frac{(C_w+6)^2}{48} + \sum_{w'\in S\setminus S'} \frac{(C_{w'}+6)^2}{48} \\
&\leq 2\sum_{w\in W} \frac{(C_w+6)^2}{48}.
\end{aligned}$$

□

It is straightforward to compute $g_d(k,l)$ for each of the possible $A(k,l,m) \subset W$ that are admissible. For example, if $k,l,m \geq 0$ such that $S := A(k,l,m) = \{1\}$ then using Figure 2.2 and (5.5),

$$\begin{aligned}
g_d(k,l) &= Q_S(k,l,m) - Q_S(k,l,m-3) \\
&= \frac{1}{48}\left((L_1(k,l,m)+6)^2 - (L_1(k,l,m-3)+6)^2\right) \\
&= \frac{1}{48}\left((3l+4m-k+6)^2 - (3l+4(m-3)-k+6)^2\right) \\
&= \frac{1}{2}\left(3l+4m-k\right).
\end{aligned}$$

Substituting $m = (d-l)/2$, $g_d(k,l) = \frac{1}{2}(l+2d-k)$. The remaining cases appear in Table 5.1. Observe that the resulting expression is linear in $d, k,$ and l despite the quadratic estimate of the branching rule.

Admissible subset $S = A(k,l,m)$	Estimate of graded multiplicity $g_d(k,l)$
$\{1\}$	$\frac{1}{2}(l+2d-k)$
$\{1,2\}$	$1+l$
$\{1,3\}$	$\frac{1}{4}(3d-k-1)$
$\{1,2,3\}$	$\frac{1}{4}(2l-d+k+3)$
$\{1,2,3,4\}$	$\frac{1}{2}(1+k)$
$\{1,2,3,5\}$	0

Table 5.1: Piecewise linear approximation for the graded multiplicities.

We extend the earlier convention as follows:

Convention 2. Whenever the symbols x, y, d and k, l, m appear together it should be understood that k, l, m are dependent on x, y, d via the equations

$$\boxed{x = k/d, \quad y = l/d, \quad d = l + 2m.} \tag{5.6}$$

Equivalently,

$$\boxed{k = dx, \quad l = dy, \quad m = (d - dy)/2.} \tag{5.7}$$

Define $\ell : \mathbb{R}^2 \to \mathbb{R}$ by

$$\ell_w(x, y) = \lim_{d\to\infty} \frac{L_w(k, l, m)}{d},$$

for $(x, y) \in \mathbb{R}^2$. Each $\ell_w(x, y)$ can be computed from the formula for $L_w(k, l, m)$ in Figure 2.2. For example, if $w = w_2 \in W$

$$\begin{aligned} L_w(k, l, m) &= l + 4m - 2 - k \\ &= dy + 2d - 2dy - 2 - dx \\ &= 2d - dx - dy - 2. \end{aligned}$$

Thus,

$$\ell_w(x, y) = \lim_{d\to\infty} (2 - x - y - 2/d) = 2 - x - y.$$

The remaining ℓ_w are given in Table 5.2.

Next, we define the asymptotic analogue of the sets $A(k, l, m)$. For $(x, y) \in \mathbb{R}^2$, let

$$A(x, y) = \{\, w \in W \mid \ell_w(x, y) \geq 0 \,\}. \tag{5.8}$$

$w \in W$	$\ell_w(x,y)$
w_1	$y - x + 2$
w_2	$2 - x - y$
w_3	$2y - x + 1$
w_4	$2y - x - 1$
w_5	$-x - 2y + 1$
w_6	$-x - 2y - 1$
w_7	$-x - 2$
w_8	$-x - y - 2$

Table 5.2: $\ell_w(x,y)$ for $w \in W$.

The classes $A(x,y)$ partition Ω. For $S \subset W$, define

$$\mathcal{R}_S = \{\, (x,y) \in \Omega \mid A(x,y) = S \,\}$$

Each non-empty $\mathcal{R}_S$ is a subset of Ω whose closure in $\overline{\Omega}$ is a convex polytope bounded by lines $\ell_w(x,y) = 0$ and the boundary of the Brion polytope. If the line $\ell_w(x,y) = 0$ is in the closure of $\mathcal{R}_S$ then it is contained in $\mathcal{R}_S$ if and only if $w \notin S$. Moreover,

$$\Omega = \biguplus \mathcal{R}_S, \tag{5.9}$$

where the union is over the admissible subsets S in the list $\{\,1\,\}$, $\{\,1,2\,\}$, $\{\,1,3\,\}$, $\{\,1,2,3\,\}$, $\{\,1,2,3,4\,\}$. The union is clearly disjoint by (5.8). Figure 5.2 shows the partitioning (5.9).

Define $g : \Omega \to \mathbb{R}$

$$g(x,y) = \lim_{d\to\infty} \frac{g_d(k,l)}{d},$$

for all $(x,y) \in \Omega$. Then

$$g(x,y) = \lim_{d\to\infty} \frac{1}{d} \left(Q_S(k,l,m) - Q_S(k,l,m-3) \right) \tag{5.10}$$

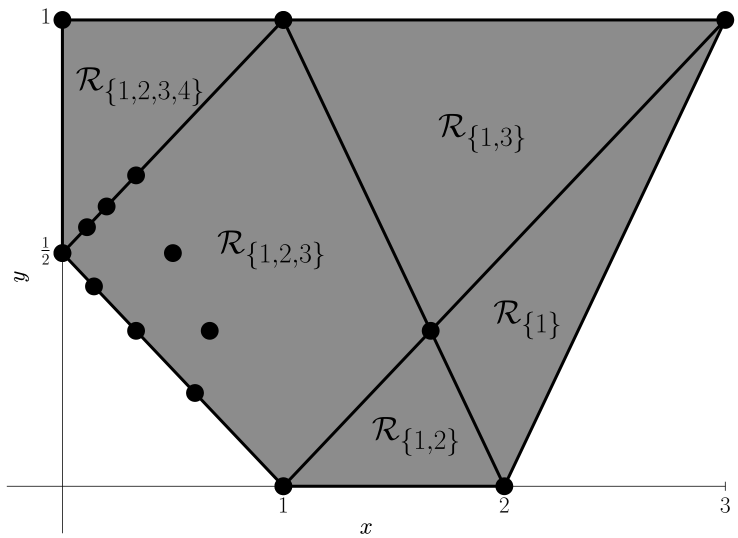

Figure 5.2: Partitioning of the Brion polytope.

where unfortunately $S = A(k,l,m)$ varies with d. The next lemma overcomes this issue.

Lemma 5.7. *For* $(x,y) \in \mathbb{R}^2$,

$$g(x,y) = \lim_{d\to\infty} \frac{1}{d} \left(Q_S(k,l,m) - Q_S(k,l,m-3) \right),$$

where $S = A(x,y)$ *can be taken independent of* d *in the limit.*

Proof. Fix $(x,y) \in \Omega$. For each $w \in W$, $L_w(k,l,m) = L_w(dx, dy, (d-dy)/2)$ is a linear polynomial in d. There are then three possibilities for w:

1. $L_w(k,l,m)$ is constant for all d. In this case, the term corresponding to w in (5.10) contributes nothing in the limit and may therefore be included or excluded as required.

2. $L_w(k,l,m) \to \infty$ as $d \to \infty$. Then $w \in A(k,l,m)$ for all d sufficiently large and $\ell_w(x,y) > 0$. Hence, $w \in A(x,y)$.

3. $L_w(k,l,m) \to -\infty$ as $d \to \infty$. Then $w \notin A(k,l,m)$ for all d sufficiently large and $\ell_w(x,y) < 0$. Hence, $w \notin A(x,y)$.

□

Lemma 5.7 allows us to use Table 5.1 to compute $g(x,y)$ on each $\mathcal{R}_S$. For example, suppose $A(x,y) = \{1,3\}$. Then

$$\begin{aligned} g(x,y) &= \lim_{d\to\infty} \frac{g_d(k,l)}{d} \\ &= \lim_{d\to\infty} \frac{1}{4}\frac{3d-k-1}{d} \\ &= \lim_{d\to\infty} \frac{1}{4}(3-x-1/d) \\ &= \frac{1}{4}(3-x). \end{aligned}$$

The remaining cases are exhibited in Table 5.3. Figure 5.3 shows the graph of the equation $z = g(x,y)$.

Admissible subset $S \subset W$	Density of graded multiplicities $g(x,y)$ for $(x,y) \in \mathcal{R}_S$
$\{1\}$	$\frac{1}{2}(y-x+2)$
$\{1,2\}$	y
$\{1,3\}$	$\frac{1}{4}(3-x)$
$\{1,2,3\}$	$\frac{1}{4}(2y-1+x)$
$\{1,2,3,4\}$	$\frac{x}{2}$

Table 5.3: Density of the graded multiplicities.

Recall that for each $d \geq 0$, $f_d(k,l)$ is defined only for integers $k,l \geq 0$. We set $f_d(k,l) = 0$, whenever k or l is not an integer. Define $f : \Omega \to \mathbb{R}$ by

$$f(x,y) = \limsup_{d\to\infty} \frac{f_d(k,l)}{d},$$

for all $(x,y) \in \Omega$. Define

$$\Omega_{\text{supp}} = \left\{ \left(\frac{k}{d}, \frac{l}{d}\right) \in \Omega \mid k,l,d \in \mathbb{Z}_{\geq 0},\, d > 0,\, k-l \in 2\mathbb{Z} \right\}. \tag{5.11}$$

Then Ω_{supp} is a dense subset of Ω. The piecewise-linear function g describes the asymptotic density of the graded multiplicities in the following sense.

Theorem 5.8. *For $(x, y) \in \Omega_{supp}$, $f(x, y) = g(x, y)$.*

Proof. By Proposition 5.6,

$$g_d(k, l) - C < f_d(k, l) < g_d(k, l) + C \tag{5.12}$$

whenever $k, l, d \in \mathbb{Z}_{\geq 0}$ and $k - l \in 2\mathbb{Z}$.

Fix $(x, y) \in \Omega_{\text{supp}}$. Write $x = k_1/d_1$ and $y = l_1/d_1$, where $k_1, l_1, d_1 \in \mathbb{Z}_{\geq 0}$, $d_1 > 0$, $k_1 - l_1 \in 2\mathbb{Z}$, and d_1 does not divide both k_1 and l_1.

$$\begin{aligned} f(x, y) &= \limsup_{d \to \infty} \frac{f_d(k, l)}{d} \\ &= \limsup_{d \to \infty} \frac{f_d\left(d\frac{k_1}{d_1}, d\frac{l_1}{d_1}\right)}{d} \\ &= \limsup_{d \to \infty} \frac{f_d(dk_1, dl_1)}{d} \end{aligned}$$

by (5.12),

$$\begin{aligned} &\leq \limsup_{d \to \infty} \frac{g_d(dk_1, dl_1)}{d} \\ &= \limsup_{d \to \infty} \frac{g_d\left(dd_1\frac{k_1}{d_1}, dd_1\frac{l_1}{d_1}\right)}{d} \\ &= \limsup_{d \to \infty} \frac{g_d\left(dd_1 x, dd_1 y\right)}{d} \\ &= \limsup_{d \to \infty} \frac{g_d\left(dx, dy\right)}{d} \\ &= g(x, y) \end{aligned}$$

Therefore, $f(x, y) \leq g(x, y)$. The reverse inequality is similar. $\square$

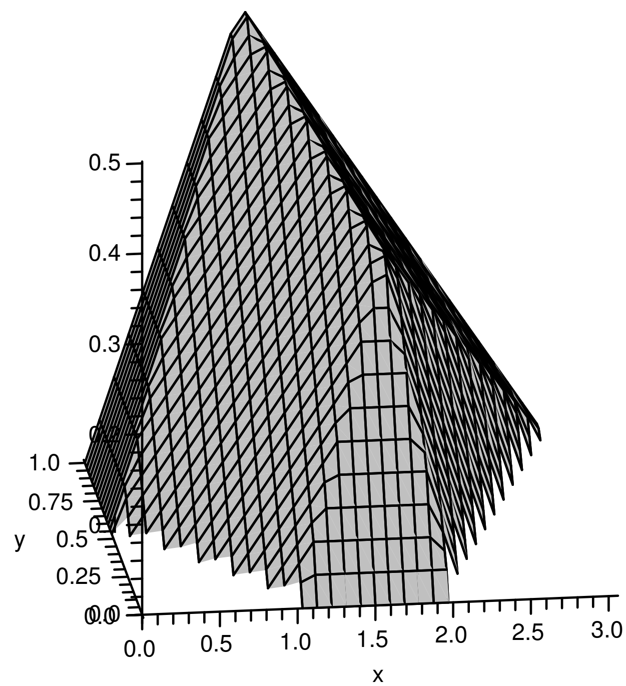

Figure 5.3: Graph of the density function $z = g(x, y)$.

Appendix A

Computer proof of Lemmas 3.10, 3.12

In this section we implement the generating functions defined in 3.3.1 in `Maple`. This is then used to prove Lemma 3.10 and Lemma 3.12.
Equations (3.35)-(3.41):

```
GFF :=  Z -> Z^6/(1-Z)/(1-Z^6):
GFL0 := Y -> GFF(Y) + (1+Y^2+Y^3+Y^4+Y^5)/(1-Y^6):
GFL2 := Y -> GFF(Y) + (2*Y+Y^2+Y^3+2*Y^4+2*Y^5)/(1-Y^6):
GFL6 := Y -> GFF(Y) + (-1-Y^3+Y^4)/(1-Y^6):
GFL8 := Y -> GFF(Y) + (Y^3+Y^5)/(1-Y^6):
GFK1 := X -> GFL8(X):
GFK2 := X -> GFF(X) + (X+X^2+2*X^4+X^5)/(1-X^6):
```

Equations (3.42)-(3.46):

```
GFD := (a,b,c,d,X,Y) -> (a+b*X+c*Y+d*X*Y)/(1-X^2)/(1-Y^2):
GFA1 := X -> GFD(q+2*q^3+q^5, 1+q^2+q^4+q^6, 0, 0, X, 0):
GFA2 := X -> GFD(1+q^2+q^4+q^6, q+2*q^3+q^5, 0, 0, X, 0):
GFA3 := (X,Y) -> GFD(q+2*q^3+q^5,
                                    1+q^2+q^4+q^6,
                                    1+q^2+q^4+q^6,
                                    q+2*q^3+q^5, X, Y):
GFB := (X,Y) -> GFD(1-q^2, 0, q^6*(1-q^2), 0, X, Y):
```

Equations (3.47)-(3.49):

```
GFP1 := (X,Y) - (1-q^8)*GFK1(q*X)/(1-q^2*Y)
                + q^2*(1-q^4)*GFK2(q*X)/(1-q^2*Y)
                + GFB(q*X,q^2*Y)
                - q^4/(1-q*X)/(1-q^4*Y)
                + q^6/(1-q^3*X)/(1-q^4*Y):
GFP2 := (X,Y) -> GFL0(q*Y)/(1-q*X)
                + q^2*GFL2(q*Y)/(1-q*X)
                + q^4/(1-q*X)/(1-q*Y)
                - q^6*GFL6(q*Y)/(1-q*X)
```

```
                - q^8*GFL8(q*Y)/(1-q*X)
                + q^6/(1-q^5*X)/(1-q^3*Y)
                - q*GFA1(q^2*X)/(1-q*Y):
GFP3 := (X,Y) -> GFA2(q*X)/(1-q*Y)
                - q*GFA3(q*X,q^2*Y)
                - q^2/(1-q^2*X)/(1-q*Y)
                + q^6/(1-q^2*X)/(1-q^5*Y):
```

Equations (3.50) and (3.51):

```
GFD11 := Z -> GFP1(Z,0):
GFD31 := Z -> GFP2(Z,0):
```

To verify the Diagonal Lemma compute:

```
simplify(GFP2(0,Z) - GFP1(Z,0));
simplify(GFP3(0,Z) - GFP2(Z,0));
```

Each returns zero which proves Lemma 3.10.

Equation (3.52):

```
GFW := (X,Y)->
         (GFP1(X*Y,Y^2)
        + GFP2(X^3*Y,X*Y)
        + GFP3(X^2,X^3*Y)
        - GFD11(X*Y)
        - GFD31(X^3*Y))/(1-q^2)/(1-q^4):
```

Right hand side of (3.53):

```
Qchar := (s-1/s)*(t-1/t)*(1-q^2)*(1-q^6):
for i in {-3,-1,1,3} do
  for j in {-1,1} do
    Qchar := Qchar/(1-q*s^i*t^j):
  od:
od:
```

Left hand side of (3.53):

```
Avg := 0:
for w1 in {-1,1} do
  for w2 in {-1,1} do
    Avg := Avg + w1 * w2 * s^w1 * t^w2 * GFW(s^w1,t^w2):
  od:
od:
```

Now subtract

```
Difference := simplify(Avg - Qchar);
```

which returns zero proving Lemma 3.12.

Appendix B

Syzygies for $(3,1)$-covariants

B.1 Program for computing syzygies

The following C++ source code computes a basis for the syzygies of the binary $(3,1)$-forms in a fixed degree. Symbolic computation is performed using the C++ library `GiNac` [GiN] To illustrate the usage of the program, we compute the syzygies in degree $(5,7,3)$.

```
$ ./syzygies 5 7 3
-X_(3,1,1)*X_(1,3,1)^2-X_(2,4,0)*X_(3,3,3)+X_(2,2,2)*X_(3,5,1),0
```

For programming simplicity the output is always terminated by 0. In this case $I_{\mathcal{G}}(5,7,3)$ is one-dimensional and the covariants satisfy the polynomial identity

$$-J_{3,1,1}J_{1,3,1}^2 - J_{2,4,0}J_{3,3,3} + J_{2,2,2}J_{3,5,1} = 0.$$

```
#include <iostream>
#include <list>
#include <fstream>
#include <sstream>
#include <string>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

symbol a("a"),b("b"),q("q"),s("s"),t("t"),X("X");
symbol x0("x_0"),x1("x_1"),y0("y_0"),y1("y_1");
symbol u0("u_0"),u1("u_1"),v0("v_0"),v1("v_1");
symbol infty("infty");

// input: a form f in symbols xx0, xx1, yy0, yy1
// output: apply the omega operator to f
ex
omega(const ex &f,
      const symbol xx0, const symbol xx1,
      const symbol yy0, const symbol yy1)
{
  return f.diff(xx0).diff(yy1) - f.diff(yy0).diff(xx1);
}

// input: two forms f and g in x0,x1,y0,y1
// output: the (m,n)-transvection of f and g
ex
transvect(const ex &f, const ex &g,
          const int m, const int n)
{
  ex ret = f * g.subs(lst(x0==u0,x1==u1,y0==v0,y1==v1));
  for (int i = 1; i <= n; i++)
    ret = omega(ret,y0,y1,v0,v1);
  for (int i = 1; i <= m; i++)
    ret = omega(ret,x0,x1,u0,u1);
  return ret.subs(lst(u0==x0,u1==x1,v0==y0,v1==y1));
}

// input: a symbol X and indexes d,k,l
// output: the indexed symbol X_{d,k,l}
ex
inline sym_gen(const symbol X,
               const int d,
               const int k,
               const int l)
{
  return indexed(X,idx(d,infty),idx(k,infty),idx(l,infty));
```

```
}

// input: a monomial term in variables x0,x1,y0,y1,a,b
// output: the coefficient of the term
ex
inline mycoeff(const ex &term)
{
  return term.subs(lst(x0==1,x1==1,y0==1,y1==1,a==1,b==1));
}

int
main(int argc, char* argv[])
{
  // Create a list of tuples (d,m,p)
  // for each generator
  list<lst> gens;
  lst g131(1,3,1);
  lst g240(2,4,0),g222(2,2,2),g200(2,0,0);
  lst g351(3,5,1),g333(3,3,3),g311(3,1,1);
  lst g440(4,4,0),g422(4,2,2),g404(4,0,4);
  lst g531(5,3,1),g513(5,1,3);
  lst g660(6,6,0),g642(6,4,2),g624(6,2,4),g600(6,0,0);
  lst g713(7,1,3);
  lst g804(8,0,4);
  lst g915(9,1,5);
  lst g1206(12,0,6);
  gens.push_back(g131);
  gens.push_back(g240);gens.push_back(g222);gens.push_back(g200);
  gens.push_back(g351);gens.push_back(g333);gens.push_back(g311);
  gens.push_back(g440);gens.push_back(g422);gens.push_back(g404);
  gens.push_back(g531);gens.push_back(g513);
  gens.push_back(g660);gens.push_back(g642);
  gens.push_back(g624);gens.push_back(g600);
  gens.push_back(g713);
  gens.push_back(g804);
  gens.push_back(g915);
  gens.push_back(g1206);

  // Compute each covariant generator as
  // an expression in a,b,x0,x1,y0,y1
  ex ff = a*pow(x0,3)*y0 + b*pow(x0,2)*x1*y1
        + b*x0*pow(x1,2)*y0 + a*pow(x1,3)*y1;
  ex FF = transvect(ff,ff,1,1)/18;
  ex hh = transvect(ff,ff,2,0)/72;
  ex I2 = transvect(ff,ff,3,1)/72;
  ex jj = transvect(ff,FF,1,0)/6;
  ex tt = transvect(ff,hh,1,0)/3;
```

```
ex pp = transvect(ff,FF,3,0)/216;
ex DD = transvect(hh,hh,2,0)/2;
ex I6 = transvect(pp,pp,1,1)/2;

// Associate to each indexed symbol X_{d,k,l}
// an expression for the corresponding covariant
exmap cov_map;
cov_map[sym_gen(X,1,3,1)] = ff;
cov_map[sym_gen(X,2,4,0)] = FF;
cov_map[sym_gen(X,2,2,2)] = hh;
cov_map[sym_gen(X,2,0,0)] = I2;
cov_map[sym_gen(X,3,5,1)] = jj;
cov_map[sym_gen(X,3,3,3)] = tt;
cov_map[sym_gen(X,3,1,1)] = pp;
cov_map[sym_gen(X,4,4,0)] = transvect(ff,pp,0,1);
cov_map[sym_gen(X,4,2,2)] = transvect(ff,pp,1,0);
cov_map[sym_gen(X,4,0,4)] = DD;
cov_map[sym_gen(X,5,3,1)] = transvect(hh,pp,0,1);
cov_map[sym_gen(X,5,1,3)] = transvect(hh,pp,1,0);
cov_map[sym_gen(X,6,6,0)] = transvect(jj,pp,0,1);
cov_map[sym_gen(X,6,4,2)] = transvect(tt,pp,0,1);
cov_map[sym_gen(X,6,2,4)] = transvect(tt,pp,1,0);
cov_map[sym_gen(X,6,0,0)] = I6;
cov_map[sym_gen(X,7,1,3)] = transvect(ff,pow(pp,2),2,0);
cov_map[sym_gen(X,8,0,4)] = transvect(hh,pow(pp,2),2,0);
cov_map[sym_gen(X,9,1,5)] = transvect(tt,pow(pp,2),2,0);
cov_map[sym_gen(X,12,0,6)] = transvect(tt,pow(pp,3),3,0);

// Associate to an indexed symbol X_{d,k,l}
// an indexed variable in M2 notation
exmap str_map;
str_map[sym_gen(X,1,3,1)] = symbol("X_(1,3,1)");
str_map[sym_gen(X,2,4,0)] = symbol("X_(2,4,0)");
str_map[sym_gen(X,2,2,2)] = symbol("X_(2,2,2)");
str_map[sym_gen(X,2,0,0)] = symbol("X_(2,0,0)");
str_map[sym_gen(X,3,5,1)] = symbol("X_(3,5,1)");
str_map[sym_gen(X,3,3,3)] = symbol("X_(3,3,3)");
str_map[sym_gen(X,3,1,1)] = symbol("X_(3,1,1)");
str_map[sym_gen(X,4,4,0)] = symbol("X_(4,4,0)");
str_map[sym_gen(X,4,2,2)] = symbol("X_(4,2,2)");
str_map[sym_gen(X,4,0,4)] = symbol("X_(4,0,4)");
str_map[sym_gen(X,5,3,1)] = symbol("X_(5,3,1)");
str_map[sym_gen(X,5,1,3)] = symbol("X_(5,1,3)");
str_map[sym_gen(X,6,6,0)] = symbol("X_(6,6,0)");
str_map[sym_gen(X,6,4,2)] = symbol("X_(6,4,2)");
str_map[sym_gen(X,6,2,4)] = symbol("X_(6,2,4)");
str_map[sym_gen(X,6,0,0)] = symbol("X_(6,0,0)");
```

```
str_map[sym_gen(X,7,1,3)] = symbol("X_(7,1,3)");
str_map[sym_gen(X,8,0,4)] = symbol("X_(8,0,4)");
str_map[sym_gen(X,9,1,5)] = symbol("X_(9,1,5)");
str_map[sym_gen(X,12,0,6)] = symbol("X_(12,0,6)");

// Read command line for (d,k,l)
size_t d = atoi(argv[1]);
size_t k = atoi(argv[2]);
size_t l = atoi(argv[3]);

// The list of all monomials of degree (d,k,l)
list<ex> *mons = new(list<ex>);

// Hilbert series of the triply-graded polynomial algebra
ex PHS = 1;
for (list<lst>::const_iterator i = gens.begin();
     i != gens.end();
     i++)
{
  PHS = PHS/(1
      -pow(q,i->op(0))
      *pow(s,i->op(1))
      *pow(t,i->op(2))
      *indexed(X,
               idx(i->op(0),infty),
               idx(i->op(1),infty),
               idx(i->op(2),infty)));
}

// get the coeff of q^d s^k t^l in PHS
ex se;
se = PHS.series(q==0,d+1).coeff(q,d);
se = se.series(s==0,k+1).coeff(s,k);
se = se.series(t==0,l+1).coeff(t,l);
se = se.expand();

// separate into terms and push into mons
if (is_a<add>(se))
  for (const_iterator i = se.begin(); i != se.end(); i++)
    mons->push_back(*i);
else
  mons->push_back(se);

// give each monomial a coeffcient A[index] and sum into mono_eqn
// keep track of the coeffcients in vars;
// number of monomials in num_mons
size_t index = 0;
```

```
size_t num_mons = 0;
ex mono_eqn=0;
lst vars;
for (list<ex>::const_iterator i = mons->begin();
     i != mons->end();
     i++)
{
  stringstream ss;
  ss << "A" << index;
  symbol newA(ss.str());
  ss.str("");
  mono_eqn += newA*(*i);
  vars.append(newA);
  index++;
  num_mons++;
}

// substitute each covariant J[d,k,l] into X[d,k,l]
// expand the result and collect coefficients
ex cov_eqn = mono_eqn.subs(cov_map,subs_options::no_pattern);
cov_eqn = cov_eqn.expand();
cov_eqn = cov_eqn.collect(lst(a,b,x0,x1,y0,y1),true);

// remove coefficients, each linear in the A[index]
// push into lin_eqns list and solve
lst lin_eqns;
if (cov_eqn != 0)
  if (is_a<add>(cov_eqn))
    for (const_iterator i = cov_eqn.begin(); i != cov_eqn.end(); i++)
      lin_eqns.append(mycoeff(*i)==0);
  else
    lin_eqns.append(mycoeff(cov_eqn)==0);
ex lin_solns = lsolve(lin_eqns,vars);

// determine a basis for the soln space
list<lst> basis;
for (const_iterator i = lin_solns.begin();
     i != lin_solns.end();
     i++)
{
  if (i->lhs() == i->rhs()) {
    lst v;
    v.append(i->lhs() == 1);
    for (lst::const_iterator j = vars.begin(); j != vars.end(); j++)
      if (*j != i->rhs())
        v.append(ex(*j==0));
    basis.push_back(v);
```

```
    }
  }

  // specialise the rhs of lin_solns for each basis element
  // and then sub this into mono_eqn to recover a syzygy
  for (list<lst>::const_iterator i = basis.begin();
       i != basis.end();
       i++)
  {
    ex syzygy = mono_eqn.subs(lin_solns).subs(*i);
    cout << syzygy.subs(str_map) << ",";
  }
  // A zero to terminate the output.
  cout << "0" << endl;
  return 0;
}
```

B.2 Computing the Hilbert series in Macaulay2

The following transcript demonstrates how to use the syzygy program of the previous section to compute the Hilbert series of $I_{\mathcal{F}}$ using Macaulay2.

```
# The triply-graded polynomial algebra with generators X_(d,k,l)
R = QQ[
    X_(1,3,1),X_(2,4,0),X_(2,2,2),X_(3,5,1),X_(3,3,3),X_(3,1,1),
    X_(4,4,0),X_(4,2,2),X_(4,0,4),X_(5,3,1),X_(5,1,3),X_(6,6,0),
    X_(6,4,2),X_(6,2,4),X_(7,1,3),X_(8,0,4),X_(9,1,5),X_(12,0,6),
    X_(2,0,0),X_(6,0,0),
    Degrees=>{
    {1,3,1},{2,4,0},{2,2,2},{3,5,1},{3,3,3},{3,1,1},{4,4,0},
    {4,2,2},{4,0,4},{5,3,1},{5,1,3},{6,6,0},{6,4,2},{6,2,4},
    {7,1,3},{8,0,4},{9,1,5},{12,0,6},{2,0,0},{6,0,0}
    }];

# Candidate degrees for a minimal generating set of syzygies
F = {{5,7,3},{6,6,2},{6,4,4},{6,6,6},{6,8,4},{6,10,2},{7,5,5},
  {7,7,3},{7,5,3},{7,9,1},{8,6,4},{8,4,4},{8,4,6},{8,8,2},
  {8,6,2},{9,3,5},{9,5,3},{9,7,1},{9,5,7},{9,7,5},{9,9,3},
  {9,11,1},{10,4,6},{10,6,4},{10,8,2},{10,4,4},{10,2,6},{10,6,2},
  {11,3,7},{11,5,5},{11,3,5},{11,5,3},{11,7,3},{11,9,1},{12,2,6},
  {12,4,4},{12,4,8},{12,6,6},{12,8,4},{12,10,2},{12,12,0},{13,3,7},
  {13,5,5},{13,3,5},{13,7,3},{14,2,8},{14,2,6},{14,4,6},{14,6,4},
  {15,1,7},{15,3,9},{15,5,7},{15,7,5},{16,2,8},{16,4,6},{17,1,9},
  {17,3,7},{18,2,10},{18,4,8},{18,6,6},{19,1,9},{21,1,11},
  {24,0,12}};

# Call the external syzygies program to process degree (d,k,l)
syzygies = (X) -> (
    command := concatenate("!./syzygies ",toString(X#0)," ",
                                          toString(X#1)," ",
                                          toString(X#2));
    value concatenate("ideal(", get command,")"));

# compute syzygies for every degree in F
li = apply(F,syzygies);
I_F = ideal(R);
for i in li do
    if i != 0 then
      I_F = I_F + i
I_F = trim I_F;

hs = hilbertSeries I_F;
```

The resulting Hilbert series agrees with (4.7). It takes serveral hours for `Macaulay2` to compute the Hilbert series.

The remainder of this section lists the resulting 104 syzygies of the binary $(3,1)$-forms which minimally generate $I_{\mathcal{G}}$. For typographical reasons, we write X_{dkl} whenever d, k, l are single digits.

$X_{131}^2X_{311} - X_{222}X_{351} + X_{240}X_{333}$
$3X_{222}X_{440} - X_{240}X_{422} - 3X_{131}X_{531}$
$3X_{333}X_{311} - 2X_{222}X_{422} + 3X_{131}X_{513}$
$3X_{351}X_{311} - 2X_{240}X_{422} - 3X_{131}X_{531}$
$X_{222}^2X_{200} - X_{222}X_{422} + X_{240}X_{404} + X_{131}X_{513}$
$4X_{222}^3 + X_{131}^2X_{404} + X_{333}^2$
$3X_{131}^2X_{222}X_{200} - 12X_{240}X_{222}^2 - X_{131}^2X_{422} - 3X_{351}X_{333}$
$X_{131}^2X_{240}X_{200} - 4X_{240}^2X_{222} + X_{131}^2X_{440} - X_{351}^2$
$3X_{222}X_{333}X_{200} - 3X_{333}X_{422} + 3X_{351}X_{404} + 2X_{131}X_{624}$
$X_{222}X_{351}X_{200} - X_{240}X_{333}X_{200} + 3X_{333}X_{440} - X_{351}X_{422} - 2X_{131}X_{642}$
$X_{131}X_{311}^2 - X_{222}X_{531} - X_{240}X_{513}$
$6X_{222}^2X_{311} + X_{333}X_{422} - X_{131}X_{624}$
$6X_{240}X_{222}X_{311} + 6X_{333}X_{440} - X_{351}X_{422} - 3X_{131}X_{642}$
$2X_{240}^2X_{311} + X_{351}X_{440} - X_{131}X_{660}$
$9X_{333}X_{531} - 3X_{351}X_{513} - 6X_{222}X_{642} + 2X_{240}X_{624}$
$X_{222}X_{422}X_{200} - 3X_{131}X_{513}X_{200} - X_{422}^2 + 3X_{440}X_{404} + X_{131}X_{713}$
$3X_{131}X_{311}X_{404} - 3X_{333}X_{513} + 2X_{222}X_{624}$
$3X_{131}X_{311}X_{422} - 6X_{351}X_{513} - 3X_{222}X_{642} + X_{240}X_{624}$
$2X_{131}X_{311}X_{440} - X_{351}X_{531} - X_{222}X_{660} + X_{240}X_{642}$
$36X_{222}X_{311}^2 + 4X_{422}^2 - 3X_{131}X_{713}$
$3X_{240}X_{311}^2 + 3X_{131}^2X_{600} + X_{440}X_{422}$
$3X_{131}X_{222}X_{311}X_{200} - 3X_{351}X_{513} - 3X_{222}X_{642} - X_{240}X_{624}$
$2X_{131}X_{240}X_{311}X_{200} + 3X_{351}X_{531} - 3X_{222}X_{660} - X_{240}X_{642}$
$4X_{311}X_{624} - X_{222}X_{713} + 3X_{131}X_{804}$
$4X_{422}X_{513} - X_{222}X_{713} - 3X_{131}X_{804}$
$6X_{422}X_{531} - 12X_{440}X_{513} - 6X_{311}X_{642} + X_{240}X_{713}36X_{131}X_{222}X_{600} + 24X_{440}X_{513} + 12X_{311}X_{642} + X_{240}X_{713}$
$2X_{131}X_{240}X_{600} - X_{440}X_{531} + X_{311}X_{660}$
$4X_{222}X_{513}X_{200} - 4X_{404}X_{531} - X_{222}X_{713} + X_{131}X_{804}$
$6X_{222}X_{531}X_{200} + 6X_{240}X_{513}X_{200} - 6X_{440}X_{513} - 6X_{311}X_{642} - X_{240}X_{713}$
$X_{131}X_{422}X_{404} + 6X_{222}^2X_{513} + X_{333}X_{624}$
$6X_{131}X_{440}X_{404} + 18X_{222}^2X_{531} + 6X_{240}X_{222}X_{513} + 3X_{333}X_{642} + X_{351}X_{624}$
$6X_{131}^2X_{513}X_{200} + 18X_{222}^2X_{531} - 6X_{240}X_{222}X_{513} - X_{131}^2X_{713} + 3X_{333}X_{642} - X_{351}X_{624}$

$$3X_{131}^2X_{531}X_{200} - 2X_{131}X_{440}X_{422} - 6X_{240}X_{222}X_{531} + 6X_{240}^2X_{513} + 3X_{333}X_{660} - 3X_{351}X_{642}$$
$$X_{131}X_{240}X_{422}X_{200} + X_{131}X_{440}X_{422} - 6X_{240}X_{222}X_{531} - 12X_{240}^2X_{513} - 6X_{333}X_{660} + 3X_{351}X_{642}$$
$$X_{131}X_{240}X_{440}X_{200} + X_{131}X_{440}^2 - 2X_{240}^2X_{531} - X_{351}X_{660}$$
$$8X_{422}X_{624} - 3X_{333}X_{713} - 3X_{131}X_{915}$$
$$18X_{222}X_{642}X_{200} + 6X_{240}X_{624}X_{200} + 72X_{131}X_{333}X_{600} + 36X_{404}X_{660} - 2X_{422}X_{642} + 22X_{440}X_{624} + X_{351}X_{713}$$
$$48X_{333}X_{513}X_{200} - 8X_{222}X_{624}X_{200} - 24X_{404}X_{642} - 9X_{333}X_{713} + 3X_{131}X_{915}$$
$$18X_{351}X_{513}X_{200} - 6X_{240}X_{624}X_{200} - 36X_{131}X_{333}X_{600} - 18X_{404}X_{660} - 8X_{422}X_{642} - 2X_{440}X_{624} - 5X_{351}X_{713}$$
$$X_{351}X_{531}X_{200} + X_{222}X_{660}X_{200} - X_{240}X_{642}X_{200} + 6X_{131}X_{351}X_{600} + X_{422}X_{660} - X_{440}X_{642}$$
$$4X_{422}^2X_{200} - 3X_{131}X_{713}X_{200} - 108X_{222}^2X_{600} - 36X_{531}X_{513} - 9X_{240}X_{804}$$
$$24X_{222}X_{311}X_{513} + X_{333}X_{713} - X_{131}X_{915}$$
$$18X_{240}X_{311}X_{513} - 18X_{131}X_{333}X_{600} - 4X_{422}X_{642} + 2X_{440}X_{624} - X_{351}X_{713}$$
$$36X_{222}X_{311}X_{531} + 4X_{422}X_{642} - 8X_{440}X_{624} + X_{351}X_{713}$$
$$3X_{240}X_{311}X_{531} + 3X_{131}X_{351}X_{600} + X_{422}X_{660}$$
$$X_{311}^2X_{404} + X_{513}^2 - X_{222}X_{804}$$
$$4X_{311}^2X_{422} + 12X_{222}^2X_{600} - 3X_{240}X_{804}$$
$$3X_{131}^2X_{200}X_{600} - 3X_{311}^2X_{440} + X_{440}X_{422}X_{200} - 12X_{240}X_{222}X_{600} + 3X_{531}^2$$
$$4X_{513}X_{624} - 3X_{333}X_{804} - X_{222}X_{915}$$
$$X_{131}X_{311}X_{713} - 4X_{531}X_{624} - 3X_{351}X_{804} - X_{240}X_{915}$$
$$36X_{311}^2X_{513} - 9X_{131}X_{804}X_{200} - 36X_{131}X_{404}X_{600} + X_{422}X_{713}$$
$$12X_{311}^2X_{531} - 4X_{131}X_{422}X_{600} - X_{440}X_{713}$$
$$4X_{311}X_{422}X_{404} - 3X_{333}X_{804} + X_{222}X_{915}$$
$$12X_{311}X_{440}X_{404} - 6X_{513}X_{642} + 6X_{531}X_{624} - 3X_{351}X_{804} + X_{240}X_{915}$$
$$8X_{311}X_{422}^2 + 36X_{222}X_{333}X_{600} - 12X_{531}X_{624} - 18X_{351}X_{804} - 3X_{240}X_{915}$$
$$4X_{131}X_{311}X_{513}X_{200} - 2X_{513}X_{642} - 2X_{531}X_{624} - 3X_{351}X_{804} - X_{240}X_{915}$$
$$X_{131}X_{311}X_{531}X_{200} + 6X_{222}X_{351}X_{600} - 6X_{240}X_{333}X_{600} + 3X_{513}X_{660} + X_{531}X_{642}$$
$$X_{240}X_{311}X_{422}X_{200} + X_{311}X_{440}X_{422} - 6X_{222}X_{351}X_{600} + 12X_{240}X_{333}X_{600} - 6X_{513}X_{660} - 3X_{531}X_{642}$$
$$X_{240}X_{311}X_{440}X_{200} + X_{311}X_{440}^2 + 2X_{240}X_{351}X_{600} - X_{531}X_{660}$$
$$X_{513}X_{713} - 2X_{422}X_{804} - X_{311}X_{915}$$
$$3X_{222}X_{804}X_{200} + 12X_{222}X_{404}X_{600} - X_{422}X_{804} - X_{311}X_{915}$$
$$3X_{240}X_{804}X_{200} + 12X_{240}X_{404}X_{600} + 12X_{131}X_{513}X_{600} + X_{531}X_{713} + 3X_{440}X_{804}$$
$$3X_{131}X_{404}X_{713} + 36X_{222}^2X_{804} + 8X_{624}^2 - 3X_{333}X_{915}$$
$$36X_{222}X_{513}^2 - 36X_{222}^2X_{804} - 4X_{624}^2 + 3X_{333}X_{915}$$

$$8X_{440}X_{422}X_{404} + 36X_{222}X_{531}X_{513} + 12X_{240}X_{222}X_{804} + 4X_{642}X_{624} + X_{351}X_{915}$$
$$4X_{440}X_{422}^2 + 18X_{222}X_{531}^2 + 27X_{240}X_{531}X_{513} - 3X_{131}X_{440}X_{713} + 9X_{240}^2X_{804} + 9X_{351}X_{333}X_{600} + 3X_{660}X_{624}$$
$$3X_{131}X_{222}X_{713}X_{200} + 72X_{222}X_{531}X_{513} - X_{131}X_{422}X_{713} + 36X_{240}X_{222}X_{804} - 3X_{351}X_{915}$$
$$X_{131}X_{240}X_{713}X_{200} - 48X_{440}^2X_{404} - 108X_{222}X_{531}^2 - 96X_{240}X_{531}X_{513} + X_{131}X_{440}X_{713} - 48X_{240}^2X_{804} - 12X_{642}^2 - 16X_{660}X_{624}$$
$$2X_{131}X_{311}X_{660}X_{200} + 12X_{240}^2X_{222}X_{600} - 3X_{240}X_{531}^2 + 3X_{351}^2X_{600} - X_{660}X_{642}$$
$$X_{240}X_{440}^2X_{200} - 4X_{240}^3X_{600} + X_{440}^3 - X_{660}^2$$
$$12X_{311}^4 - 4X_{222}X_{422}X_{600} + 12X_{131}X_{513}X_{600} + 3X_{440}X_{804}$$
$$9X_{333}X_{804}X_{200} + 36X_{333}X_{404}X_{600} - X_{624}X_{713} - X_{422}X_{915} + X_{131}X_{12,0,6}$$
$$8X_{531}X_{624}X_{200} - 3X_{351}X_{804}X_{200} + 2X_{240}X_{915}X_{200} - 36X_{351}X_{404}X_{600} + 12X_{131}X_{624}X_{600} - X_{642}X_{713} + 3X_{440}X_{915}$$
$$18X_{222}X_{311}X_{804} + X_{624}X_{713} - X_{422}X_{915}$$
$$12X_{240}X_{311}X_{804} - 3X_{351}X_{804}X_{200} - 12X_{351}X_{404}X_{600} - 12X_{131}X_{624}X_{600} - X_{642}X_{713} - X_{440}X_{915}$$
$$X_{311}^2X_{713} + 12X_{222}X_{513}X_{600} + 3X_{531}X_{804}$$
$$36X_{311}X_{513}^2 + X_{624}X_{713} - X_{131}X_{12,0,6}$$
$$12X_{311}X_{531}^2 + 24X_{333}X_{440}X_{600} - 8X_{351}X_{422}X_{600} - 12X_{131}X_{642}X_{600} - X_{660}X_{713}$$
$$3X_{624}X_{804} - 3X_{513}X_{915} + X_{222}X_{12,0,6}$$
$$36X_{311}^2X_{804} - 12X_{422}X_{804}X_{200} - 48X_{422}X_{404}X_{600} + X_{713}^2$$
$$3X_{311}X_{404}X_{713} - 3X_{513}X_{915} + 2X_{222}X_{12,0,6}$$
$$X_{311}X_{422}X_{713} + 72X_{333}X_{513}X_{600} - 36X_{222}X_{624}X_{600} + 9X_{642}X_{804} - 6X_{531}X_{915} - 2X_{240}X_{12,0,6}$$
$$X_{311}X_{440}X_{713} + 12X_{351}X_{513}X_{600} - 4X_{240}X_{624}X_{600} + 3X_{660}X_{804}$$
$$3X_{131}X_{311}X_{804}X_{200} + 36X_{333}X_{513}X_{600} - 24X_{222}X_{624}X_{600} + 3X_{642}X_{804} - 3X_{531}X_{915} - X_{240}X_{12,0,6}$$
$$3X_{222}X_{311}X_{713}X_{200} + 72X_{333}X_{513}X_{600} - 12X_{222}X_{624}X_{600} + 9X_{642}X_{804} - 3X_{531}X_{915} - 2X_{240}X_{12,0,6}$$
$$18X_{513}X_{804}X_{200} + 72X_{404}X_{513}X_{600} - 3X_{713}X_{804} - 2X_{311}X_{12,0,6}$$
$$X_{422}X_{404}X_{713} + 18X_{222}X_{513}X_{804} + X_{624}X_{915}$$
$$4X_{422}^2X_{713} - 6X_{440}X_{404}X_{713} - 3X_{131}X_{713}^2 + 36X_{240}X_{513}X_{804} + 72X_{333}X_{624}X_{600} + 6X_{642}X_{915} + 2X_{351}X_{12,0,6}$$
$$36X_{513}^3 - 36X_{222}X_{513}X_{804} - X_{624}X_{915} + X_{333}X_{12,0,6}$$
$$6X_{131}^2X_{713}X_{600} - 36X_{531}^2X_{513} + 2X_{440}X_{422}X_{713} - 36X_{240}X_{531}X_{804} + 12X_{351}X_{624}X_{600} + 3X_{660}X_{915}$$
$$3X_{131}X_{311}X_{915}X_{200} - 3X_{440}X_{404}X_{713} + 18X_{240}X_{513}X_{804} - 3X_{642}X_{915} - 2X_{351}X_{12,0,6}$$
$$18X_{513}X_{915}X_{200} - 6X_{222}X_{12,0,6}X_{200} + 72X_{404}X_{624}X_{600} - 3X_{713}X_{915} + 2X_{422}X_{12,0,6}$$
$$108X_{311}X_{513}X_{804} + 3X_{713}X_{915} - 4X_{422}X_{12,0,6}$$
$$24X_{222}X_{624}X_{200}X_{600} + 36X_{311}X_{531}X_{804} + 6X_{531}X_{915}X_{200} + 2X_{240}X_{12,0,6}X_{200} - 9X_{333}X_{713}X_{600} + 15X_{131}X_{915}X_{600} + 2X_{440}X_{12,0,6}$$
$$18X_{311}X_{404}X_{804}X_{200} + 72X_{311}X_{404}^2X_{600} - 3X_{804}X_{915} + 2X_{513}X_{12,0,6}$$
$$12X_{311}X_{422}X_{804}X_{200} + 6X_{311}^2X_{915}X_{200} - X_{311}X_{713}^2 + 18X_{333}X_{804}X_{600} + 6X_{222}X_{915}X_{600} + 2X_{531}X_{12,0,6}$$

$$108X_{513}^2X_{804} - 108X_{222}X_{804}^2 - 3X_{915}^2 + 4X_{624}X_{12,0,6}$$
$$X_{404}X_{713}^2 + 36X_{222}X_{804}^2 + X_{915}^2$$
$$6X_{311}X_{422}X_{915}X_{200} - 4X_{131}X_{311}X_{12,0,6}X_{200} - 108X_{222}^2X_{804}X_{600} + 27X_{240}X_{804}^2 - 24X_{624}^2X_{600} + 36X_{333}X_{915}X_{600} + 2X_{642}X_{12,0,6}$$
$$54X_{311}X_{804}^2 + 9X_{804}X_{915}X_{200} + 36X_{404}X_{915}X_{600} - X_{713}X_{12,0,6}$$
$$9X_{404}X_{713}X_{804}X_{200} + 36X_{404}^2X_{713}X_{600} + 54X_{513}X_{804}^2 + X_{915}X_{12,0,6}$$
$$81X_{404}X_{804}^2X_{200}^2 + 648X_{404}^2X_{804}X_{200}X_{600} + 1296X_{404}^3X_{600}^2 + 81X_{804}^3 + X_{12,0,6}^2$$

$$48X_{440}X_{422}X_{804}X_{200} + 24X_{311}X_{440}X_{915}X_{200} + 576X_{240}X_{513}^2X_{600} + 12X_{131}X_{422}X_{713}X_{600}$$
$$-432X_{240}X_{222}X_{804}X_{600} - X_{440}X_{713}^2 + 108X_{531}^2X_{804} - 48X_{642}X_{624}X_{600} + 36X_{351}X_{915}X_{600} + 8X_{660}X_{12,0,6}$$

References

[BB05] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR **2133266 (2006d:**05001)

[Blo79] David M. Bloom, *Linear algebra and geometry*, Cambridge University Press, Cambridge, 1979. MR **643240 (83a:**15001)

[Bri87] Michel Brion, *Sur l'image de l'application moment*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986), Lecture Notes in Math., vol. 1296, Springer, Berlin, 1987, pp. 177–192 (French). MR **932055 (89i:**32062)

[CM93] David H. Collingwood and William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993. MR **1251060 (94j:**17001)

[Đok98] Dragomir Ž. Đoković, *Explicit Cayley triples in real forms of G_2, F_4 and E_6*, Pacific J. Math. **184** (1998), no. 2, 231–255. MR **1628587 (99e:**17005)

[Dol03] Igor Dolgachev, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series, vol. 296, Cambridge University Press, Cambridge, 2003. MR **2004511 (2004g:**14051)

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[Ful97] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry. MR **1464693 (99f:**05119)

[GiN] *GiNac, an open framework for symbolic computation within the C++ programming language.*

[Gra] Daniel R. and Stillman Grayson Michael E., *Macaulay 2, a software system for research in algebraic geometry.*

[GW98] Roe Goodman and Nolan R. Wallach, *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications, vol. 68, Cambridge University Press, Cambridge, 1998. MR **1606831 (99b:**20073)

[Hec82] G. J. Heckman, *Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups*, Invent. Math. **67** (1982), no. 2, 333–356. MR **665160 (84d:**22019)

[Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original. MR **1834454 (2002b:**53081)

[Hes80] Wim H. Hesselink, *Characters of the nullcone*, Math. Ann. **252** (1980), no. 3, 179–182. MR **593631 (82c:**17004)

[How95] Roger Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proc., vol. 8, Bar-Ilan Univ., Ramat Gan, 1995, pp. 1–182. MR **1321638 (96e:**13006)

[Hum78] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978. Second printing, revised. MR **499562 (81b:**17007)

[Hum90] ______, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR **1066460 (92h:**20002)

[Joh76] Kenneth D. Johnson, *Composition series and intertwining operators for the spherical principal series. II*, Trans. Amer. Math. Soc. **215** (1976), 269–283. MR 0385012 (52 #5882)

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, 2nd ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002. MR **1920389 (2003c:**22001)

[Kos59] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032. MR 0114875 (22 #5693)

[KR71] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809. MR 0311837 (47 #399)

[Olv99] Peter J. Olver, *Classical invariant theory*, London Mathematical Society Student Texts, vol. 44, Cambridge University Press, Cambridge, 1999. MR **1694364 (2001g:**13009)

[Pap98] Yannis Y. Papageorgiou, SL_2*, the cubic and the quartic*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 1, 29–71 (English, with English and French summaries). MR **1614910 (99f:**20071)

[Pea82] G. Peano, *Formazioni invariantive delle corrispondenze*, Giornale Mat. Univ. Ital. **20** (1882), 79–100.

[Sam90] Hans Samelson, *Notes on Lie algebras*, 2nd ed., Universitext, Springer-Verlag, New York, 1990. MR **1056083 (91h:**17006)

[Smi04] A. V. Smirnov, *Decomposition of symmetric powers of irreducible representations of semisimple Lie algebras, and the Brion polytope*, Tr. Mosk. Mat. Obs. **65** (2004), 230–252 (Russian, with Russian summary); English transl., Trans. Moscow Math. Soc. (2004), 213–234. MR **2193441 (2006i:**17011)

[Tod46a] J. A. Todd, *The complete system of the binary (3,1) form*, Proc. Cambridge Philos. Soc. **42** (1946), 196–205. MR 0017270 (8,129g)

[Tod46b] ______, *The complete system of the binary (4,1) form*, Proc. Cambridge Philos. Soc. **42** (1946), 206–216. MR 0017271 (8,129h)

[Vin89] Ernest B. Vinberg, *Linear representations of groups*, Basler Lehrbücher [Basel Textbooks], vol. 2, Birkhäuser Verlag, Basel, 1989. Translated from the Russian by A. Iacob. MR **1013789 (90h:**22001)

[WW00] N. R. Wallach and J. Willenbring, *On some q-analogs of a theorem of Kostant-Rallis*, Canad. J. Math. **52** (2000), no. 2, 438–448. MR **1755786 (2001j:**22020)

[Wal05] Nolan R. Wallach, *The Hilbert series of measures of entanglement for 4 qubits*, Acta Appl. Math. **86** (2005), no. 1-2, 203–220. MR **2134319 (2006c:**81020)

[Tur22] H. W. Turnbull, *Gordan's theorem for double binary forms*, Proc. Edinburgh Math Soc. **41** (1922/23), 116–127.

[Slo] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences.*

Curriculum Vitae

Anthony Paul van Groningen
Place of Birth: Kew, Victoria, Australia
Education

- Doctor of Philosophy in Mathematics, May 2007
  University of Wisconsin—Milwaukee
  Dissertation Title: Graded multiplicities of the nullcone
  for the algebraic symmetric pair of type G
- Master of Science in Mathematics, May 2000
  University of Wisconsin—Milwaukee
- Bachelor of Science in Applied Science, May 1998
  Major: Computer Science; Minor: Mathematics
  University of Wisconsin—Milwaukee

Major Professor Date

Major Professor Date