

MULTIVARIATE HILBERT SERIES OF LATTICE CONES
AND HOMOGENEOUS VARIETIES

by

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A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY
in
MATHEMATICS

at

The University of Wisconsin–Milwaukee

August 2015

ABSTRACT

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The University of Wisconsin-Milwaukee, 2015
Under the Supervision of Dr. Jeb F. Willenbring

We consider the dimensions of irreducible representations whose highest weights lie on a given finitely generated lattice cone. We present a rational function representing the multivariate formal power series whose coefficients encode these dimensions. This result generalizes the formula for the Hilbert series of an equivariant embedding of an homogeneous projective variety. We use the multivariate generating function to compute Hilbert series for the Kostant cones and other affine and projective varieties of interest in representation theory. As a special case, we show how the multivariate series can be used to compute the Hilbert series of the three classical families of determinantal variety.

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ACKNOWLEDGMENTS

I would first like to thank my advisor, Dr. Jeb F. Willenbring. He has been an excellent teacher and mentor to me from my first year in the Masters program to today. This work would not have been possible without his patience, guidance, and support. I would also like to thank the rest of my committee: Drs. Allen Bell, Craig Guilbault, Peter Hinow, and Yi Ming Zou. Thank you for taking the time to read this dissertation.

I am grateful to all the wonderful professors and teachers whose courses have brought me to this point in my education. In particular, I would like to thank Drs. David Ellis and Benjamin Newton of Beloit College for supporting me and teaching me how beautiful pure mathematics can be. Without your help and support, I would never have decided that a career in pure mathematics was for me. I am lucky enough to have had the opportunity to attend courses given by many passionate and exemplary instructors, both in and out of the mathematics curriculum, who have set the example for me as I teach my own courses.

Finally, I would like to thank my friends, family, and fellow graduate students for all their help and support. I could not have done this without you.

To Josh, Rachel, Amy, Renee, Nick, and Amanda. In the order I met you.

Chapter 1

Introduction

Let G be a semisimple linear algebraic group over \mathbb{C} . The dimension of any finite dimensional irreducible representation of G is given by the Weyl Dimension Formula. We present a suitable rational expression for the multivariate power series which encodes in its coefficients the dimensions of irreducible representations of G whose highest weights lie in a finitely generated lattice cone in $P_+(\mathfrak{g})$.

When the cone is generated by a single dominant integral weight, the generating function is the Hilbert series for an equivariant embedding of G/P into a projective space for some parabolic subgroup P . In fact, this case describes the Hilbert series for all such equivariant embeddings. A closed form for these Hilbert series was established by Gross and Wallach in [12]. Note that these embeddings include the most important projective embeddings of homogeneous projective varieties, including the Segre embedding of products of projective spaces, the Veronese embeddings of projective space, and the Plücker embeddings of Grassmannians. Another important variety whose coordinate ring behaves much like that of G/P is the closure of the G -orbit of a highest weight vector in a finite dimensional irreducible representation of G . The coordinate ring was studied in this sense by Vinberg and Popov in [24], and has a decomposition into irreducible highest weight representations similar to that of G/P .

We study the case of a general finitely generated lattice cone, and show that there is a multivariate extension of the results in [12] and [24]. This extension explicitly

describes multivariate Hilbert series on many of the most interesting homogeneous varieties. As a special case, we recover the closed form for the Hilbert series of an equivariant embedding of G/P given in [12]. We also obtain explicit rational functions for the multivariate Hilbert series on any variety whose coordinate ring decomposes as a G -representation over a finitely generated lattice cone, such as the symmetric, antisymmetric, and standard determinantal varieties.

In addition to these varieties, we obtain a closed form for the multivariate Hilbert series of the Kostant cones, which generalize the orbit of a single highest weight vector. These varieties go back to Kostant, who proved that their ideals are generated by quadratic elements. See [11], [18], [19], and the upcoming book by Wallach on the subject [25]. We give a proof that the coordinate ring of a Kostant cone has a multiplicity-free decomposition as a G -representation over a particular finitely generated lattice cone in $P_+(\mathfrak{g})$.

1.1 Description of main result

Here, we briefly describe the main theorem in the dissertation. Let G be a semisimple algebraic group, and assume X is a homogeneous projective variety. In [12], Gross and Wallach computed an explicit rational function describing the Hilbert series of X that holds in any embeddings of X into projective space using the Weyl Dimension Formula. We recall their result here.

Assume $T \subset B \subset G$ is a fixed choice of maximal torus and Borel subgroup inside G . Let $L(\lambda)$ be an irreducible highest weight representation of G , and assume that P is the parabolic subgroup of G stabilizing the unique hyperplane in $L(\lambda)$ fixed by B . Then the homogeneous projective variety G/P embeds in the projective space $\mathbb{P}(L(\lambda))$ of hyperplanes in $L(\lambda)$ via the map

$$\pi_\lambda : G/P \rightarrow \mathbb{P}(L(\lambda))$$

given by $\pi_\lambda(gP) = g.H$. Then the coordinate ring of $\pi_\lambda(G/P)$ is \mathbb{N} -graded. In particular,

$$\mathbb{C}[\pi_\lambda(G/P)] = \bigoplus_{n \geq 0} L(n\lambda),$$

where $\mathbb{C}[\pi_\lambda(G/P)]$ is the homogeneous coordinate ring of $\pi_\lambda(G/P)$. The Hilbert series of $\pi_\lambda(G/P)$ is then given by

$$HS_{\pi_\lambda}(q) = \sum_{n \geq 0} \dim(L(n\lambda))q^n.$$

Gross and Wallach computed the following rational function expressing the Hilbert series for this embedding. Note that since all embeddings of homogeneous projective varieties are of the form discussed above, this rational expression represents the Hilbert series for any homogeneous projective variety.

Theorem (Gross and Wallach). A closed form for the Hilbert series of the embedding π_λ of G/P in $\mathbb{P}(L(\lambda))$ is given by

$$\prod_{\alpha \in \Phi^+} \left(c_\lambda(\alpha)q \frac{d}{dq} + 1 \right) \frac{1}{1-q},$$

where $c_\lambda(\alpha) := \frac{(\lambda, \alpha)}{(\rho, \alpha)}$, ρ is the sum of the dominant integral weights of \mathfrak{g} , and (\cdot, \cdot) is the bilinear form on \mathfrak{h}^* induced by the Killing form on \mathfrak{g} .

Forgetting the geometric motivation for a moment, the formal power series $HS_{\pi_\lambda}(q)$ encodes in its coefficients the dimensions of irreducible highest weight representations of G lying along the ray generated by λ , $\mathbb{N}\lambda$ in the dominant chamber of the weight lattice $P_+(\mathfrak{g})$. We present a generalization of this. Consider all irreducible highest weight representations of G whose highest weights lie on the finitely generated lattice cone $\langle \lambda_1, \dots, \lambda_k \rangle := \mathbb{N}\lambda_1 \oplus \dots \oplus \mathbb{N}\lambda_k$ in $P(\mathfrak{g})$. The multivariate formal power series

$$HS_{\mathfrak{q}}\langle \lambda_1, \dots, \lambda_k \rangle := \sum_{\mathbf{a} \in \mathbb{N}^k} \dim(L(a_1\lambda_1 + \dots + a_k\lambda_k))\mathfrak{q}^{\mathbf{a}},$$

where $\mathfrak{q}^{\mathbf{a}}$ encodes in its coefficients the dimensions of the irreducible highest weight representations of G whose highest weight lies somewhere in the cone $\langle \lambda_1, \dots, \lambda_k \rangle$. Then we prove that the theorem proved in [12] generalizes to the following (keeping the numbering it appears with later in the dissertation).

Theorem 4.2.1. Let $\lambda_1, \dots, \lambda_k$ be dominant integral weights. Then

$$HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle = \prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_1}(\alpha)q_1 \frac{\partial}{\partial q_1} + \dots + c_{\lambda_k}(\alpha)q_k \frac{\partial}{\partial q_k} \right) \prod_{i=1}^k \frac{1}{1 - q_i},$$

where $c_{\lambda}(\alpha) := \frac{(\lambda, \alpha)}{(\rho, \alpha)}$.

An explanation for the notation in Theorem 4.2.1 can be found in §2.5.

Returning to the geometry, for specific choices of $\lambda_1, \dots, \lambda_k$, the series $HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle$ describes the multivariate Hilbert series of the most interesting varieties from the perspective of representation theory. We spend a good deal of time going through examples in detail, including using Theorem 4.2.1 to find explicit rational expressions for both multivariate and single variable Hilbert series on the three classical types of determinantal varieties.

1.2 Note on organization

The dissertation is organized into five chapters, the first of which is introductory in nature. Chapter 2 describes the necessary preliminary results needed to understand the proof of Theorem 4.2.1. In Chapter 3, we present results from Classical Invariant Theory intended to motivate the examples covered in Chapter 5. Chapter 4 contains an overview of the geometric setting of partial flag varieties and a proof of Theorem 4.2.1. In Chapter 5, we discuss specific examples and specializations of Theorem 4.2.1 to compute Hilbert series (both singly- and multi-graded) of many classes of varieties of interest in representation theory.

Chapter 2

Preliminaries

In this chapter, we present the basic results needed throughout the dissertation. There are many great references for these results, including [4], [9], [10], and [21]. Sometimes we will want a more specific reference. Good references on the structure theory of semisimple Lie algebras are [14] and [16]. For a readable introduction to algebraic geometry and linear algebraic groups, see [15]. We also use this chapter as a way to introduce the notation that will be standard throughout the dissertation.

We begin with a section describing basic facts about linear algebraic groups and certain special subgroups. Then follows a section on assigning a Lie algebra to an algebraic group and discussing the correspondence between groups and algebras. Sections 2.3, 2.4, and 2.5 discuss three fundamental theorems in the representation theory of linear algebraic groups, which will be used throughout the dissertation: the Theorem of the Highest Weight, the Borel-Weil Theorem, and the Weyl Dimension Formula. We then switch gears from representation theory and conclude with a brief discussion of multivariate Hilbert series in algebraic geometry.

2.1 Linear algebraic groups

Throughout this dissertation, we assume that G is a linear algebraic group over \mathbb{C} , unless explicitly stated otherwise. By a linear algebraic group, we mean a Zariski closed subgroup of $GL(n, \mathbb{C})$. For example, the special linear group $SL(n, \mathbb{C})$ is the zero locus of the polynomial $\det(g) - 1$, where $\det(g)$ is the determinant of the matrix

$g \in GL(n, \mathbb{C})$, which is polynomial in the matrix entries of g . In fact, it is often a good idea to keep this example in mind when trying to understand the definitions in this chapter.

We begin by defining some special subgroups of G . These classes of subgroups will both allow us to discuss many of the algebraic properties of G and provide us with the most interesting algebro-geometric examples of quotients of G .

Definition 2.1.1. A *Borel subgroup*, $B \subset G$, is a maximal solvable subgroup of G .

Note that a Borel subgroup is not necessarily normal in G . Nevertheless, the quotient G/B , which can be given the structure of a projective variety, is intensely studied. The geometry of G/B is closely related to the representation theory of G . In the case where $G = SL(n, \mathbb{C})$ and B is the subgroup of upper triangular matrices, B is the stabilizer of the full flag

$$0 \subset \langle e_1 \rangle \subset \cdots \subset \langle e_1, \dots, e_{n-1} \rangle \subset \mathbb{C}^n,$$

where e_1, \dots, e_n are the standard basis vectors of \mathbb{C}^n , under the standard action of $SL(n, \mathbb{C})$ on \mathbb{C}^n . For this reason, G/B is often called *the full flag variety*, or just *the flag variety*, even when G is not assumed to be the special linear group. Note also that all Borel subgroups B of G are conjugate. In particular, they are all isomorphic as algebraic groups.

Definition 2.1.2. A subgroup, $P \subset G$, is called *parabolic* if it contains a Borel subgroup.

In the case where P is a parabolic subgroup of G , the quotient G/P can be given the structure of a projective variety. In fact, the converse is also true. If H is a subgroup of G and G/H can be given the structure of a projective variety, then H must have been a parabolic subgroup of G (cf., p.384 in [9]). As a reminder, a parabolic subgroup P is not necessarily normal in G . Therefore, the quotient G/P does not have the structure of a group. However, since G/P is a projective variety, it comes equipped with quite a bit of geometric structure, including a homogeneous coordinate ring and a Hilbert series. Determining the Hilbert series of G/P as a

rational function is one of the main motivations behind this dissertation. We discuss Hilbert series in more depth in §2.5.

The varieties G/P are called *partial flag varieties* or sometimes just *flag varieties*. The motivation behind these terms is similar to that of G/B . Assume G is the special linear group $SL(n, \mathbb{C})$ and let B be the subgroup of upper triangular matrices as before. Then any parabolic subgroup $P \supset B$ of G fixes a partial flag

$$V_0 \subset V_1 \subset \cdots \subset V_k \text{ in } \mathbb{C}^n,$$

under the standard action of $SL(n, \mathbb{C})$ on \mathbb{C}^n .

Definition 2.1.3. A *unipotent* subgroup, $U \subset G$, is a subgroup of G consisting of matrices of the form $I + N$ for some nilpotent matrix N .

Definition 2.1.4. An *algebraic torus*¹, T , in G is a subgroup isomorphic to $(\mathbb{C}^\times)^l$ for some $l \in \mathbb{N}$. The integer l is called the *rank* of T .

Maximal toral subgroups play an important role in the representation theory of G , which we will discuss more thoroughly in §2.3. One of the nice facts about Borel subgroups is that every Borel subgroup is the product of a maximal torus and a unipotent subgroup. In other words, if B is a Borel subgroup, then $B = T \cdot U$, where T is a maximal torus in G and U is a maximal unipotent subgroup of G . If B , T , and U are as above, then we have a natural map

$$\pi : G/U \rightarrow G/B.$$

Note that, since $B = T \cdot U$, the fibres of this map are isomorphic as varieties to T . In particular, $\pi^{-1}(1B) = \{tU \mid t \in T\}$. Actually, something much stronger is true. Under the map π , G/U has the structure of a torus bundle over G/B .

2.2 Lie algebras of linear algebraic groups

In this section, we discuss the basic definitions and facts necessary to introduce the dictionary between the category of finite dimensional (real) Lie algebras and the

¹A group that is an algebraic torus is sometimes referred as a *toral group* or simply a *torus*. We will use these terms interchangeably to mean an algebraic torus.

category of simply connected (real) Lie groups. This dictionary does not completely hold in the setting of linear algebraic groups. We will discuss when the correspondence fails, and why this failure does not pose too much of a problem for the study of algebraic groups. On the way, we will discuss the tangent space of an algebraic group and use it to define the Lie algebra, \mathfrak{g} , of a linear algebraic group, G . A readable introduction to the basics of Lie algebras can be found in [14], and a more in depth discussion of the process of assigning a finite dimensional Lie algebra to a linear algebraic group can be found in [15].

2.2.1 Lie algebras

Let L be a vector space over a field K . For the sake of simplicity, assume that K has characteristic zero.

Definition 2.2.1. L , equipped with a bilinear operation $[\cdot, \cdot]$ called the *bracket* of L , is called a *Lie algebra* over K if the following conditions are satisfied:

1. the bracket operation is K -bilinear,
2. $[X, Y] = -[Y, X]$ for all $X, Y \in L$, and
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, for all $X, Y, Z \in L$.

Property 2 says that the bracket operation is anti-commutative, and Property 3 is referred to as the *Jacobi identity*. Note that the bracket operation is, in general, neither commutative nor associative. In this way, the bracket operation does not give L the structure of a ring.

Any associative algebra A over K can be made into a Lie algebra by defining a bracket operation on A by the commutator: $[X, Y] = XY - YX$. In this way, we are led to our first example of a Lie algebra. Let $M_n(\mathbb{C})$ be the vector space of all $n \times n$ complex matrices. This space is an associative algebra under matrix multiplication. When we wish to refer to the Lie algebra structure on $M_n(\mathbb{C})$, we denote it by $\mathfrak{gl}(n, \mathbb{C})$. This is the Lie algebra of $n \times n$ matrices under the commutator bracket. Many of the most interesting examples of finite dimensional Lie algebras

can be realized as subalgebras of $\mathfrak{gl}(n, \mathbb{C})$. From now on, we will assume that K is the field of complex numbers, unless stated otherwise.

Definition 2.2.2. A vector subspace W in a Lie algebra L is called a *Lie subalgebra* if W is closed under the bracket operation: $[X, Y] \in W$ for all $X, Y \in W$. W is called an *ideal* in L if it is closed under the bracket operation by all elements of L : $[X, Y] \in W$ for all $X \in L, Y \in W$.

Note that if W is an ideal in L , then the quotient vector space L/W is naturally a Lie algebra under the bracket $[X + W, Y + W] := [X, Y] + W$.

A Lie algebra L is called *abelian* if $[X, Y] = 0$ for all $X, Y \in L$. In this case, the bracket of L is called *trivial*. Note that any vector space V can be made into a Lie algebra by assigning the trivial bracket to V . A non-abelian Lie algebra is called *simple* if it contains no ideals, save itself and the zero subspace.

Simple Lie algebras are, in a sense, the building blocks of the category of Lie algebras we will be studying further in this section. A Lie algebra L is called *semisimple* if it can be decomposed

$$L = L_1 \oplus \cdots \oplus L_k$$

into a direct sum of simple ideals L_i .

Definition 2.2.3. A linear map $T : L_1 \rightarrow L_2$ between Lie algebras L_1 and L_2 is called a *Lie algebra homomorphism* if T preserves the bracket operations on L_1 and L_2 , ie. $T([X, Y]_{L_1}) = [T(X), T(Y)]_{L_2}$.

We then have the following theorem on semisimple Lie algebras. This is Corollary 5.2 of [14].

Theorem 2.2.4. *If L is semisimple, then $L = [L, L]$, and all ideals and homomorphic images of L are semisimple.*

In this way, we may think of the finite dimensional, semisimple Lie algebras as a full subcategory of the category of finite dimensional Lie algebras. The representation theory of this category of Lie algebras is particularly nice when $K = \mathbb{C}$.

When $K = \mathbb{R}$, there is an equivalence of categories between the finite dimensional semisimple Lie algebras and a certain category of real Lie groups. We will revisit both of these settings later in this chapter.

2.2.2 The tangent space of an algebraic group

Our motivation for studying Lie algebras in this dissertation is in using them as a tool to understand the representation theory of complex linear algebraic groups. Therefore, we wish to understand the relationship between an algebraic group and its Lie algebra. In order to do so, we define the notions of tangent spaces and vector fields in the category of algebraic groups. This subsection draws heavily from Chapters I and III in [15]. This subsection assumes some familiarity with elementary concepts from algebraic geometry. Some good sources for this background are [3] and [15].

Let X be an affine algebraic variety. We wish to introduce the notion of a tangent space at a point $x \in X$. We will then show that the tangent space at the identity of a linear algebraic group naturally has the structure of a Lie algebra. In this way, we are able to mimic the Lie group case in the algebraic category. We begin with an extrinsic definition.

Definition 2.2.5. Assume $X \subset \mathbb{A}^n(\mathbb{C})$ is an affine variety defined by the vanishing of the polynomials $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$. Fix a point $a \in X$. Set

$$d_a f := \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i).$$

The *geometric tangent space* of X at the point a is defined to be the linear variety in $\mathbb{A}^n(\mathbb{C})$ defined by the vanishing of all $d_a f$, for all f in the ideal of X . Denote the geometric tangent space of X at a as $Tan(X)_a$.

It can be shown that for any finite set f_1, \dots, f_k of generators for the ideal of X , $Tan(X)_a$ is defined by the vanishing of the corresponding linear functionals $d_a f_1, \dots, d_a f_k$. We now wish to give an intrinsic formulation of the geometric tangent space at a point in a variety.

Let M be the maximal ideal of the coordinate ring $\mathbb{C}[X]$ given by polynomial functions which vanish at the point x . The quotient ring $\mathbb{C}[X]/M$ can be identified with \mathbb{C} , and thus, the quotient M/M^2 is a complex vector space. Since M is finitely generated as a $\mathbb{C}[X]$ -module, the vector space M/M^2 is finite dimensional. Now, let \mathcal{O}_a be the localization $\mathbb{C}[X]_M$, with maximal ideal $\mathcal{M}_a := M\mathbb{C}[X]_M$. There is a canonical isomorphism of from M/M^2 to $\mathcal{M}_a/\mathcal{M}_a^2$ induced by the inclusion $\mathbb{C}[X] \rightarrow \mathbb{C}[X]_M$. We then define the tangent space of X at a point $a \in X$ to be the dual space $(\mathcal{M}_a/\mathcal{M}_a^2)^*$ over the field $\mathbb{C} = \mathcal{O}_a/\mathcal{M}_a$. It can be shown that this space can be explicitly identified with $Tan(X)_a$.

The space $(\mathcal{M}_a/\mathcal{M}_a^2)^*$ does not depend on any embedding in affine space for its definition. This formulation of the tangent space is then intrinsic to the variety X . From now on, we will set $Tan(X)_a = (\mathcal{M}_a/\mathcal{M}_a^2)^*$.

We return to the case where $X = G$ is a linear algebraic group over \mathbb{C} , and consider the space $Tan(X)_e$, where e is the identity in G . We wish to have a natural assignment of a finite dimensional Lie algebra to the algebraic group G . Following what we know for the case of real Lie groups, we wish to identify the tangent space at the identity as a Lie algebra in a functorial way. To this end, let $\mathbb{C}[G]$ be the coordinate ring of G . G acts on $\mathbb{C}[G]$ via $g.f(h) := f(g^{-1}h)$. This action extends to the space $Der(G)$ of all derivations of $\mathbb{C}[G]$. The commutator bracket $[f, g] = fg - gf$ of two derivations $f, g \in Der(G)$ is again a derivation. Thus, $Der(G)$ is a Lie algebra. However, this algebra is not finite dimensional. Note, the product of two derivations is not necessarily a derivation. The Lie algebra structure on $Der(G)$ is therefore the natural ‘multiplication’ structure to consider.

The action of G on $\mathbb{C}[G]$ extends to an action on the algebra $Der(G)$ of derivations on $\mathbb{C}[G]$. Let \mathfrak{g} be the set of all *left invariant* derivations on $\mathbb{C}[G]$. The bracket of two left invariant derivations is again left invariant. Thus \mathfrak{g} is a Lie subalgebra of $Der(G)$. We call \mathfrak{g} the *Lie algebra of G* . For any homomorphism $\phi : G_1 \rightarrow G_2$ of linear algebraic groups, we have a natural linear map $d\phi_{e_1} : Tan(G_1)_{e_1} \rightarrow Tan(G_2)_{e_2}$, where e_1, e_2 are the identity elements of G_1, G_2 , resp., and $d\phi_{e_1}$ is the differential of the map ϕ at the identity. The following theorem (Theorem 9.1 in [15]) guarantees

that \mathfrak{g} is the natural Lie algebra to assign to G .

Theorem 2.2.6. *Let G be a linear algebraic group, and let $\text{Tan}(G)_e$ be the tangent space to G at the identity. Then $\text{Tan}(G)_e$ and \mathfrak{g} are naturally isomorphic. In particular, \mathfrak{g} is finite dimensional and $\dim(G) = \dim(\mathfrak{g})$. If $\phi : G_1 \rightarrow G_2$ is a regular homomorphism of algebraic groups, then the map $d\phi_e : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a homomorphism of Lie algebras.*

Note that the differential $d\phi_{e_1}$ behaves functorially. Thus, Theorem 2.2.6 guarantees that, to every linear algebraic group G , we can naturally assign a finite dimensional Lie algebra \mathfrak{g} , which is the tangent space of G at the identity.

2.2.3 Correspondence between groups and Lie algebras

We now turn to the question of whether the assignment of a Lie algebra to a linear algebraic group from the previous subsection is ‘unique’, in the sense that, under a suitable restriction of the category of groups we consider, we can reverse the assignment in a natural way. In other words, we want to recall the equivalence of categories from a suitable subcategory of algebraic groups to a the category of finite dimensional (complex) Lie algebras. We recall the situation for real Lie groups before discussing why this process does not work for algebraic groups. However, the natural assignment from the previous subsection will prove good enough for the purpose of discussing the representation theory of algebraic groups in terms of the representation theory of their Lie algebras.

Let G be a finite dimensional Lie group. Then the tangent space at the identity is identified with the finite dimensional real Lie algebra of left invariant vector fields on G . If we assume that the objects in the category of Lie groups are simply connected, then there is an equivalence of categories between the category of finite dimensional, (real) Lie groups and the category finite dimensional real Lie algebras. In particular, there is an equivalence of categories between the category of finite dimensional, semisimple, simply connected real Lie groups and the category of finite dimensional semisimple real Lie algebras. This is Theorem 3.28 in [26]. The group

assigned by this correspondence to a finite dimensional Lie algebra \mathfrak{g} is called the *adjoint group* of \mathfrak{g} .

We could naively hope that there is a suitable category of linear algebraic groups under which a similar result holds. To this end, we make the following definition.

Definition 2.2.7. A connected algebraic group is called *semisimple* if it contains no closed connected abelian normal subgroup except the trivial subgroup.

The following theorem gives the relationship between semisimple algebraic groups and semisimple *complex* Lie algebras (cf., p. 89 in [15]).

Theorem 2.2.8. *A connected linear algebraic group G is semisimple if and only if its Lie algebra, \mathfrak{g} , is semisimple.*

Therefore, if we start with a semisimple group, we are guaranteed to assign a semisimple Lie algebra to it, as in the case of Lie groups. When we venture out of semisimple complex Lie algebras, we run into a problem. Let $\mathfrak{g} = \mathbb{C}$ be the one dimensional complex abelian Lie algebra. This algebra has no ‘adjoint group’ in the category of linear algebraic groups. The group \mathbb{C}^\times has Lie algebra \mathfrak{g} , but \mathbb{C}^\times is not simply connected. Any finite covering group of \mathbb{C}^\times will have the same Lie algebra. Also, the universal covering group of \mathbb{C}^\times is *not* a linear algebraic group. Note that this is actually quite subtle. The additive group \mathbb{C} can be given the structure of a linear algebraic group with Lie algebra the abelian Lie algebra \mathbb{C} . Since \mathbb{C} is simply connected, we would hope that this could be our adjoint group. However, this is *not* functorial. The covering map (in this case the exponential) from \mathbb{C}^\times to \mathbb{C} is *not* regular, and thus not a morphism in the category of algebraic groups. In this way, we do not have a natural adjoint group to assign to \mathfrak{g} .

Even with this ‘pathology’, we can study the representation theory of a semisimple algebraic group by studying that of a finite dimensional semisimple complex Lie algebra. In fact, in this setting, the two objects have ‘equivalent’ representation theories, as we would desire. We will discuss this in greater detail in the following section.

2.3 The Theorem of the Highest Weight

In this section, we discuss the representation theory of semisimple linear algebraic groups, culminating in the Theorem of the Highest Weight, which characterizes the equivalence classes of finite dimensional irreducible representations of an algebraic group G by the combinatorial data of the weight lattice of its Lie algebra \mathfrak{g} . In this section, and for the rest of the dissertation, we assume G is a semisimple, simply connected linear algebraic group, unless explicitly stated otherwise. More in depth discussions of the material in this section can be found in Chapters 1 and 3 of [10] and Chapters X and XI of [15].

2.3.1 Representation theory of algebraic groups

Let V be a complex vector space. By a *regular representation* of an algebraic group G , we mean a regular homomorphism

$$\pi : G \rightarrow GL(V),$$

where $GL(V)$ denotes the general linear group of invertible linear transformations on V . Note, the word *regular* here is used in the sense of algebraic geometry. That is, we assume the representation is polynomial in the matrix entries on G . Unless stated otherwise, when we discuss representations of G , we mean *regular* representations. Note that a regular representation is necessarily finite dimensional. We denote a representation as a pair (π, V) .

On the other hand, if \mathfrak{g} is a Lie algebra and V is a complex vector space, then we call a Lie algebra homomorphism

$$\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$$

a *representation* of \mathfrak{g} , where $\mathfrak{gl}(V)$ is the general linear Lie algebra of all linear transformations of V under the commutator bracket. We also denote representations of \mathfrak{g} by pairs (ϕ, V) .

A regular representation (π, V) of an algebraic group G can be differentiated to obtain a representation $(d\pi, V)$ of its Lie algebra. The representation $(d\pi, V)$

is called the *differential* of the representation (π, V) . Thus, every representation of an algebraic group G induces a representation on its Lie algebra \mathfrak{g} . When G is semisimple, we can go the other way, and “integrate” a representation of \mathfrak{g} to a representation of G . In this way, studying the representation theory of the Lie algebra \mathfrak{g} of an algebraic group G yields information about the representation theory of the group G .

A representation (π, V) of G is said to be *irreducible* if it contains no non-trivial G -invariant subspaces. That is, if $W \subset V$ is a subspace such that $\pi(G)W \subset W$, then W is either the trivial subspace or $W = V$. In a sense, the irreducible representations of G are the building blocks of all representations of G , at least when G is semisimple. This is made precise in the following theorem (cf., p. 88 in [15]).

Theorem 2.3.1. *Let G be a semisimple linear algebraic group. Then every representation (π, V) of G decomposes as a direct sum of representations of G*

$$V = \bigoplus_i V_i,$$

such that each V_i is irreducible.

Our first goal will be to classify all finite dimensional irreducible representation of a semisimple algebraic group G . This is the content of the Theorem of the Highest Weight. Note that it can be shown that all irreducible representations of a semisimple, simply connected algebraic group are finite dimensional. Therefore, the Theorem of the Highest Weight completely classifies all irreducible representations of a semisimple, simply connected linear algebraic group G .

2.3.2 Root systems and highest weight theory

Let \mathfrak{g} be a finite dimensional, semisimple complex Lie algebra. The *adjoint representation* of \mathfrak{g} is the representation

$$ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$$

given by $adX = [X, \cdot]$ for all $X \in \mathfrak{g}$. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called *toral* if adX is a semisimple transformation for all $X \in \mathfrak{h}$. A toral subalgebra with maximal

dimension is called a *Cartan subalgebra*. It can be shown (cf., Lemma 2.5.17 in [10]) that every toral subalgebra is abelian and that every finite dimensional, semisimple complex Lie algebra contains nonzero Cartan subalgebras (cf., Corollary 2.5.16 in [10]). Further, all Cartan subalgebras of a given Lie algebra are isomorphic as Lie algebras.

Fix a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For each λ in the dual space \mathfrak{h}^* , set

$$\mathfrak{g}_\lambda := \{Y \in \mathfrak{g} \mid adX(Y) = \lambda(X)Y \text{ for all } X \in \mathfrak{h}\}.$$

Note that $\mathfrak{g}_0 = \{Y \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } X \in \mathfrak{h}\}$. This subspace of \mathfrak{g} is called the *centralizer* of \mathfrak{h} in \mathfrak{g} . Define Φ to be the set of all $\lambda \in \mathfrak{h}^* \setminus 0$ such that $\mathfrak{g}_\lambda \neq 0$. Elements of Φ are called the *roots* of the Lie algebra \mathfrak{g} . The nonzero \mathfrak{g}_λ with $\lambda \in \Phi$ are called the *root spaces* of \mathfrak{g} .

Since, for all $X \in \mathfrak{h}$, the transformation adX are all mutually commuting and semisimple, we have a *root space decomposition* of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda.$$

It can be shown that a Cartan subalgebra is its own centralizer in \mathfrak{g} (cf., Proposition 2.5.18 in [10]), and thus $\mathfrak{g}_0 = \mathfrak{h}$.

We collect the basic facts on roots and root spaces in the following theorem. This is Theorem 2.5.20 in [10].

Theorem 2.3.2. *Let \mathfrak{g} be a semisimple complex Lie algebra. The roots and root spaces of \mathfrak{g} satisfy the following properties:*

1. Φ spans \mathfrak{h}^* .
2. If $\alpha \in \Phi$, then $\dim[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 1$, and there is a unique element $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$. The element h_α is called the *coroot* of α .
3. If $\alpha \in \Phi$ and $z \in \mathbb{C}$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$. Also, $\dim \mathfrak{g}_\alpha = 1$.
4. Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Let p be the largest integer non-negative integer with $\beta + p\alpha \in \Phi$ and let q be the smallest non-negative integer with $\beta - q\alpha \in \Phi$. Then

$$\beta(h_\alpha) = q - p \in \mathbb{Z}$$

and $\beta + r\alpha \in \Phi$ for all integers r with $-q \leq r \leq p$. Hence $\beta - \beta(h_\alpha)\alpha \in \Phi$.

5. If $\alpha, \beta \in \Phi$ and $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Define a bilinear form on \mathfrak{g} by $(X, Y) = \text{Tr}(adXadY)$. This form is called the *Killing form* for \mathfrak{g} . It can be shown (cf., Theorem 2.5.11 in [10]) that the Killing form of a Lie algebra \mathfrak{g} is nondegenerated if and only if \mathfrak{g} is semisimple. Since (\cdot, \cdot) is nondegenerate, it induces an isomorphism between \mathfrak{g} and \mathfrak{g}^* . We will often abuse notation and denote by (\cdot, \cdot) the bilinear form induced by the Killing form on \mathfrak{g}^* (and, in particular, \mathfrak{h}^*).

Using the Killing form, we can define positive roots. Let E be the real span of $\{h_\alpha \mid \alpha \in \Phi\}$. A *regular* element of E is a vector $h \in E$ such that $(\alpha, h) \neq 0$ for all $\alpha \in \Phi$. Regular elements of E exist. Fix a regular element h , and set

$$\Phi^+ = \{\alpha \in \Phi \mid (\alpha, h) > 0\}.$$

The set Φ^+ is called the set of *positive roots* of \mathfrak{g} . It can be shown that $\Phi = \Phi^+ \cup (-\Phi^+)$.

A positive root α which cannot be written as the sum of two other positive roots is called *simple*. Let Δ be the set of all simple roots of \mathfrak{g} . Then we have the following (cf., Proposition 2.5.23 in [10]).

Proposition 2.3.3. *Let Δ be the set of simple roots of \mathfrak{g} . Then every positive root is a linear combination of the elements of Δ with nonnegative integer coefficients.*

In this sense, the simple roots generate all of the positive roots. Note that these definitions depend on our choice of regular element $h \in E$. However, it can be shown that a set of simple roots always exists, and each set of simple roots for a given semisimple complex Lie algebra has the same cardinality.

We now turn to highest weight theory. Begin by fixing a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Definition 2.3.4. Let \mathfrak{g} be a semisimple complex Lie algebra with Cartan subalgebra \mathfrak{h} . The *weight lattice* for \mathfrak{g} is the set

$$P(\mathfrak{g}) = \{\mu \in \mathfrak{h}^* \mid \mu(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

By the linearity of $\mu \in \mathfrak{h}^*$, $P(\mathfrak{g})$ is an additive subgroup of \mathfrak{h}^* .

Definition 2.3.5. The *root lattice* of \mathfrak{g} , $Q(\mathfrak{g})$, is the additive subgroup of \mathfrak{h}^* generated by $\Phi \cup \{0\}$.

The root lattice $Q(\mathfrak{g})$ is a subgroup of the weight lattice $P(\mathfrak{g})$. Let (ϕ, V) be a finite dimensional representation of \mathfrak{g} . For each $\mu \in \mathfrak{h}^*$, set

$$V(\mu) = \{v \in V \mid \phi(Y)v = \mu(Y)v \text{ for all } Y \in \mathfrak{h}\}.$$

If $V(\mu) \neq 0$, call μ a *weight* of V . Let $\mathfrak{X}(V) \subset \mathfrak{h}^*$ be the set of all weights of V . Then we have the following theorem (cf., Theorem 3.1.16 in [10]).

Theorem 2.3.6. *Let (ϕ, V) be a finite dimensional representation of \mathfrak{g} . Then $\mathfrak{X}(V) \subset P(\mathfrak{g})$, and*

$$V = \bigoplus_{\mu \in \mathfrak{X}(V)} V(\mu).$$

Let Δ be a fixed choice of simple roots for \mathfrak{g} with positive roots Φ^+ . Define the *fundamental dominant weights* in $P(\mathfrak{g})$ to be the weights dual to the coroots h_α under the form induced by the Killing form on \mathfrak{h} for all $\alpha \in \Delta$. That is, if $\Delta = \{\alpha_1, \dots, \alpha_k\}$, then the fundamental dominant weights are the weights $\{\omega_1, \dots, \omega_k\} \subset P(\mathfrak{g})$ such that

$$(\omega_i, h_{\alpha_j}) = \delta_{ij}.$$

Denote by $P_+(\mathfrak{g})$ the subsemigroup of $P(\mathfrak{g})$ generated by the fundamental dominant weights. In other words,

$$P_+(\mathfrak{g}) = \mathbb{N}\omega_1 \oplus \dots \oplus \mathbb{N}\omega_k \subset P(\mathfrak{g}).$$

Weights in $P_+(\mathfrak{g})$ are referred to as *dominant integral weights*. Further, a dominant integral weight is called *regular* if none of its integral coefficients are zero.

Fundamental dominant weights play an important role in the representation theory of semisimple algebraic groups and semisimple complex Lie algebras. They are precisely the data needed to parametrize the finite dimensional irreducible representations of an algebraic group G . This is the content of the Theorem of the Highest Weight, which can be found in any introductory text on Lie theory (including [9], [10], [14], etc.).

Let (ϕ, V) be a nonzero irreducible, finite dimensional representation of \mathfrak{g} . Then there exists a unique weight $\lambda \in P_+(\mathfrak{g})$ such that $\dim(V(\lambda)) = 1$. This weight λ is called the *highest weight* of the representation V . Any nonzero vector v_λ in the (one dimensional) weight space $V(\lambda)$ is called a *highest weight vector*. We state the following version of the Theorem of the Highest Weight for the finite dimensional irreducible representations of \mathfrak{g} .

The Theorem of the Highest Weight. Let λ be a dominant integral weight of \mathfrak{g} . Then the following hold.

1. There exists a finite dimensional irreducible representation, $(\sigma, L(\lambda))$, of \mathfrak{g} with highest weight λ .
2. Let (ϕ, V) be an finite dimensional irreducible representation of \mathfrak{g} with highest weight λ . Then (ϕ, V) is equivalent to $(\sigma, L(\lambda))$.

Note that, due to the way the representation theory of G and that of \mathfrak{g} are intertwined, we could have written this discussion in terms of the group G . In particular, the finite dimensional irreducible representations of G are also parametrized by the dominant integral weights of \mathfrak{g} . We will fix the notation $L(\lambda)$ to mean the irreducible highest weight representations of highest weight λ for the remainder of the dissertation.

2.4 The Borel-Weil Theorem

The Borel-Weil Theorem can be thought of as a reformulation of the Theorem of the Highest weight in the setting of line bundles on the flag variety G/B . While beautiful in its own right, this point of view leads to many of the geometric applications of representation theory, including the motivation behind the results from [12] generalized in this dissertation. The Borel-Weil Theorem realizes irreducible highest weight representations of G concretely as the spaces of sections on certain line bundles on G/B .

To begin, we fix a choice of Borel subgroup $B \subset G$. Let V be an irreducible representation of G , and let $\mathbb{P}(V)$ be the projective space of lines through the origin in V . The action of G extends to an action on $\mathbb{P}(V)$, and (cf., p. 392 in [9]) there is a unique closed orbit of G on $\mathbb{P}(V)$. Since the unique closed orbit can be given the structure of a projective variety, it must be of the form G/P for some parabolic subgroup $P \subseteq G$.

Consider the tautological line bundle on $\mathbb{P}(V)$. Since G/P is a closed orbit of G in $\mathbb{P}(V)$, we may pull back the tautological line bundle to the projective variety G/P . In this sense, to every irreducible representation of G , we assign a projective variety, $X = G/P$, and a G -equivariant line bundle, L , on X . We use the terms *line bundle* and *G -equivariant line bundle* interchangeably. Now, let π be the natural projection from the flag variety G/B to G/P . We can further pull back the line bundle L via π to a line bundle π^*L on G/B .

Under this construction, the weight lattice $P(\mathfrak{g})$ is isomorphic to the group of line bundles on G/B . Assume λ is a dominant integral weight of \mathfrak{g} . Then we can realize the space $H^0(G/B, L_\lambda)$ of holomorphic sections of the associated line bundle L_λ as an irreducible highest weight representation of G . We make this precise below (cf., p. 392 in [9]).

We now realize the finite dimensional irreducible representations of G using G -equivariant line bundles. Starting with the group G , we consider the flag variety G/B . Recall that B is a product of a maximal torus T and a unipotent subgroup U . Let \mathfrak{b} , \mathfrak{h} , and \mathfrak{n} be the Lie algebras of B , T , and U , resp. Since T is an abelian

subgroup of G , its Lie algebra is also abelian. Also, the Lie algebra of a unipotent group is nilpotent. So, on the level of Lie algebras, the product $B = T \cdot U$ corresponds to the decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ of the algebra \mathfrak{b} into an abelian and a nilpotent Lie algebra. The nilpotent algebra \mathfrak{n} is the sum of the positive root spaces

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

Every dominant integral weight λ corresponds to a one dimensional representation of the torus T . Since B is the product of T and the unipotent group U , this one dimensional representation extends trivially to a representation of B . Denote this representation by \mathbb{C}_λ . Then define

$$L_\lambda = G \times_B \mathbb{C}_\lambda$$

to be the fibre product of G and \mathbb{C}_λ over B . In other words,

$$L_\lambda = (G \times \mathbb{C}_\lambda) / \sim,$$

where \sim is the equivalence relation given by $(g, v) \sim (gx, x^{-1}v)$ for all $x \in B$. L_λ has a natural projection π to G/B given by $\pi[(g, v)] = gB$, where $[(g, v)]$ is the equivalence class of (g, v) in $G \times_B \mathbb{C}_\lambda$. This map is clearly well-defined, since $(gx)B = gB$ for all $x \in B$. Equipped with the map π , L_λ is a holomorphic line bundle on G/B . We have the following (cf., p.393 in [9]).

Borel-Weil Theorem. For a dominant integral weight λ , the space of sections $H^0(G/B, L_{-\lambda})$ is equivalent to the irreducible representation of G with highest weight λ .

In our previous notation, the Borel-Weil Theorem states that $L(\lambda) \cong H^0(G/B, L_{-\lambda})$ as G -representations. In this way, the Borel-Weil Theorem gives us a concrete realization of the irreducible highest weight representations of G . We will use the Borel-Weil Theorem to help construct an explicit description of the homogeneous coordinate ring of the orbit G/P in Chapter 3.

2.5 The Weyl Dimension Formula

In this section, we present the Weyl Dimension Formula, which can be used to compute the dimension of an irreducible finite dimensional highest weight representation of G . We begin by presenting some basics of character theory, culminating in the Weyl Character Formula. We then deduce the Weyl Dimension Formula from the Weyl Character Formula. Standard references for these topics include [10], [15], and [16].

2.5.1 The Weyl Character Formula

Let (π, V) be a representation of G . We define the *character* of V to be the function

$$\chi_V : G \rightarrow \mathbb{C}$$

given by $\chi_V(g) = \text{Tr}(\pi(g))$. Note that, since π is a homomorphism, the character of V is a *class function* on G , ie. χ_V is constant on conjugacy classes of G : $\chi_V(hgh^{-1}) = \chi_V(g)$ for all $h \in G$. Also, let e be the identity element of G . Then $\chi_V(e) = \dim(V)$, again since π is a homomorphism.

Some basic properties of characters are given below. This is Proposition 2.1 in [9].

Proposition 2.5.1. *Let V and W be representations of G . Then $\chi_{V \oplus W} = \chi_V + \chi_W$, $\chi_{V \otimes W} = \chi_V \chi_W$, and $\chi_{V^*} = \overline{\chi_V}$, where \bar{z} denotes the complex conjugate of z .*

Motivated by the fact that $\chi_V(1) = \dim(V)$, we wish to compute the character of V , when G is a semisimple linear algebraic group. As before, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set Φ^+ of positive roots. Let $\Delta = \{\alpha_1, \dots, \alpha_k\}$ be the simple roots of \mathfrak{g} , and let $\omega_1, \dots, \omega_k$ be the fundamental dominant weights in \mathfrak{h}^* dual to the simple roots. Denote the semigroup of dominant integral weights by $P_+(\mathfrak{g})$.

To every $\alpha_i \in \Delta$, define the *simple root reflection* $s_{\alpha_i} : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ to be the map

$$s_{\alpha_i}(\beta) = \beta - \beta(h_{\alpha_i})\alpha_i,$$

where h_α is the coroot in \mathfrak{h} of α . Then $s_{\alpha_i}^2$ is the identity function on \mathfrak{h}^* , and s_{α_i} can be the reflection of \mathfrak{h} through the hyperplane $(h_\alpha)^\perp$. Define the *Weyl group*, W , of G to be the finite group generated by the simple root reflections. It is a basic fact that the Weyl group of G is isomorphic to $Norm(T)/T$, where $Norm(T)$ is the normalizer of the torus T in G .

Let R be the integral group ring $\mathbb{Z}[P(\mathfrak{g})]$ of the weight lattice $P(\mathfrak{g})$. For every $\lambda \in P(\mathfrak{g})$, let e^λ denote the basis element of R corresponding to λ . Define the *Weyl function* of G to be the element $\Delta_G \in R$ given by

$$\Delta_G = e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}).$$

Note that, by considering weights as characters of the maximal torus T , Δ_G can be considered a function on T . Define the *adjoint representation* of G to be the representation $(\sigma, GL(\mathfrak{g}))$ given by $\sigma(g)(X) = gXg^{-1}$ for all $g \in G$ and $X \in \mathfrak{g}$. Then, since W can be realized as $Norm(T)/T$, the adjoint representation of G gives a representation σ of W on \mathfrak{h}^* . For any $s \in W$, define

$$sgn(s) = \det(\sigma(s)).$$

Note that, since W is generated by reflections, $sgn(s) = \pm 1$. We can, of course, restrict the action of W on \mathfrak{h}^* to an action on the weight lattice $P(\mathfrak{g}) \subset \mathfrak{h}^*$. We will denote this action simply $s\lambda$ for each $s \in W$, $\lambda \in P(\mathfrak{g})$.

Let (ϕ, V) be a finite dimensional representation of \mathfrak{g} . Then V decomposes into weight spaces of \mathfrak{h} :

$$V = \bigoplus_{\lambda \in P(\mathfrak{g})} V(\lambda).$$

Define

$$\chi_V = \sum_{\lambda \in P(\mathfrak{g})} \dim(V(\lambda))e^\lambda.$$

Then χ_V is an element of the integral group ring R . There is a regular representation (π, V) of G whose differential is (ϕ, V) . Restricting this representation to the subgroup T , we have

$$\chi_V(h) = \text{Tr}(\pi(h)), \text{ for all } h \in T,$$

so that χ_V is a character on T (cf., p. 331 in [10]). We are now ready to state the Weyl Character Formula.

Weyl Character Formula. Let $\lambda \in P_+(\mathfrak{g})$ and let $L(\lambda)$ be the finite dimensional irreducible representation of G with highest weight λ . Then

$$\chi_{L(\lambda)} = \frac{\sum_{s \in W} \text{sgn}(s) e^{s(\lambda + \rho)}}{\Delta_G},$$

where $\rho := \omega_1 + \cdots + \omega_k$ and k is the rank of T .

2.5.2 Deriving the Weyl Dimension Formula from the Weyl Character Formula

Note that, in the context of the Weyl Character Formula, since χ_V is a character of the torus T , $\chi_V(e) = \dim(V)$, where e is the identity element of G . The Weyl Dimension Formula uses this fact and the Weyl Character Formula to compute the dimension of V , where (π, V) is an irreducible, finite dimensional regular representation of G . We will derive the Weyl Dimension Formula from the Weyl Character Formula explicitly. For a more in depth reference, see §7.1 in [10].

For the rest of the dissertation, if $\{\omega_1, \dots, \omega_k\}$ is a set of fundamental dominant weights for \mathfrak{g} , then set

$$\rho = \omega_1 + \cdots + \omega_k.$$

Note also that it can be shown that $\Delta_G = \sum_{s \in W} \text{sgn}(s) e^{s\rho}$. This fact is known as the Weyl Denominator Formula, and is Corollary 7.1.3 in [10].

Weyl Dimension Formula. Let $\lambda \in P_+(\mathfrak{g})$. The dimension of $L(\lambda)$ is a polynomial of degree $|\Phi^+|$ in λ given by

$$\dim(L(\lambda)) = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where (\cdot, \cdot) is the bilinear form on \mathfrak{h}^* induced by the Killing form on \mathfrak{g} .

Proof. Note that $\Delta_G(e) = 0$, since $\Delta_G(shs^{-1}) = \text{sgn}(s)\Delta(G)(h)$ holds for all $h \in T$. Similarly, the numerator in the Weyl Character Formula also vanishes at the identity. We must apply an algebraic version of l'Hospital's rule to compute the dimension of $L(\lambda)$.

In order to do this, begin by defining a linear functional $\epsilon : R \rightarrow \mathbb{C}$ by

$$\epsilon\left(\sum_{\lambda} n_{\lambda} e^{\lambda}\right) = \sum_{\lambda} n_{\lambda}.$$

Under the action $se^{\lambda} = e^{s\lambda}$ of W on R , $\epsilon(sf) = \epsilon(f)$ for all $f \in R$. Let (\cdot, \cdot) be the bilinear form on \mathfrak{h}^* induced by the Killing form on \mathfrak{g} , and define a derivation ∂_{α} on R by

$$\partial_{\alpha}(e^{\lambda}) = (\alpha, \lambda)e^{\lambda}.$$

Then $s(\partial_{\alpha}f) = \partial_{s(\alpha)}(sf)$ holds for all $s \in W$ and $f \in R$.

Define $D = \prod_{\alpha \in \Phi^+} \partial_{\alpha}$, and for simplicity, let N_{λ} denote the denominator

$$\sum_{w \in W} \text{sgn}(s) e^{s(\lambda + \rho)}$$

in the Weyl Character Formula. Note that $s(Df) = \text{sgn}(s)D(sf)$ holds for all $s \in W$. Then,

$$D.N_{\lambda} = \sum_{w \in W} s(D(e^{\lambda + \rho})) = \prod_{\alpha \in \Phi^+} (\rho + \lambda, \alpha) \sum_{s \in W} e^{s(\rho + \lambda)}.$$

Also, by the above and the Weyl Denominator Formula,

$$D\Delta_G = \prod_{\alpha \in \Phi^+} (\rho, \alpha) \sum_{s \in W} e^{s\rho}. \quad (2.1)$$

Applying the function ϵ , we obtain

$$\epsilon(D(N\lambda)) = |W| \prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha).$$

In particular, by the Weyl Character Formula, $\Delta_G \chi_V = N_{\lambda}$. Hence

$$\epsilon(D(\Delta_G \chi_V)) = |W| \prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha). \quad (2.2)$$

It can be shown that, for every $f \in R$, $\epsilon(D(f\Delta_G)) = \epsilon(fD(\Delta_G))$. In particular, this holds when $f = \chi_V$. Thus, (2.2) implies that

$$\epsilon(\chi_V D(\Delta_G)) = |W| \prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha). \quad (2.3)$$

But, (2.1) implies that

$$\epsilon(\chi_V D(\Delta_G)) = \prod_{\alpha \in \Phi^+} (\rho, \alpha) \epsilon\left(\sum_{w \in W} \sum_{\mu \in \mathfrak{h}^*} m_\lambda(\mu) e^{\mu + s\rho}\right) = |W| \dim(L(\lambda)) \prod_{\alpha \in \Phi^+} (\rho, \alpha) \quad (2.4)$$

Setting (2.3) and (2.4) equal to each other and solving for $\dim(L(\lambda))$ implies the result. \square

Following the notation given by Gross and Wallach in [12], set $c_\lambda(\alpha) = \frac{(\lambda, \alpha)}{(\rho, \alpha)}$. Then the Weyl Dimension Formula states that

$$\dim(L(\lambda)) = \prod_{\alpha \in \Phi^+} (1 + c_\lambda(\alpha)),$$

by the bilinearity of (\cdot, \cdot) .

We include the proof of the Weyl Dimension Formula in this section to emphasize the role certain differential operators play in the representation theory of linear algebraic groups. In the proof as given, the operator

$$D = \prod_{\alpha \in \Phi^+} \partial_\alpha$$

plays the role of ‘derivative’ in an argument which resembles l’Hospital’s Rule in order to compute the indeterminate form of the Weyl Character Formula at the identity. The main result in Chapter 4 can be thought of as a method of computing dimensions of finite dimensional irreducible representations of G using certain differential operators. The setting is quite different, as are the operators used, but the use of differential operators to compute the dimensions of representations hearkens back to the proof of the Weyl Dimension Formula.

2.6 Multivariate Hilbert series

In this section, we switch gears from the theory of Lie algebras to *associative* algebras, in order to discuss multivariate Hilbert series on graded algebras and modules. We begin by defining what it means for a complex algebra to be graded over \mathbb{N}^k and give some examples. We introduce the Hilbert function and Hilbert series of an \mathbb{N} -graded algebra, and their generalizations to the \mathbb{N}^k -graded case. We then discuss the Hilbert series of projective varieties by considering their coordinate rings to be graded by degree. This is a singly graded case of the more general multivariate Hilbert series. We conclude with a discussion of the multivariate case and some specializations of multivariate Hilbert series to the singly graded case. A readable introduction to the theory of multivariate Hilbert series is [20]. There are many good references for the single variable case, including [1], [3], and [23].

2.6.1 Graded algebras

Throughout this section, let A be an associative algebra over \mathbb{C} . Note that, in the definitions to follow, the *algebra* A could be replaced with a *module* M over A by changing all instances of the word “subspace” to “submodule”.

Definition 2.6.1. An algebra A is called \mathbb{N}^k -*graded* if

$$A = \bigoplus_{\mathbf{a} \in \mathbb{N}^k} A_{\mathbf{a}},$$

where the components $A_{\mathbf{a}}$ are subspaces of A such that $A_{\mathbf{a}}A_{\mathbf{b}} = A_{\mathbf{a}+\mathbf{b}}$ holds for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^k$. The components $A_{\mathbf{a}}$ are called the *homogeneous components* of A . Elements of A which lie in one of the homogeneous components of A are called *homogeneous*.

As an example of a graded \mathbb{C} -algebra, consider the polynomial ring $A = \mathbb{C}[z_1, \dots, z_n]$ in n indeterminates. Let $\mathbf{z}^{\mathbf{a}}$ denote the monomial $z_1^{a_1} \dots z_n^{a_n}$ for all $\mathbf{a} \in \mathbb{N}^n$. Then $\mathbf{z}^{\mathbf{a}}\mathbf{z}^{\mathbf{b}} = \mathbf{z}^{\mathbf{a}+\mathbf{b}}$. Then we have the decomposition

$$\mathbb{C}[x_1, \dots, x_n] = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} \mathbb{C}\mathbf{z}^{\mathbf{a}}.$$

This decomposition gives A the structure of an \mathbb{N}^n -graded \mathbb{C} -algebra.

Let A be an \mathbb{N} -graded (associative) algebra. Assume further that the homogeneous components of A are finite dimensional subspaces of A . Note that this is the case for the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, where instead of the grading we used previously, we grade by degree, ie. $A_k = \{f \mid f \text{ is homogeneous of degree } k\}$. Define the *Hilbert function* of A to be the formal function

$$HF_A : \mathbb{N} \rightarrow \mathbb{N}$$

such that $HF_A(k) = \dim(A_k)$. We further define the *Hilbert series* of A to be the formal power series

$$HS_A(q) = \sum_{i=0}^{\infty} HF_A(i)q^i.$$

In other words, $HS_A(q)$ is the generating function of the dimensions of the homogeneous components of A . The Hilbert series of A is an element of the ring of formal power series $\mathbb{Z}[[q]]$ with integer coefficients in the indeterminate q . In the case where A is \mathbb{N} -graded, we sometimes call the series $HS_A(q)$ *singly-graded*.

In the singly-graded case, $HS_A(q)$ has nice properties. For instance, if A is generated in degree one (as will be the case for the homogeneous coordinate rings of flag varieties G/P , cf. [12]), then

$$HS_A(q) = \frac{P(q)}{(1-q)^d},$$

where $P(q)$ is a polynomial in q with integer coefficients. Finding explicit rational functions that represent the series $HS_A(q)$ is a broad topic of research.

Now let A be an \mathbb{N}^k -graded (associative) algebra, again with finite dimensional homogeneous components. Define the *Hilbert series* of A to be the formal power series

$$HS_A(q_1, \dots, q_k) = \sum_{\mathbf{a} \in \mathbb{N}^k} \dim(A_{\mathbf{a}}) \mathbf{q}^{\mathbf{a}}.$$

The series $HS_A(q_1, \dots, q_k)$ resides in the ring of formal power series $\mathbb{Z}[[q_1, \dots, q_k]]$ with integer coefficients in the indeterminates q_1, \dots, q_k .

Returning to the example given above with $A = \mathbb{C}[z_1, \dots, z_n]$ under the \mathbb{N}^n -grading, we can easily compute the multivariate Hilbert series of A . Since the dimension of each homogeneous component $A_{\mathbf{a}} = \mathbb{C}\mathbf{z}^{\mathbf{a}}$ is one, the Hilbert series of A is

$$HS_A(\mathbf{q}) = \sum_{\mathbf{a} \in \mathbb{N}^n} \mathbf{q}^{\mathbf{a}} = \prod_{i=1}^n \frac{1}{1 - q_i}.$$

Note that in this case, $HS_A(\mathbf{q})$ is simply the sum of all monomials in $\mathbb{C}[q_1, \dots, q_n]$.

2.6.2 The Hilbert series of a projective variety

Hilbert series play a central role in the study of graded algebras, especially when the algebra is the homogeneous coordinate ring of a projective variety. In the case where the coordinate ring is \mathbb{N} -graded, we can read off certain geometric information about the variety from the expression of $HS_A(q)$ as a rational function.

Let $\mathbb{C}[z_1, \dots, z_n]$ be the algebra of polynomials with complex coefficients in the indeterminates z_1, \dots, z_n . As previously discussed, this algebra has a natural \mathbb{N} -grading by degree. An ideal in $\mathbb{C}[z_1, \dots, z_n]$ is called *homogeneous* if it can be generated by homogeneous elements of $\mathbb{C}[z_1, \dots, z_n]$. If $\mathcal{I} \subseteq \mathbb{C}[z_1, \dots, z_n]$ is such an ideal, then the quotient algebra

$$\mathbb{C}[z_1, \dots, z_n]/\mathcal{I}$$

inherits the gradation by degree. Note that the ideal of a projective varieties is an homogeneous ideal. Thus, the coordinate ring of a projective variety has a natural gradation by degree.

Definition 2.6.2. Let X be a projective variety. We define the *Hilbert series* of X to be the Hilbert series of its homogeneous coordinate ring $\mathbb{C}[X]$, under the gradation of $\mathbb{C}[X]$ by degree.

The Hilbert series of a projective variety X encodes geometric information about the variety X . For instance, the degree of the variety X , which is a measure of how X is embedded in projective space, can be computed by plugging 1 into the

numerator of the Hilbert series. Note that the Hilbert series of a projective variety is not uniquely determined by the variety. It depends on the particular way that X is embedded in a projective space. Isomorphic varieties may have different Hilbert series, depending on what space they are embedded in.

Throughout this dissertation, we will be particularly interested in those projective varieties which come equipped with a transitive action of a semisimple linear algebraic group G . Such a projective variety is called *G -homogeneous* or simply *homogeneous* if the group is clear from context. As mentioned in §2.1, homogeneous projective varieties are characterized by embeddings of G/P into a projective space, where P is a parabolic subgroup of G . That is, if G acts transitively on a variety X , and X can be given the structure of a projective variety, then the stabilizer P of X in G is a parabolic subgroup of G . Therefore, the study of the geometry of homogeneous projective varieties is really the study of the various ways G/P can be embedded into a projective space. Explicit rational functions representing the Hilbert series of any homogeneous projective variety were computed in [12]. We recall their results in §4.1.

2.6.3 Multivariate Hilbert series and specializations

We now turn our focus to the case where A is an \mathbb{N}^k -graded algebra over \mathbb{C} . There are many ways to specialize a multivariate Hilbert series $HS_A(q_1, \dots, q_k)$ to a single variate series. Each specialization corresponds to a different \mathbb{N} -gradation on A . For instance, if A is the Stanley-Reisner ring of a simplicial complex Δ , then the specialization $q_i \mapsto q$ for all $i = 1, \dots, k$ is called the *course Hilbert series* on A . This specializes the multivariate Hilbert series to the standard Hilbert series of the simplicial complex Δ (cf., §1.2 in [20]). In this sense, the multivariate Hilbert series $HS_A(q_1, \dots, q_k)$ encodes more information than a singly-graded Hilbert series on A . Specializing the variables leads to a description of many different singly-graded Hilbert series for A .

This idea will be used in some of the examples in Chapter 5, especially when considering multivariate Hilbert series of determinantal varieties. Determinantal

varieties come equipped with a standard structure of affine variety, which leads to a singly-graded Hilbert series that has been studied extensively by both representation theorists and commutative algebraists. In §5.4, we present a method of computing this singly-graded Hilbert series as a specialization of the rational function given in Theorem 4.2.1.

Chapter 3

Classical Invariant Theory

In this chapter, we review results from Classical Invariant Theory (CIT), which is the study of the polynomial invariants of the classical families of algebraic groups, $GL(n, \mathbb{C})$, $O(n)$, and $Sp(n)$, acting on a vector space V . These invariants inherit a gradation by degree from the ring of polynomial functions on V , and computing explicit rational functions that represent their Hilbert series is a subject of study. Much of the genesis of CIT was due to Brauer, Frobenius, Schur, Weyl, and others, who studied the G -invariant elements, $(\otimes^k V)^G$, of the k -fold tensor product of V . Solutions to this problem when G is a classical group and $V = W^k \oplus (W^*)^k$ are known as the First Fundamental Theorem for G . We discuss this situation in §2 of this chapter. Much of the exposition in this chapter can be found in [10]. Other standard references include [6] and [22].

Note that this chapter requires considerably more background than Chapter 2. This survey is meant to motivate the types of examples we will revisit in Chapter 5. For example, we present determinantal varieties as objects of classical interest in CIT when we discuss the Second Fundamental Theorems in §3 of this chapter. This is meant to motivate our study of their multivariate Hilbert series in Chapter 5. The main theorem in Chapter 4 may be understood without using this chapter. In this sense, this chapter may be skipped by a reader who wishes to understand the main result without worrying about where the examples fit in the classical literature.

3.1 Reductive algebraic groups and finite generation

In [10], the authors state that the *basic problem of invariant theory* is to describe the G -invariant elements $(\otimes^k V)^G$ of a regular representation (π, V) of a linear algebraic group G . In general, this problem is quite difficult. The problem is much easier to solve if we restrict the groups we consider to *reductive* algebraic groups.

Definition 3.1.1. A linear algebraic group G is called *reductive* if every regular representation (π, V) of G is completely reducible.

Note that by *completely reducible*, we mean that if W is a G -invariant subspace of V , then V has a G -invariant subspace W' such that $V = W \oplus W'$.

Let \widehat{G} be the set of all isomorphism classes of irreducible regular representations of G . If W is an irreducible regular representation of G , denote by $[W]$ the isomorphism class of W in \widehat{G} . For every $\lambda \in \widehat{G}$, we have a *G -isotypic subspace* W_λ of V given by

$$W_\lambda := \sum_{W \subset V, [W]=\lambda} W.$$

Since any representation (π, V) of a reductive group G is completely reducible, we have a decomposition of V into G -isotypic subspaces

$$V = \bigoplus_{\lambda \in \widehat{G}} W_\lambda.$$

We now fix a representation (π, V) of G . Let $\mathbb{C}[V]$ be the algebra of all polynomial functions on V . Then G acts on $\mathbb{C}[V]$ by

$$\pi(g)f(v) := f(g^{-1}v).$$

Under this action, the spaces $\mathbb{C}[V]_k$ of homogeneous polynomials of degree k are G -invariant. Then, since G is reductive, the coordinate ring $\mathbb{C}[\pi_k(G)]$ is completely reducible, where π_k denotes the restriction of π to $\mathbb{C}[V]_k$. This implies (cf., p. 226 in [10]) that $\mathbb{C}[V]_k$ has a primary decomposition into G -isotypic subspaces. In other words,

$$\mathbb{C}[V]_k = \bigoplus_{\lambda \in \widehat{G}} W_\lambda.$$

Since any polynomial $f \in \mathbb{C}[V]$ can be written as a sum of homogeneous polynomials, the decomposition of $\mathbb{C}[V]_k$ allows us to decompose f over the isotypic subspaces. We write

$$f = \sum_{\lambda \in \widehat{G}} f_\lambda,$$

where $f_\lambda \in W_\lambda$. Let f^\sharp denote the isotypic component of f corresponding to the trivial representation of G .

Let $\mathbb{C}[V]^G$ denote the algebra of G invariant polynomials in $\mathbb{C}[V]$. This algebra is often referred to as the *algebra of G -invariants*. Note that multiplying by a G -invariant polynomial leaves every G -isotypic subspace of $\mathbb{C}[V]_k$ invariant. This in turn implies that $(\phi f)^\sharp = \phi f^\sharp$ for all $f \in \mathbb{C}[V]$ and G -invariant ϕ . Therefore, the projection map $f \mapsto f^\sharp$ is a $\mathbb{C}[V]^G$ -module homomorphism. We are now ready to state the following celebrated result of Hilbert. This is Theorem 5.1.1 in [10].

Theorem 3.1.2. *Suppose G is a reductive linear algebraic group acting by a regular representation on a vector space V . Then the algebra $\mathbb{C}[V]^G$ is finitely generated as a \mathbb{C} -algebra.*

The proof of this theorem uses the Hilbert Basis Theorem to construct a finite set of generators for the ideal of G -invariant polynomials without a constant term in $\mathbb{C}[V]$ and then uses the homomorphism $f \mapsto f^\sharp$ to show that this finite set of generators actually generates $\mathbb{C}[V]^G$ as a \mathbb{C} -algebra. For the details, see [10].

Note that Theorem 4.2.1 does not say that the algebra $\mathbb{C}[V]^G$ is a polynomial ring over \mathbb{C} . It says nothing about the relations between the generators. However, finding an appropriate finite generating set can often lead to explicit computations inside the algebra of G -invariants. This theorem is one of the main motivations for studying reductive groups rather than a larger class of groups in CIT.

If we have a set $\{f_1, \dots, f_n\}$ of generators for $\mathbb{C}[V]^G$, where n is as small as possible, we call $\{f_1, \dots, f_n\}$ a set of *basic invariants* for $\mathbb{C}[V]^G$. Theorem 4.2.1

guarantees that we always have a finite set of basic invariants for $\mathbb{C}[V]^G$, whenever G is a reductive group. This allows us to concretely determine the Hilbert series of $\mathbb{C}[V]^G$ when G is reductive.

For instance, assume $H(q)$ denotes the Hilbert series of $\mathbb{C}[V]^G$. Namely,

$$H(q) = \sum_{k=0}^{\infty} \dim(\mathbb{C}[V]_k^G) q^k.$$

Then, in the case that the basic invariants are algebraically independent, we have

$$H(q) = \prod_{i=1}^n \frac{1}{1 - t^{d_i}},$$

where d_1, \dots, d_n are assumed to be the degrees of the basic invariants. Explicit formulations of the Hilbert series as a rational function when the basic invariants are not necessarily algebraically independent is one of the fundamental problems of invariant theory.

3.2 First Fundamental Theorems

For a reductive group G , the First Fundamental Theorem is an explicit description of the basic invariants of the ring of invariants $\mathbb{C}[(V^*)^n \oplus V^m]^G$ for any regular representation (π, V) of G . In other words, the First Fundamental Theorem describes a generating set $\{f_1, \dots, f_n\}$ for the ring of invariants $\mathbb{C}[(V^*)^n \oplus V^m]^G$, when G is reductive, and n is as small as possible. In this section, we will describe the First Fundamental Theorem for the classical families $GL(V)$, $O(n)$, and $Sp(n)$. We focus on these groups because, as we will see in the next section, the invariant theory for these groups leads naturally to a description of the coordinate rings of the three classical families of determinantal variety. We continue to follow the discussion in [10].

3.2.1 The First Fundamental Theorem for $GL(V)$

Assume G is a reductive group, and (π, V) is a regular representation of G . We begin by describing a geometric construction of the polynomial $GL(V)$ -invariants of $\mathbb{C}[(V^*)^n \oplus V^m]$. This is motivated by the fact that

$$\mathbb{C}[(V^*)^n \oplus V^m]^{GL(V)} \subseteq \mathbb{C}[(V^n)^* \oplus V^m]^G.$$

Thus, the First Fundamental Theorem for $GL(V)$ will give some information about the invariants of the group G .

Note that we have a natural isomorphism $(V^*)^k \cong \text{Hom}(V, \mathbb{C}^k)$, where the ordered k -tuple (v_1^*, \dots, v_k^*) of linear functionals on V corresponds to the linear map

$$v \mapsto (v_1^*(v), \dots, v_k^*(v))$$

from V to \mathbb{C}^k . Similarly, we have the natural isomorphism $V^m \cong \text{Hom}(\mathbb{C}^m, V)$, where the ordered m -tuple (v_1, \dots, v_m) of vectors in V corresponds to the linear map

$$(z_1, \dots, z_m) \mapsto z_1 v_1 + \dots + z_m v_m$$

from \mathbb{C}^m to V . Therefore, the polynomial algebra $\mathbb{C}[(V^*)^k \oplus V^m]$ is naturally isomorphic as a \mathbb{C} -algebra to the algebra

$$\mathbb{C}[\text{Hom}(V, \mathbb{C}^k) \oplus \text{Hom}(\mathbb{C}^m, V)],$$

and $GL(V)$ acts on the above via the action

$$g \cdot f(x, y) = f(x\pi(g^{-1}), \pi(g)y).$$

Let $M_{k,m}$ denote the space of $k \times m$ complex matrices. We define a map $\mu : \text{Hom}(V, \mathbb{C}^k) \oplus \text{Hom}(\mathbb{C}^m, V) \rightarrow M_{k,m}$, by

$$\mu(x, y) = xy,$$

where xy denotes composition of the linear transformations x and y . Then, by construction, μ is $GL(V)$ -invariant, in the sense that

$$\mu(g \cdot (x, y)) = \mu(x\pi(g^{-1}), \pi(g)y) = x\pi(g)^{-1}\pi(g)y = xy = \mu(x, y),$$

and thus the induced map μ^* on the coordinate ring $\mathbb{C}[M_{k,m}]$ of $M_{k,m}$ has range in the $GL(V)$ -invariant polynomials on $\text{Hom}(V, \mathbb{C}^k) \oplus \text{Hom}(\mathbb{C}^m, V)$,

$$\mu^* : \mathbb{C}[M_{k,m}] \rightarrow \mathbb{C}[\text{Hom}(V, \mathbb{C}^k) \oplus \text{Hom}(\mathbb{C}^m, V)]^{GL(V)}.$$

Let z_{ij} denote the image $\mu^*(x_{ij})$ of the matrix entry function x_{ij} on $M_{k,m}$. Then z_{ij} is the *contraction*

$$z_{ij}(v_1^*, \dots, v_k^*, v_1, \dots, v_m) = v_i^*(v_j).$$

We are now ready to state the First Fundamental Theorem for $GL(V)$. This is Theorem 5.2.1 in [10].

Theorem 3.2.1. *The map*

$$\mu^* : \mathbb{C}[M_{k,m}] \rightarrow \mathbb{C}[\text{Hom}(V, \mathbb{C}^k) \oplus \text{Hom}(\mathbb{C}^m, V)]^{GL(V)}$$

is surjective. Hence $\mathbb{C}[\text{Hom}(V, \mathbb{C}^k) \oplus \text{Hom}(\mathbb{C}^m, V)]^{GL(V)}$ is generated (as a \mathbb{C} -algebra) by the contractions $\{v_i^(v_j) \mid i = 1, \dots, k, j = 1, \dots, m\}$.*

3.2.2 First Fundamental Theorems for $O(n)$ and $Sp(n)$

We now modify the story for $GL(V)$ to get a similar theorem for the classical families $O(n)$ and $Sp(n)$. We begin with $O(n)$.

Let $V = \mathbb{C}^n$, and denote by (\cdot, \cdot) the symmetric form

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i \text{ for all } \mathbf{x}, \mathbf{y} \in V.$$

Let $O(n)$ be the orthogonal group of this form, ie. $g \in O(n)$ if and only if $g^t g = I_n$, where I_n is the identity matrix in GL_n . Denote by SM_k the set of symmetric matrices in M_k , and define the map $\tau : M_{n,k} \rightarrow SM_k$ by

$$\tau(X) = X^t X.$$

Then τ is $O(n)$ -invariant, in the sense that

$$\tau(gX) = X^t g^t g X = X^t I_n X = X^t X = \tau(X)$$

holds for all $g \in O(n)$ and $X \in M_{n,k}$. Therefore, the induced map on the coordinate ring $\mathbb{C}[SM_k]$ has range in the $O(n)$ -invariant polynomials in $\mathbb{C}[M_{n,k}]$,

$$\tau^* : \mathbb{C}[SM_k] \rightarrow \mathbb{C}[M_{n,k}]^{O(n)}.$$

Since $M_{n,k} \cong V^k$ as vector spaces, we can consider the range of τ^* to lie in $\mathbb{C}[V^k]^{O(n)}$.

Let $v_1, \dots, v_k \in \mathbb{C}^n$, and let X be the matrix $[v_1, \dots, v_k]$ in $M_{k,n}$ where we consider the vectors v_i to be as column vectors. Then the matrix $X^t X$ is the symmetric $k \times k$ matrix with ij -entry (v_i, v_j) . By *symmetric*, we mean that $X = X^t$. Then, under the map τ^* , the matrix entry functions x_{ij} on the symmetric matrices SM_k pulls back to the quadratic polynomial

$$\tau^*(x_{ij})(v_1, \dots, v_k) = (v_i, v_j),$$

which is then $O(n)$ -invariant since the range of τ^* is in the $O(n)$ -invariant polynomials on V^k . We are then led to the following First Fundamental Theorem for $O(n)$. This is Theorem 5.2.2 (Part 1) in [10].

Theorem 3.2.2. *The homomorphism τ^* is surjective. Hence the algebra of $O(n)$ -invariant polynomials in k vector arguments is generated by the quadratic polynomials $\{(v_i, v_j) \mid 1 \leq i \leq j \leq k\}$.*

Now assume that $n = 2m$ is a non-negative, even integer. Let J_n be the $n \times n$ block-diagonal matrix

$$J_n = \text{diag}[\kappa, \dots, \kappa],$$

consisting of m copies of κ , where κ is the 2×2 matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We define an anti-symmetric bilinear form ω on \mathbb{C}^n by

$$\omega(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, J_n \mathbf{y}),$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Let $Sp(n)$ be the invariance group of this form, ie. $g \in Sp(n)$ if and only if $g^t J_n g = J_n$.

Let AM_k denote the set of all $k \times k$ anti-symmetric matrices, and let γ be the map from $M_{n,k}$ to AM_k given by

$$\gamma(X) = X^t J_n X.$$

By *anti-symmetric*, we mean that $X = -X^t$. By construction, we know that γ is $Sp(n)$ -invariant, in the sense that

$$\gamma(gX) = X^t g^t J_n g X = X^t J_n X = \gamma(X),$$

for every $g \in Sp(n)$ and $X \in M_{n,k}$. Therefore, γ induces a map γ^* on the coordinate ring of AM_k whose range lies in the $Sp(n)$ invariant polynomials on $M_{n,k}$,

$$\gamma^* : \mathbb{C}[AM_k] \rightarrow \mathbb{C}[M_{n,k}]^{Sp(n)}.$$

Again, since $M_{n,k} \cong V^k$, we can consider the range of γ^* to be in $\mathbb{C}[V^k]^{Sp(n)}$.

Now, as in the orthogonal case, let $v_1, \dots, v_k \in V (= \mathbb{C}^n)$, and consider the matrix X in $M_{n,k}$ given by $[v_1, \dots, v_k]$. Then the matrix $X^t J_n X$ is the $k \times k$ anti-symmetric matrix with ij -entry $(v_i, J_n v_j) = \omega(v_i, v_j)$. Therefore, the matrix entry functions x_{ij} pull back to the quadratic polynomials

$$(v_i, J_n v_j),$$

which are $Sp(n)$ -invariant. We then have the following First Fundamental Theorem for $Sp(n)$. This is Theorem 5.2.2 (Part 2) in [10].

Theorem 3.2.3. *Suppose n is even. Then the homomorphism γ^* is surjective. Hence the algebra of $Sp(n)$ -invariant polynomials in k vector arguments is generated by the quadratic polynomials $\{\omega(v_i, v_j) \mid 1 \leq i < j \leq k\}$.*

As we have seen in this section, the invariant theory of $O(n)$ is closely tied to the vector space SM_k of symmetric $k \times k$ complex matrices. Similarly, the invariant theory of $Sp(n)$ is closely tied to the vector space AM_k of anti-symmetric $k \times k$ complex matrices. We will see in the next section another connection between these families of classical groups and these spaces of matrices.

3.3 Second Fundamental Theorems

Let G be a reductive group. Homogeneous varieties X whose coordinate ring $\mathbb{C}[X]$ have a multiplicity-free decomposition as a G -representation are called *multiplicity-free G -spaces*. The term *multiplicity-free* means that the finite dimensional irreducible representations of G which appear in the irreducible decomposition of $\mathbb{C}[X]$ have multiplicity one. An important example of a multiplicity-free space is the group G itself, thought of as a $G \times G$ -representation. We will explore this space in §5.2. In this section, we will be interested in a special multiplicity-free space for each family of classical group we studied in the previous section. These families are called *determinantal varieties*, and they play an important role in representation theory, commutative algebra, and other areas of mathematics. For $G = GL_n, O(n)$, or $Sp(n)$, the multiplicity-free decomposition of the determinantal variety will be called the *Second Fundamental Theorem* for G .

3.3.1 The Second Fundamental Theorem for GL_n

We begin by defining a representation of $GL_n \times GL_m$ on the coordinate ring $\mathbb{C}[M_{n,m}]$, where, as before, $M_{n,m}$ is the vector space of $n \times m$ complex matrices. Define the representation $\pi_{n,m}$ on $\mathbb{C}[M_{n,m}]$ as follows:

$$\pi_{n,m}(g, h)f(X) = \pi_{n,m}(g^{-1}Xh),$$

for all $(g, h) \in GL_n \times GL_m$ and $X \in M_{n,m}$. Recall from §3.2.1 that, the representation π of GL_k on $\mathbb{C}[M_{n,k} \times M_{k,m}]$ given by

$$\pi(g)f(x, y) = f(xg, g^{-1}y),$$

for all $(x, y) \in M_{n,k} \times M_{k,m}$ and $g \in GL_n$. Then, the First Fundamental Theorem for GL_n states that the multiplication map μ induces a surjective \mathbb{C} -algebra homomorphism

$$\mu^* : \mathbb{C}[M_{n,m}] \rightarrow \mathbb{C}[(V^*)^n \times (V^m)]^{GL_n},$$

where $V = \mathbb{C}^k$.

In fact (cf., Corollary 5.2.1 in [10]), if $k \geq \min(n, m)$, then the kernel of μ^* is trivial. The Second Fundamental Theorem for GL_n is a description of the kernel of μ^* when $k < \min(n, m)$. In this case, $\mu(M_{n,k} \times M_{k,m})$ is the space of all matrices in $M_{n,m}$ of rank at most k . Denote this set by $\mathcal{D}_{n,m}^{\leq k}$. The set $\mathcal{D}_{n,m}^{\leq k}$ has the structure of an affine variety. It is often called the *determinantal variety*. To differentiate from the determinantal varieties in the next two subsections, we call $\mathcal{D}_{n,m}^{\leq k}$ the *standard determinantal variety*.

Let $\mathcal{J}_{n,m}^k$ be the ideal of polynomials in $\mathbb{C}[M_{n,m}]$ vanishing on the determinantal variety $\mathcal{D}_{n,m}^{\leq k}$. Then $\mathcal{J}_{n,m}^k = \ker(\mu^*)$, and $\mathbb{C}[\mathcal{D}_{n,m}^{\leq k}] = \mathbb{C}[M_{n,m}]/\mathcal{J}_{n,m}^k$. The Second Fundamental Theorem for GL_n contains three parts: generators for the ideal $\mathcal{J}_{n,m}^k$, the decomposition of $\mathcal{J}_{n,m}^k$ as a $GL_n \times GL_m$ -representation, and the decomposition of $\mathbb{C}[\mathcal{D}_{n,m}^{\leq k}]$ as a $GL_n \times GL_m$ -representation. This is Theorem 12.2.12 in [10].

Theorem 3.3.1. *Assume $k < \min(n, m)$.*

1. *The set of all $(k+1) \times (k+1)$ minors is a minimal generating set for the ideal $\mathcal{J}_{n,m}^k$.*
2. *As a $GL_n \times GL_m$ -representation, the determinantal ideal $\mathcal{J}_{n,m}^k$ decomposes as*

$$\mathcal{J}_{n,m}^k \cong \bigoplus_{\lambda} (L_n(\lambda))^* \otimes L_m(\lambda),$$

where $L_n(\lambda)$ is an irreducible highest weight representation of GL_n , $L_m(\lambda)$ is an irreducible highest weight representation of GL_m , and $k < \text{depth}(\lambda) \leq \min(n, m)$.

3. *As a $GL_n \times GL_m$ -representation, the coordinate ring of the determinantal variety decomposes as*

$$\mathbb{C}[\mathcal{D}_{n,m}^{\leq k}] \cong \bigoplus_{\lambda} (L_n(\lambda))^* \otimes L_m(\lambda),$$

where $\text{depth}(\lambda) \leq k$.

We will be particular interested in Part 3 of the above, which we will use to compute a rational function expressing the multivariate Hilbert series of $\mathcal{D}_{n,m}^{\leq k}$ and specializing to the standard single variable Hilbert series in §5.4.3.

3.3.2 The Second Fundamental Theorem for $O(n)$

We now turn to the Second Fundamental Theorem for $O(n)$. As in §3.2.2, let SM_n denote the space of all $n \times n$ symmetric complex matrices. Then we have a map

$$\tau : M_{k,n} \rightarrow SM_n,$$

given by $\tau(X) = X^t X$. By the First Fundamental Theorem for $\mathbb{O}(\times)$, the induced homomorphism

$$\tau^* : \mathbb{C}[SM_n] \rightarrow \mathbb{C}[M_{k,n}]$$

is surjective. As in the case for GL_n , when $k \geq n$, τ^* is also injective (cf., Corollary 5.2.4 in [10]). The Second Fundamental Theorem for $O(n)$ concerns the case where $k < n$.

We begin by defining an action of GL_n on $\mathbb{C}[SM_n]$. Let π be the representation of GL_n on $\mathbb{C}[SM_n]$ given by

$$\pi(g)f(X) = f(g^t X g),$$

for all $g \in GL_n$ and $X \in SM_n$.

Let $k < n$. It can be shown (cf., Lemma 5.2.4 in [10]) that the range of τ consists of all symmetric matrices of rank at most k . This set has the structure of an affine variety, called the *symmetric determinantal variety*. Let $\mathcal{SD}_n^{\leq k}$ denote the symmetric determinantal variety, and let $\mathcal{S}\mathcal{J}_n^k$ denote the ideal of polynomials in $\mathbb{C}[SM_n]$ that vanish on $\mathcal{SD}_n^{\leq k}$. The Second Fundamental Theorem for $O(n)$ describes three things: a set of generators for $\mathcal{S}\mathcal{J}_n^k$, a decomposition for $\mathcal{S}\mathcal{J}_n^k$ as a GL_n -representation, and a decomposition for the coordinate ring $\mathbb{C}[\mathcal{SD}_n^{\leq k}] = \mathbb{C}[SM_n]/\mathcal{S}\mathcal{J}_n^k$ as a GL_n -representation. This is Theorem 12.2.14 in [10]. Here, an *even* dominant integral weight in $P_+(\mathfrak{g})$ is a weight which lies in the lattice cone $\langle 2\omega_1, \dots, 2\omega_l \rangle$, where $\omega_1, \dots, \omega_l$ are the fundamental dominant weights of G .

Theorem 3.3.2. *Assume $k < n$.*

1. *The restriction to SM_n of the $(k+1) \times (k+1)$ minors is a minimal generating set for $\mathcal{S}\mathcal{J}_n^k$.*
2. *As a GL_n -representation, the symmetric determinantal ideal $\mathcal{S}\mathcal{J}_n^k$ decomposes as*

$$\mathcal{S}\mathcal{J}_n^k \cong \bigoplus_{\lambda} L(\lambda),$$

where λ runs over all even dominant integral weights that satisfy $k < \text{depth}(\lambda) \leq n$.

3. *As a GL_n -representation, the coordinate ring of the symmetric determinantal variety decomposes as*

$$\mathbb{C}[\mathcal{SD}_n^{\leq k}] \cong \bigoplus_{\lambda} L(\lambda),$$

where λ runs over all even dominant integral weights of depth at most k .

As before, we will be most concerned with Part 3 of the above Theorem. In §5.4.1, we will use this theorem to compute an explicit rational function representing the multivariate Hilbert series on $\mathcal{SD}_n^{\leq k}$ and use this multivariate series to present an algorithm for computing explicit rational functions representing the standard Hilbert series of $\mathcal{SD}_n^{\leq k}$ for any n and k .

3.3.3 The Second Fundamental Theorem for $Sp(n)$

As in §3.2.3, let AM_n denote the set of all $n \times n$ anti-symmetric complex matrices. Assume that k is even. We have a map

$$\gamma : M_{k,n} \rightarrow AM_n,$$

given by $\gamma(X) = X^t J_k X$, where J_k is the matrix defined in §3.2.3. The induced homomorphism on the coordinate rings

$$\gamma^* : \mathbb{C}[AM_n] \rightarrow \mathbb{C}[M_{k,n}]$$

is surjective by the First Fundamental Theorem for $Sp(n)$. By Corollary 5.2.4 in [10], γ^* is also injective when $k \geq n$. The Second Fundamental Theorem for $Sp(n)$ concerns the case where $k < n$.

We begin by defining an action of GL_n on $\mathbb{C}[AM_n]$. Let π be the representation of GL_n on $\mathbb{C}[AM_n]$ given by

$$\pi(g)f(X) = f(g^t X g),$$

for all $g \in GL_n$ and $X \in AM_n$.

Assume $k < n$. By Lemma 5.2.5 in [10], the range of γ consists of all rank at most k matrices in AM_n . This set has the structure of an affine variety. Call this variety the *anti-symmetric determinantal variety* and denote it by $\mathcal{AD}_n^{\leq k}$. Let \mathcal{AJ}_n^k denote the ideal of polynomials in $\mathbb{C}[AM_n]$ which vanish on $\mathcal{AD}_n^{\leq k}$. The Second Fundamental Theorem for $Sp(n)$ describes a set of generators for \mathcal{AJ}_n^k , a decomposition of \mathcal{AJ}_n^k as a GL_n -representation, and a decomposition of the coordinate ring of $\mathcal{AD}_n^{\leq k}$ as a GL_n -representation. The generators are more complicated in this case than in the GL_n and $O(n)$ cases. For this reason, we focus solely on the decomposition of the coordinate ring $\mathbb{C}[\mathcal{AD}_n^{\leq k}]$. This is Theorem 12.2.15 (Part 3) in [10].

Theorem 3.3.3. *Assume $k < n$. As a GL_n -representation, the coordinate ring of the anti-symmetric determinantal variety decomposes as*

$$\mathbb{C}[\mathcal{AD}_n^{\leq k}] \cong \bigoplus_{\lambda} L(\lambda),$$

where $\text{depth}(\lambda) \leq k$.

We will use this theorem to compute both singly- and multi-graded Hilbert series on the anti-symmetric determinantal variety in Chapter 5. The computation in this case is more complicated than that of the symmetric or standard determinantal varieties, but the same methods may be used.

Chapter 4

Generalization of a theorem of Gross and Wallach

In this chapter, we begin with an exposition of the formulation by Benedict H. Gross and Nolan R. Wallach of the Hilbert series of a partial flag variety. This exposition is taken from [12], using the Borel-Weil theorem and other standard facts on the representation theory of algebraic groups. The author recommends the text [9] for a clear presentation of these facts. We then present a multivariate generalization of the Hilbert series from [12], and prove that this generalization holds for all semisimple, simply connected linear algebraic groups over \mathbb{C} . We conclude with a brief description of the diagonal embedding of a partial flag variety, whose multivariate Hilbert series may be computed using our generalization. Much of this material may be found in the author's ArXiv e-print [17].

4.1 Results from Gross and Wallach

Let $\lambda \in P_+(\mathfrak{g})$, and let L be the unique line in $L(\lambda)$ fixed by B . Equivalently, let H be the unique hyperplane in $L(\lambda)^*$ fixed by B . Since the stabilizer of H contains all elements of G which fix H , the stabilizer is a parabolic subgroup. Denote this subgroup by P_λ . For any fixed Borel subgroup B , there are finitely many parabolic subgroups containing B . Therefore, many different dominant weights correspond to the same subgroup P_λ . For instance, whenever the weight is *regular*,

ie. $\lambda \in \mathbb{Z}^+\omega_1 \oplus \cdots \oplus \mathbb{Z}^+\omega_k$, where \mathbb{Z}^+ denotes all positive integers, $P_\lambda = B$. However, each different weight will correspond to a different embedding of the partial flag variety into a projective space. Namely, the partial flag variety $X_\lambda := G/P_\lambda$ embeds in the projective space $\mathbb{P}(L(\lambda))$ of all hyperplanes in $L(\lambda)^*$ via the map

$$\begin{aligned} \pi_\lambda : G/P_\lambda &\rightarrow \mathbb{P}(L(\lambda)) \\ gP &\mapsto g.H, \end{aligned}$$

which embeds G/P_λ onto the unique closed orbit of G on $\mathbb{P}(L(\lambda))$ (cf., p. 392 in [9]). Recall that the Hilbert series of a projective variety is dependent on its embedding in a projective space. In [12], the authors prove the following theorem.

Theorem (Gross and Wallach). The Hilbert series of the embedding π_λ of G/P_λ in $\mathbb{P}(L(\lambda))$ is given by

$$\prod_{\alpha \in \Phi^+} \left(c_\lambda(\alpha)q^{\frac{d}{dq}} + 1 \right) \frac{1}{1-q}.$$

The first step to understanding this result is to understand the homogeneous coordinate ring $A(X_\lambda)$. In order to do this, we realize irreducible representations of G as sections of line bundles over the partial flag variety G/P_λ . This realization is the celebrated Borel-Weil Theorem (cf., p. 393 in [9]). Note that our statement below of the Borel-Weil Theorem is slightly different than that given in §2.4. This is because the projective space we are considering is the space of *hyperplanes* and not *lines*. This version is ‘dual’ to the previous version, in that the bundle \mathcal{L}_λ corresponds to the representation $L(\lambda)$. Let $\mathcal{O}(1)$ be the tautological line bundle on $\mathbb{P}(L(\lambda))$. Then the pullback $\pi_\lambda^*\mathcal{O}(1) := \mathcal{L}_\lambda$ is a G -equivariant line bundle on X . Let $H^0(X_\lambda, \mathcal{L}_\lambda)$ be the space of sections from X_λ into \mathcal{L}_λ .

Borel-Weil Theorem. Let λ be a dominant integral weight of \mathfrak{g} . The space of sections $H^0(X_\lambda, \mathcal{L}_\lambda)$ is equivalent to the irreducible highest weight representation $L(\lambda)$.

The Borel-Weil theorem is a concrete realization of the irreducible highest weight representations of G . Recall from §2.4 that using this theorem, we can cook up an

isomorphism between the weight lattice $P_+(\mathfrak{g})$ and the group of line bundles on G/B . Under this isomorphism, the bundle $\mathcal{L}_\lambda^n := \pi_\lambda^* \mathcal{O}(n)$ has sections $H^0(X_\lambda, \mathcal{L}_\lambda^n) = L(n\lambda)$ for all $n \geq 0$. All of these bundles are G -equivariant.

Following [12], we consider the restriction homomorphism $H^0(\mathbb{P}(L(\lambda)), \mathcal{O}(n)) \rightarrow H^0(X_\lambda, \mathcal{L}_\lambda^n)$. This homomorphism is both G -equivariant and nonzero. Since $L(n\lambda)$ is an irreducible representation, this means that the restriction is surjective. Thus, the embedding π_λ of X_λ is projectively normal. Therefore, the coordinate ring of X_λ is given by

$$A(X_\lambda) = \bigoplus_{n \in \mathbb{N}} L(n\lambda).$$

This decomposition is graded, since $L(\lambda)L(\mu) \subseteq L(\lambda + \mu)$ for all dominant weights λ, μ . The Hilbert series of this embedding is then given by

$$HS_q(\pi_\lambda) = \sum_{n \in \mathbb{N}} \dim(L(n\lambda))q^n.$$

Using the fact that the Weyl Dimension Formula implies that

$$\dim(L(n\lambda)) = \prod_{\alpha \in \Phi^+} (1 + nc_\lambda(\alpha)),$$

Gross and Wallach show via direct computation that the Hilbert series has the form given in this section.

One of the reasons this result is interesting is that the projective varieties G/P_λ include many of the most interesting varieties from classical algebraic geometry. We give the details for one such example.

The following exposition can be found in Lecture 6 in [13]. We begin by defining the Plücker embedding of the Grassmannian $G(k, n)$ consisting of all $(n - k)$ -planes in \mathbb{C}^n . That is we define $G(k, n)$ to consist of all subspaces of \mathbb{C}^n of codimension k . The Plücker embedding of $G(k, n)$ into $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ is an object of classical interest in algebraic geometry¹, and it can be realized as a certain quotient of $SL(n, \mathbb{C})$. In

¹Recall that, in the notation of [12], $\mathbb{P}(V)$ is the projective space of all *hyperplanes* in V . This is isomorphic to the classical projective space of all *lines* through the origin in V^* .

order to do this, let $W \in G(k, n)$. Assume the vectors v_1, \dots, v_{n-k} are a basis for W . Assign to W the multivector $v_1 \wedge \dots \wedge v_{n-k} \in \bigwedge^{n-k} \mathbb{C}^n$. This defines an embedding of $G(n, k)$ into $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$, called the Plücker embedding.

In §7 of [12], they present the Plücker embedding as the quotient $SL(n, \mathbb{C})/P_\lambda$, where λ is the weight $e_1 + e_2 + \dots + e_k$. They use their formulation of the Hilbert series to compute the dimension and degree of the Plücker embedding. In a similar way, they are able to compute geometric information about the Veronese, Segre, and flag varieties by first realizing them as a quotient of an algebraic group.

4.2 The multivariate case

Let $\lambda_1, \dots, \lambda_k$ be a finite collection of weights in $P_+(\mathfrak{g})$. Consider the finitely generated lattice cone $\langle \lambda_1, \dots, \lambda_k \rangle := \mathbb{N}\lambda_1 \oplus \dots \oplus \mathbb{N}\lambda_k \subset P_+(\mathfrak{g})$. Consider the formal power series

$$HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle = \sum_{\mathbf{a} \in \mathbb{N}^k} \dim(L(a_1\lambda_1 + \dots + a_k\lambda_k)) \mathbf{q}^{\mathbf{a}},$$

where $\mathbf{q}^{\mathbf{a}} := q_1^{a_1} \dots q_k^{a_k}$. The coefficient of $\mathbf{q}^{\mathbf{a}}$ is the dimension of the irreducible highest weight representation of G with highest weight $a_1\lambda_1 + \dots + a_k\lambda_k$. Therefore, $HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle$ is a generating function for the dimensions of the irreducible highest weight representations with highest weight lying in the finitely generated lattice cone $\langle \lambda_1, \dots, \lambda_k \rangle$.

In the case that $k = 1$, this is the ray $\mathbb{N}\lambda$ in the dominant chamber of the weight lattice through the weight λ . We can then think of the Hilbert series considered by Gross and Wallach as the generating function for the dimensions of highest weight representations whose highest weight lies on the ray $\langle \lambda \rangle$. Namely, since

$$HS_q(\pi_\lambda) = HS_q\langle \lambda \rangle = \sum_{a \in \mathbb{N}} \dim(L(a\lambda)) q^a,$$

the Hilbert series of G/P_λ can be interpreted combinatorially as the generating function of the dimensions of irreducible highest weight representations whose highest weights lie on a particular lattice cone. The formal power series $HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle$ can

the be thought of as an immediate generalization of this idea to an arbitrary finitely generated lattice cone. The “ HS ” in our notation is not by accident. In the next section, we will give a geometric interpretation of $HS_{\mathfrak{q}}\langle\lambda_1, \dots, \lambda_k\rangle$ as a multivariate Hilbert series on a class of projective varieties related to the partial flag varieties.

We will next be concerned with proving a generalization of the formulation given by Gross and Wallach of the closed form of the Hilbert series for π_{λ} to the multivariate series. We will prove the following theorem. This theorem is the heart of the dissertation, and will be used throughout Chapter 5 to compute single variable and multivariate Hilbert series for many classes of homogeneous varieties.

Theorem 4.2.1. *Let $\lambda_1, \dots, \lambda_k$ be dominant integral weights. Then*

$$HS_{\mathfrak{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_1}(\alpha)q_1 \frac{\partial}{\partial q_1} + \dots + c_{\lambda_k}(\alpha)q_k \frac{\partial}{\partial q_k} \right) \prod_{i=1}^k \frac{1}{1 - q_i},$$

where $c_{\lambda}(\alpha) := \frac{(\lambda, \alpha)}{(\rho, \alpha)}$.

Note that, in the case where $k = 1$, we recover the result from [12] in the previous section. The multivariate series can be thought of as encoding the Hilbert series of many different embeddings of partial flag varieties simultaneously. We can recover the individual embeddings by specializing the multivariate series in the appropriate way. For example, fix some $j \in \{1, \dots, k\}$ and set $q_i = 0$ for all $i \neq j$. Then we have

$$HS_{(0, \dots, q_j, \dots, 0)}\langle\lambda_1, \dots, \lambda_k\rangle = \prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_j}(\alpha)q_j \frac{\partial}{\partial q_j} \right) \frac{1}{1 - q_j} = HS_{q_j}(\pi_{\lambda_j}).$$

In other words, the above specialization gives the Hilbert series of the embedding π_{λ_k} of the partial flag variety G/P_{λ_k} .

In the case where k is the rank of \mathfrak{g} , and $\lambda_i = \omega_i$ for all i , the cone is the entire weight lattice. In this case, the series $HS_{\mathfrak{q}}\langle\lambda_1, \dots, \lambda_k\rangle$ is a generating function for *all* the dimensions of the finite dimensional irreducible representations of G , where each representation is indexed by its coordinates in the basis $\{\omega_1, \dots, \omega_k\}$ for the dominant chamber of the weight lattice.

There are many other specializations which produce interesting single variable and multivariate Hilbert series on interesting varieties. We will revisit many of these specializations in the next chapter.

4.3 Proof of the generalization

We will now prove the theorem from the previous section. This is our main theorem, and we will spend the rest of the dissertation applying this theorem to compute single variable and multivariate Hilbert series on many varieties of particular interest in representation theory.

Theorem 4.2.1. Let $\lambda_1, \dots, \lambda_k$ be dominant integral weights. Then

$$HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle = \prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_1}(\alpha)q_1 \frac{\partial}{\partial q_1} + \dots + c_{\lambda_k}(\alpha)q_k \frac{\partial}{\partial q_k} \right) \prod_{i=1}^k \frac{1}{1 - q_i},$$

where $c_{\lambda}(\alpha) := \frac{(\lambda, \alpha)}{(\rho, \alpha)}$.

Note: In the formula in the above theorem, we are applying the partial differential operator

$$\prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_1}(\alpha)q_1 \frac{\partial}{\partial q_1} + \dots + c_{\lambda_k}(\alpha)q_k \frac{\partial}{\partial q_k} \right)$$

to the rational function

$$\prod_{i=1}^k \frac{1}{1 - q_i}.$$

Proof. Consider $L(a_1\lambda_1 + \dots + a_k\lambda_k)$. By the Weyl Dimension Formula, we have

$$\dim(L(a_1\lambda_1 + \dots + a_k\lambda_k)) = \prod_{\alpha \in \Phi^+} \frac{(a_1\lambda_1 + \dots + a_k\lambda_k + \rho, \alpha)}{(\rho, \alpha)}.$$

Since the Killing form (\cdot, \cdot) is bilinear, the above product can be rewritten as

$$\dim(L(a_1\lambda_1 + \dots + a_k\lambda_k)) = \prod_{\alpha \in \Phi^+} (1 + a_1c_{\lambda_1}(\alpha) + \dots + a_kc_{\lambda_k}(\alpha)).$$

Therefore, we can rewrite the series as

$$HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \sum_{\mathbf{a} \in \mathbb{N}^k} \prod_{\alpha \in \Phi^+} (1 + a_1 c_{\lambda_1}(\alpha) + \dots + a_k c_{\lambda_k}(\alpha)) \mathbf{q}^{\mathbf{a}}, \quad (4.1)$$

where $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$, and $\mathbf{q}^{\mathbf{a}} := q_1^{a_1} \dots q_k^{a_k}$. Consider the product

$$\prod_{\alpha \in \Phi^+} (1 + a_1 c_{\lambda_1}(\alpha) + \dots + a_k c_{\lambda_k}(\alpha)).$$

Note that the above is a polynomial in the a_i for $1 \leq i \leq k$. Let $d := |\Phi^+|$ and $|\mathbf{i}| := i_1 + \dots + i_k$, for any $\mathbf{i} \in \mathbb{N}^k$. Then, expanding the product of sums into a sum of products, we have

$$\prod_{\alpha \in \Phi^+} (1 + a_1 c_{\lambda_1}(\alpha) + \dots + a_k c_{\lambda_k}(\alpha)) = \sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \mathbf{a}^{\mathbf{i}}, \quad (4.2)$$

where $b_{\mathbf{i}}$ is the coefficient of the monomial $\mathbf{a}^{\mathbf{i}}$ for each \mathbf{i} with $|\mathbf{i}| \leq d$. Combining (4.1) and (4.2), we have

$$HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \sum_{\mathbf{a} \in \mathbb{N}^k} \sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \mathbf{a}^{\mathbf{i}} \mathbf{q}^{\mathbf{a}}.$$

The coefficients of the polynomial (4.2) do not depend on \mathbf{a} . Therefore, we may rearrange the order of summation as follows:

$$HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \sum_{\mathbf{a} \in \mathbb{N}^k} \mathbf{a}^{\mathbf{i}} \mathbf{q}^{\mathbf{a}}. \quad (4.3)$$

We now find a rational function representing the series $\sum_{\mathbf{a} \in \mathbb{N}^k} \mathbf{a}^{\mathbf{i}} \mathbf{q}^{\mathbf{a}}$. Fix a k -tuple $(i_1, \dots, i_k) \in \mathbb{N}^k$, and define $f_{(i_1, \dots, i_k)}(\mathbf{q})$ to be the formal power series

$$f_{(i_1, \dots, i_k)}(\mathbf{q}) := \sum_{\mathbf{a} \in \mathbb{N}^k} \mathbf{a}^{\mathbf{i}} \mathbf{q}^{\mathbf{a}}.$$

Then applying the partial differential operator $q_j \frac{\partial}{\partial q_j}$ to $f_{(i_1, \dots, i_k)}(\mathbf{q})$ increases the integer in the j th coordinate by one. To see this, note that for each summand $\mathbf{a}^{\mathbf{i}} \mathbf{q}^{\mathbf{a}}$, we have

$$\frac{\partial}{\partial q_j} (a_1^{i_1} \dots a_j^{i_j} \dots a_k^{i_k} q_1^{a_1} \dots q_j^{a_j} \dots q_k^{a_k}) = a_1^{i_1} \dots a_j^{i_j+1} \dots a_k^{i_k} q_1^{a_1} \dots q_j^{a_j-1} \dots q_k^{a_k}.$$

Multiplying both sides by q_j , we have

$$q_j \frac{\partial}{\partial q_j} (f_{(i_1, \dots, i_k)}(\mathbf{q})) = f_{(i_1, \dots, i_j+1, \dots, i_k)}(\mathbf{q}).$$

Define $f_{(0, \dots, 0)}(\mathbf{q}) := \prod_{j=1}^k \frac{1}{1 - q_j}$. Because the differential operators $q_j \frac{\partial}{\partial q_j}$ commute for all j , we have

$$f_{(i_1, \dots, i_k)}(\mathbf{q}) = \left(q_1 \frac{\partial}{\partial q_1} \right)^{i_1} \cdots \left(q_k \frac{\partial}{\partial q_k} \right)^{i_k} \prod_{j=1}^k \frac{1}{1 - q_j}$$

Consider the k -tuple $\left(q_1 \frac{\partial}{\partial q_1}, \dots, q_k \frac{\partial}{\partial q_k} \right)$, and define

$$\left(\mathbf{q} \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}} := \left(q_1 \frac{\partial}{\partial q_1} \right)^{i_1} \cdots \left(q_k \frac{\partial}{\partial q_k} \right)^{i_k}.$$

Then, we have

$$f_{(i_1, \dots, i_k)}(\mathbf{q}) = \left(\mathbf{q} \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}} \prod_{j=1}^k \frac{1}{1 - q_j}.$$

Therefore, (4.3) becomes

$$\left[\sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \left(\mathbf{q} \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}} \right] \prod_{j=1}^k \frac{1}{1 - q_j}. \quad (4.4)$$

The crucial point in this proof is to note that the sum

$$\sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \left(\mathbf{q} \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}} \quad (4.5)$$

is the same polynomial as in (4.2), after making the substitution $a_i \mapsto q_i \frac{\partial}{\partial q_i}$, for each $i = 1, \dots, k$. Therefore, the polynomial in (4.5) factors in the same way. Namely,

$$\sum_{|\mathbf{i}| \leq d} b_{\mathbf{i}} \left(\mathbf{q} \frac{\partial}{\partial \mathbf{q}} \right)^{\mathbf{i}} = \prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_1}(\alpha) q_1 \frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha) q_k \frac{\partial}{\partial q_k} \right).$$

Combining this fact with (4.4), we have shown that

$$HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle = \prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_1}(\alpha)q_1 \frac{\partial}{\partial q_1} + \dots + c_{\lambda_k}(\alpha)q_k \frac{\partial}{\partial q_k} \right) \prod_{j=1}^k \frac{1}{1 - q_j}$$

□

4.4 A geometric interpretation of $HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle$

In the single variable case, studied in [12], the problem was well motivated by computing an explicit rational function representing the Hilbert series of all G -equivariant embeddings of a partial flag variety G/P_λ into projective space. We wish to present a similar geometric motivation for the series $HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle$. There are a few different choices of varieties with multivariate Hilbert series given by $HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle$. In this section we present the one most closely related to the setting described in [12]. In the next chapter (namely §5.1 and §5.3), we discuss two more settings where Theorem 4.2.1 explicitly computes multivariate Hilbert series on classes of varieties.

Let G be a linear algebraic group, as before, with Borel subgroup B . Let P be the parabolic subgroup that simultaneously stabilizes the hyperplane H_i in $L(\lambda_i)$ fixed by B for the dominant weights $\lambda_1, \dots, \lambda_k$ in $P_+(\mathfrak{g})$. In particular, $P = P_1 \cap \dots \cap P_k$, where P_i is the stabilizer in G of λ_i for $i = 1, \dots, k$. Then, we have a map

$$\pi : G/P \rightarrow \mathbb{P}(L(\lambda_1)) \times \dots \times \mathbb{P}(L(\lambda_k)),$$

where, as before, $\mathbb{P}(L(\lambda_i))$ is the projective space of hyperplanes in $L(\lambda_i)$ for each $i = 1, \dots, k$, and $\pi(gP) = (g.H_1, \dots, g.H_k)$ is the diagonal embedding of G/P into $\mathbb{P}(L(\lambda_1)) \times \dots \times \mathbb{P}(L(\lambda_k))$.

The coordinate ring of this embedding can be described in the language of line bundles, similarly to the discussion in §4.1. Let $X = \pi(G/P)$. Then, we have the following description of the coordinate ring of X (cf., [2])

$$A(X) = \bigoplus_{\mathbf{a} \in \mathbb{N}^k} H^0(X, \mathcal{L}^{a_1, \dots, a_k}),$$

where $\mathcal{L}^{a_1, \dots, a_k}$ is the line bundle on G/P given by the pullback of $\mathcal{O}(1)$ on

$$\mathbb{P}(L(\lambda_1)) \times \cdots \times \mathbb{P}(L(\lambda_k))$$

along π . The space of sections $H^0(X, \mathcal{L}^{a_1, \dots, a_k})$ is isomorphic as a G -representation to the highest weight representation $L(a_1\lambda_1 + \cdots + a_k\lambda_k)$.

The above discussion can be summarized in the following way: the coordinate ring of the diagonal embedding of the partial flag variety $X = G/P$ is given by:

$$A(X) = \bigoplus_{\mathbf{a} \in \mathbb{N}^k} L(a_1\lambda_1 + \cdots + a_k\lambda_k).$$

Therefore, the series $HS_{\mathbf{q}}\langle \lambda_1, \dots, \lambda_k \rangle$ is the multivariate Hilbert series of the embedding π of the partial flag variety G/P . This is another way in which Theorem 4.2.1 can be thought of as a direct generalization of the ideas in [12].

Chapter 5

Examples of multivariate Hilbert series on homogeneous varieties

In this chapter, we compute explicit rational forms for the multivariate Hilbert series of a suite of examples of varieties of great interest in representation theory. We begin by discussing the coordinate ring of the variety $U \backslash G$, where U is a maximal unipotent subgroup of G . Other examples include the coordinate ring, $\mathbb{C}[G]$, of the group G , considered as a $G \times G$ -representation, as well as the three families of determinantal varieties. We present computations for the coordinate ring of the Kostant cone, as well as some series which behave particularly nicely.

5.1 The homogeneous variety $U \backslash G$

We begin our examples by considering the coordinate ring of the variety $U \backslash G$, where U is a maximal unipotent subgroup of G such that $B = T \cdot U$.

Following the discussion in §3.3 of [27], consider the action of $U \times G \subset G \times G$ on $\mathbb{C}[G]$ given by $(u, g) \cdot f(x) = f(u^{-1}xg)$. The U -invariants $\mathbb{C}[G]^U$ are isomorphic to the ring $\mathbb{C}[U \backslash G]$, where we consider U acting on the left. We wish to decompose the ring $\mathbb{C}[U \backslash G]$ as a G -representation.

Since the torus T normalizes U , we have a left torus action on $\mathbb{C}[U \backslash G]$ given by $t \cdot f(x) = f(t^{-1}x)$. We then have a Peter-Weyl decomposition

$$\mathbb{C}[U \backslash G] = \bigoplus_{\lambda \in P_+(\mathfrak{g})} (L(\lambda)^*)^U \otimes L(\lambda).$$

The Theorem of the Highest Weight implies that $(L(\lambda)^*)^U$ is one dimensional. To emphasize this, we replace $(L(\lambda)^*)^U$ with \mathbb{C}_λ . Therefore, we have

$$\mathbb{C}[U \backslash G] = \bigoplus_{\lambda \in P_+(\mathfrak{g})} \mathbb{C}_\lambda \otimes L(\lambda).$$

The action of the torus T equips $\mathbb{C}[U \backslash G]$ with a gradation by the character group $\mathfrak{X}(T) \cong \mathbb{Z}^k$, where k is the rank of \mathfrak{g} . To see this, note that every weight $\lambda \in P^+\mathfrak{g}$ defines a character e^λ of T . If we look at the weight spaces $\mathbb{C}[U \backslash G]_\lambda := \{f \in \mathbb{C}[U \backslash G] \mid f(t^{-1}x) = e^\lambda(t)f(x) \forall x \in U \backslash G, t \in T\}$, under the right action of G on $\mathbb{C}[U \backslash G]$ given by $f(x).g = f(xg)$, $\mathbb{C}[U \backslash G]_\lambda \cong L(\lambda)$. Note also that $\mathbb{C}[U \backslash G]_\lambda \mathbb{C}[U \backslash G]_\mu = \mathbb{C}[U \backslash G]_{\lambda+\mu}$. Thus, the algebra $\mathbb{C}[U \backslash G]$ is graded by $\mathfrak{X}(T)$ via the weight space decomposition.

As graded algebras, we have

$$\mathbb{C}[U \backslash G] \cong \bigoplus_{\lambda \in P_+(\mathfrak{g})} L(\lambda).$$

In other words, the graded components of the homogeneous coordinate ring of the full flag variety $U \backslash G$ are irreducible highest weight representations. Further, every irreducible highest weight representation of G appears in the decomposition with multiplicity one. Thus, as a G -representation, $\mathbb{C}[U \backslash G]$ decomposes over the lattice cone $\langle \omega_1, \dots, \omega_k \rangle$, where k is the rank of \mathfrak{g} , i.e.

$$\mathbb{C}[U \backslash G] \cong \bigoplus_{\lambda \in \langle \omega_1, \dots, \omega_k \rangle} L(\lambda),$$

and under this gradation, the multivariate Hilbert series of $U \backslash G$ is given by

$$HS_{\mathfrak{q}} \langle \omega_1, \dots, \omega_k \rangle.$$

Note that this settings produces a class of varieties which have multivariate Hilbert series given by Theorem 4.2.1. Since the coordinate ring of $U \backslash G$ decomposes over the cone $\langle \omega_1, \dots, \omega_k \rangle$ generated by the fundamental dominant weights it contains all algebras of the form

$$\bigoplus_{\lambda \in \langle \lambda_1, \dots, \lambda_k \rangle} L(\lambda)$$

as subalgebras. In other words, all algebras which are graded by irreducible highest weight representations of G are contained in $\mathbb{C}[U \setminus G]$. Then, in order to find a suitable variety, X , whose coordinate ring has the form given above, we need only take X to be the spectrum of the algebra, equipped with the Zariski topology. In this case, X will have multivariate Hilbert series

$$HS_X(q_1, \dots, q_k) = \sum_{\mathbf{a} \in \mathbb{N}^k} \dim(L(a_1 \lambda_1 + \dots + a_k \lambda_k)) \mathbf{q}^{\mathbf{a}},$$

which can be explicitly computed as a rational function using Theorem 4.2.1. This settings motivates the study of formal power series of the form $HS_{\mathbf{q}} \langle \lambda_1, \dots, \lambda_k \rangle$ from a geometric point of view.

5.2 The coordinate ring of an algebraic group

In this section, we consider the coordinate ring $\mathbb{C}[G]$ of a semisimple linear algebraic group as a representation of the semisimple group $G \times G$ under the action given in the previous section:

$$(g, h).f(x) := f(g^{-1}xh).$$

Under this action, the coordinate ring has the following decomposition into irreducible $G \times G$ -representations:

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in P_+(\mathfrak{g})} (L(\lambda))^* \otimes L(\lambda).$$

The highest weight of the irreducible $G \times G$ -representation $(L(\lambda))^* \otimes L(\lambda)$ can be represented by the ordered pair of weights $(-s\lambda, \lambda)$, where s is the longest element of the Weyl group of G , acting on the weight λ as discussed in §2.5. We can then use Theorem 4.2.1 to compute a multivariate Hilbert series for $\mathbb{C}[G]$.

Since we are assuming G is linear algebraic, we can simplify the above by augmenting the action of $G \times G$ on $\mathbb{C}[G]$. Let g^t denote the transpose of the matrix $g \in G$. Consider the action of $G \times G$ on $\mathbb{C}[G]$ given by

$$(g, h).f(x) := f(g^t x h).$$

In this case, the coordinate ring has the following decomposition into irreducible $G \times G$ -representations:

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in P_+(\mathfrak{g})} L(\lambda) \otimes L(\lambda).$$

Note that, since $\dim(L(\lambda)) = \dim((L(\lambda))^*)$, the tensor products in the two decompositions have the same dimension. In particular, the graded subspaces in both decompositions have equal dimension. Therefore, they have the same multivariate Hilbert series. However, the second decomposition is simpler from the point of view of lattice cones, since λ runs through the cone $\langle(\omega_1, \omega_1), \dots, (\omega_k, \omega_k)\rangle$, where $\omega_1, \dots, \omega_k$ are the fundamental dominant weights of G . This decomposition avoids the action of the Weyl group on the weight lattice.

We now compute some examples. Begin by considering the coordinate ring of $G = SL_2$. The group G has only one fundamental dominant weight. Label this weight as ω (often times this weight is labeled as simply 1, due to the fact that $P_+(\mathfrak{sl}_2) = \mathbb{N}$). Then, by the discussion above, the coordinate ring decomposes as an $SL_2 \times SL_2$ -representation in the following way:

$$\mathbb{C}[SL_2] \cong \bigoplus_{\lambda \in \langle(\omega, \omega)\rangle} L(\lambda).$$

Therefore, the series $HS_q\langle(\omega, \omega)\rangle$ is singly-graded. Note that we do not need the full machinery of Theorem 4.2.1 to compute a singly-graded series. This could be computed using the results in [12]. However, this example is interesting as a simple base case for more complicated algebraic groups.

We now compute $HS_q\langle(\omega, \omega)\rangle$. Let $\Phi^+ = \{(\alpha, 0)\} \cup \{(0, \alpha)\}$ denote the positive roots of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, where α is the positive root of \mathfrak{sl}_2 . Then

$$c_{(\omega, \omega)}(\alpha, 0) = 1 = c_{(\omega, \omega)}(0, \alpha),$$

and the Hilbert series is particularly simple to compute. We have

$$HS_q\langle(\omega, \omega)\rangle = (1 + q \frac{\partial}{\partial q})^2 \frac{1}{1 - q} = \frac{1 + q}{(1 - q)^3}.$$

For another example, this time with multivariate Hilbert series, let $G = SL_3$. In this case, the multivariate Hilbert series of $\mathbb{C}[G]$ is given by

$$HS_{\mathbf{q}}\langle(\omega_1, \omega_1), (\omega_2, \omega_2)\rangle.$$

We then have the following figure, after computing the coefficients of the differential operators that contribute non-trivial terms in the formula from Theorem 4.2.1.

| α | $c_{(\omega_1, \omega_1)}(\alpha)$ | $c_{(\omega_2, \omega_2)}(\alpha)$ |
|----------------------------|------------------------------------|------------------------------------|
| $(\alpha_1, 0)$ | 1 | 0 |
| $(\alpha_2, 0)$ | 0 | 1 |
| $(\alpha_1 + \alpha_2, 0)$ | 1/2 | 1/2 |
| $(0, \alpha_1)$ | 1 | 0 |
| $(0, \alpha_2)$ | 0 | 1 |
| $(0, \alpha_1 + \alpha_2)$ | 1/2 | 1/2 |

Figure 5.1: Coefficients for the differential operators in $HS_{\mathbf{q}}\langle(\omega_1, \omega_1), (\omega_2, \omega_2)\rangle$.

Note that the differential operators that contribute to the product formula in Theorem 4.2.1 for the semisimple group $SL_3 \times SL_3$ are precisely the same as those which would contribute in $HS_{\mathbf{q}}\langle\omega_1, \omega_2\rangle$ for $G = SL_3$, counted once for each copy of SL_3 . This holds true in any semisimple group.

Since there are only two variables in this series, denote them by q and r . Computing the series $HS_{q,r}\langle(\omega_1, \omega_1), (\omega_2, \omega_2)\rangle$ yields the following rational function:

$$\frac{1 + 4q + 4r + q^2 - qr + r^2 - 10q^2r - 10qr^2 + q^3r - q^2r^2 + qr^3 + 4q^3r^2 + 4q^2r^3 + q^3r^3}{(1 - q)^5(1 - r)^5}.$$

Note that the numerator in this series looks complicated. The denominator, on the other hand, is much simpler. It is often easy to compute the denominator of these multivariate series by counting the number of differential operators of a given variable that occur in the rational function.

If we plot the coefficients of the monomials in the numerator along axes corresponding to the powers of the variables involved, we get a very nice picture. See

Figure 5.2. Note that there are two symmetries when we plot these coefficients: one along the diagonal and one along the anti-diagonal. We wish to find a combinatorial description of the symmetries in the numerator of $HS_{\mathbf{q}}\langle\lambda_1, \dots, \lambda_k\rangle$ for a general group. This will be a topic of future study.

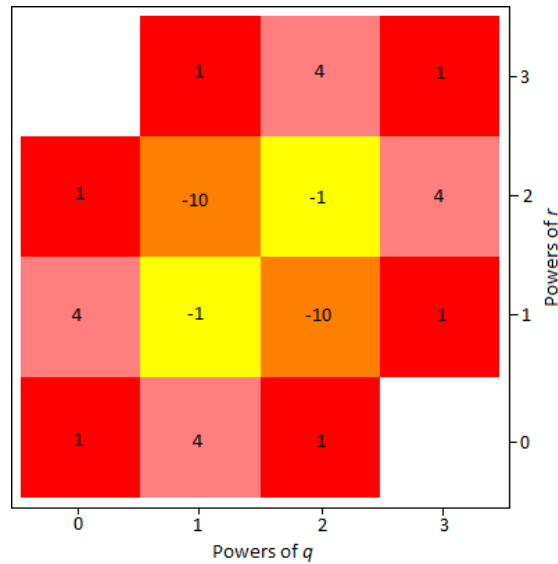


Figure 5.2: The coefficients of the numerator of $HS_{q,r}\langle(\omega_1, \omega_1), (\omega_2, \omega_2)\rangle$.

5.3 The Kostant cone

This section is split into three subsections. The first of which presents without proof the basic definitions and theorems about the Kostant cone, as given in the upcoming book “Basic Geometric Invariant Theory” by Wallach. The next subsection will present a construction of the coordinate ring of a Kostant cone as an infinite dimensional G -module that decomposes into finite dimensional irreducible representations whose highest weights trace out a lattice cone in $P_+(\mathfrak{g})$. The final subsection makes explicit the relationship between the Kostant cone and the multivariate generating function given in the main theorem.

5.3.1 Basic definitions and theorems

We begin by defining the Kostant cone given by a k -tuple of highest weight representations of G . The following definitions and the results from this section can be found in Chapter 4 §3 of [25]. Let $L(\lambda_1), \dots, L(\lambda_k)$ be highest weight representations of G of highest weight λ_i , and choose a highest weight vector $v_i \in L(\lambda_i)$ for each i . Let V be the direct sum $L(\lambda_1) \oplus \dots \oplus L(\lambda_k)$. Then G acts on V diagonally. Let $S(V)$ be the symmetric algebra on V . Then $S(V)$ is \mathbb{N}^k -graded. Following the notation in [25], let $S^{n_1, \dots, n_k}(V) \cong S^{n_1}(V) \otimes \dots \otimes S^{n_k}(V)$ be a multi-homogeneous component of $S(V)$. By Lemma 32 in [25], the multiplicity of the irreducible representation with highest weight $\sum_i n_i \lambda_i$ in $S^{n_1, \dots, n_k}(V)$ is one. Denote this subrepresentation V^{n_1, \dots, n_k} . Then we have the following definition.

Definition 5.3.1. The *Kostant cone* X of V is the set of $v \in V$ such that

$$v^n \in \sum_{n_1 + \dots + n_k = n} V^{n_1, \dots, n_k}.$$

There is a nice, concrete way of describing X in terms of an ‘augmented’ G -action on V . We let the group $G \times (\mathbb{C}^\times)^k$ act on V via

$$(g, z_1, \dots, z_k) \cdot (v_1, \dots, v_k) = (z_1 g \cdot v_1, \dots, z_k g \cdot v_k).$$

This is a quasi-affine variety, and by Theorem 63 in [25], the Zariski closure of this variety is equal to the Kostant cone X on V . Note that, if the highest weights λ_i are all linearly independent, then the action of $G \times (\mathbb{C}^\times)^k$ on V is the same as the diagonal action of G on V . Theorem 63 requires augmenting the action with an additional torus action in order to hold for the case where the weights are dependent.

Note that the variety X is a direct generalization of the orbit of a highest weight vector. In the case of a highest weight λ , with highest weight vector v , $G \times \mathbb{C}^\times \cdot v = G \cdot v$, and the closure of the orbit of the highest weight vector v is the Kostant cone of $V = L(\lambda)$.

One of the interesting properties of the Kostant cone, originally due to Kostant, is the fact that the ideal of polynomials vanishing on X is always generated by

quadratic polynomials. This result is referred to as Kostant's quadratic generation theorem, and proofs can be found in [11], [18], and [19] for the case of a single weight and Chapter 4 of [25] for the general case.

5.3.2 The coordinate ring of the Kostant cone

Note that, in the case that $k = 1$, the Kostant cone is the closure $G.v_\lambda$ in $L(\lambda)$. These varieties have been intensely studied, and their coordinate rings are computed in [24]. Here, we generalize the results in [24] to the multivariate case. Let $L(\lambda_1), \dots, L(\lambda_k)$ be finite dimensional irreducible representations of G with highest weights $\lambda_1, \dots, \lambda_k$, and choose a highest weight vector v_i from each $L(\lambda_i)$. Define $V := L(\lambda_1) \oplus \dots \oplus L(\lambda_k)$ as before. Let X be the Zariski closure of $G \times (\mathbb{C}^\times)^k.(v_1, \dots, v_k)$. In other words, X is the Kostant cone corresponding to $\lambda_1, \dots, \lambda_k$. In this section, we adapt the computation in [24] of the coordinate ring of the orbit of a highest weight vector to find an explicit description of the coordinate ring of X , and we use their notation.

We consider G and its subgroups as subgroups of $G \times (\mathbb{C}^\times)^k$ via the map $g \mapsto (g, 1, \dots, 1)$. We will abuse notation and write $g.(v_1, \dots, v_k)$ when we mean $(g, 1, \dots, 1).(v_1, \dots, v_k)$. We then have

$$b.(v_1, \dots, v_k) = (\lambda_1(b)v_1, \dots, \lambda_k(b)v_k), \forall b \in B. \quad (5.1)$$

Let O be the orbit of (v_1, \dots, v_k) , π the canonical mapping of $V/\{0\}$ onto $\mathbb{P}(V)$, and P the isotropy subgroup of $\pi(v_1, \dots, v_k)$.

Proposition 5.3.2. *P is a parabolic subgroup, containing B .*

Proof. Let π_i be the canonical mapping of $L(\lambda_i)/\{0\}$ onto $\mathbb{P}(L(\lambda_i))$, and let P_i be the isotropy subgroup of $\pi_i v_i$. Then it follows from (3.1) that P_i is a parabolic subgroup containing B . We will show that $P = P_1 \cap \dots \cap P_k$.

To this end, assume $p \in P$. Then p , acting diagonally, stabilizes the line through the origin in V containing (v_1, \dots, v_k) . In particular, p stabilizes the line through the origin in $L(\lambda_i)$ containing v_i . Thus, $P \subset P_1 \cap \dots \cap P_k$.

Now assume $p \in P_1 \cap \cdots \cap P_k$. Then p stabilizes the line through the origin in each $L(\lambda_i)$ that contains v_i . Since P acts diagonally on $\pi(v_1, \dots, v_k)$, this gives $p \in P$. \square

Note that, as characters of B , the weights $\lambda_1, \dots, \lambda_k$ extend uniquely to characters of P . Let H be the isotropy subgroup of (v_1, \dots, v_k) .

Proposition 5.3.3. $H = \{p \in P \mid \lambda_i(p) = 1, i = 1, \dots, k\}$.

Proof. Note that if $\lambda_i(p) = 1$, for all $i = 1, \dots, k$, we have

$$p \cdot (v_1, \dots, v_k) = (\lambda(p)v_1, \dots, \lambda(p)v_k) = (v_1, \dots, v_k),$$

so H contains $\{p \in P \mid \lambda_i(p) = 1, i = 1, \dots, k\}$.

For the opposite inclusion, let $h \in H$. Then, since h fixes (v_1, \dots, v_k) and acts linearly, it must fix the line $\pi(v_1, \dots, v_k)$. So $H \subset P$. Thus $h \cdot (v_1, \dots, v_k) = (\lambda_1(h)v_1, \dots, \lambda_k(h)v_k)$. This implies that $\lambda_i(h) = 1$, for all $i = 1, \dots, k$, since H is the isotropy group of (v_1, \dots, v_k) . \square

We have the following further characterization of X .

Theorem 5.3.4. $X = O \cup \{0\}$.

Proof. O is invariant under multiplication by any element of \mathbb{C}^\times . To see this, let $z \in \mathbb{C}^\times$. Then $(g, z_1, \dots, z_k) \cdot (zv_1, \dots, zv_k) = (zz_1g.v_1, \dots, zz_kg.v_k)$. But this is just $(g, zz_1, \dots, zz_k) \cdot (v_1, \dots, v_k)$, which is in O . Thus, the claim follows from the fact that πO is closed in $\mathbb{P}(V)$. This is true, since $\pi O \cong G/P$, and P is a parabolic subgroup of G . \square

The maps $G \xrightarrow{\tau} O \xrightarrow{\iota} X$, where $\tau(g) = g \cdot (v_1, \dots, v_k)$ and ι is the canonical inclusion generate inclusions

$$\mathbb{C}[X] \hookrightarrow \mathbb{C}[O] \hookrightarrow \mathbb{C}[G]$$

on the level of homogeneous coordinate rings. Further, the maps ι and τ commute with left translations. To see that τ commutes with left translations, consider $\tau(gh)$.

Then

$$\tau(gh) = gh.(v_1, \dots, v_k) = g.(hv_1, \dots, hv_k) = g.\tau(h).$$

Since τ and ι commute with left translations, $\mathbb{C}[X]$ and $\mathbb{C}[O]$ are left-invariant subalgebras of $\mathbb{C}[G]$. Thus, $\mathbb{C}[O] = \mathbb{C}[G/H] = \mathbb{C}[G]^H$, where H is considered to act on the right. H is normal in P . Thus, $\mathbb{C}[O]$ is right-invariant with respect to P . The action of P by right translation then reduces to the action of the torus P/H . Therefore, we have a decomposition into weight spaces

$$\mathbb{C}[O] = \bigoplus_{\lambda \in \mathfrak{X}(P)} \mathbb{C}[O]_\lambda,$$

where $\mathbb{C}[O]_\lambda := \{f \in \mathbb{C}[O] \mid f(gp) = \lambda(p)f(g), g \in G, p \in P\}$.

We define $S(\lambda) := \{f \in \mathbb{C}[G] \mid f(gb) = \lambda(b)f(g), g \in G, b \in B\}$. Note that this is a finite dimensional, left-invariant subspace of $\mathbb{C}[G]$. The set

$$\mathfrak{X}(B)^+ := \{\lambda \in \mathfrak{X}(B) \mid S(\lambda) \neq 0\}$$

is the set of all highest weights of irreducible representations of G . The representation $S(\lambda)$, when $\lambda \in \mathfrak{X}(B)^+$, is dual to the irreducible representation of G of highest weight λ . The duality can be expressed explicitly by

$$\langle f, g.v_\lambda \rangle = f(g) \tag{5.2}$$

for all $f \in S(\lambda)$, $g \in G$, and v_λ a highest weight vector of weight λ .

It is clear that $\mathbb{C}[O]_\lambda \subset S(\lambda)$. Let $\langle \lambda_1, \dots, \lambda_k \rangle$ be the set of all non-negative integer combinations of $\lambda_1, \dots, \lambda_k$.

Proposition 5.3.5. $\mathbb{C}[O]_\lambda \neq 0$ implies λ is an integer combination of $\lambda_1, \dots, \lambda_k$.

Proof. Assume that $\mathbb{C}[O]_\lambda \neq 0$. Take a nonzero $f \in \mathbb{C}[O]$ such that $f(gp) = \lambda(p)f(g)$ for all $g \in G, p \in P$. In particular, $f(gh) = \lambda(h)f(g)$ for all $g \in G, h \in H$. By the way we identify $\mathbb{C}[O]$ as a subalgebra of $\mathbb{C}[G]$, we have

$$f(gh) = f \circ \tau(gh) = f(gh.(v_1, \dots, v_k)) = f(g.(\lambda_1(h)v_1, \dots, \lambda_k(h)v_k)).$$

By Proposition 2,

$$H = \{p \in P \mid \lambda_i(p) = 1, i = 1, \dots, k\}.$$

Thus, $f(gh) = f(g \cdot (v_1, \dots, v_k)) = f \circ \tau(g) = f(g)$. Summarizing, we have

$$f(g) = f(gh) = \lambda(h)f(g).$$

Then $f \neq 0$, implies $\lambda(h) = 1$. Therefore, we have

$$\text{Ker}(\lambda) \supset \text{Ker}(\lambda_1) \cap \dots \cap \text{Ker}(\lambda_k).$$

There exists a bijective correspondence between subgroups of T and subgroups of $\mathfrak{X}(T)$, given by $(\Gamma \leq \mathfrak{X}(T))$

$$\Gamma \mapsto T^\Gamma := \{t \in T \mid \chi(t) = 1 \text{ for all } \chi \in \Gamma\}.$$

This correspondence reverses inclusions: if $T^{\Gamma_1} \subset T^{\Gamma_2}$, then $\Gamma_1 \supset \Gamma_2$. To see this, note that Γ consists of ALL characters whose value on T^Γ is one. If $T^{\Gamma_1} \subset T^{\Gamma_2}$, then any element of Γ_2 has value one on T^{Γ_1} . Thus, $\Gamma_2 \subset \Gamma_1$. Now, let Γ_1 be the subgroup generated by $\lambda_1, \dots, \lambda_r$, and let Γ_2 be the subgroup generated by λ .

We have $T^{\Gamma_1} = \text{Ker}(\lambda_1) \cap \dots \cap \text{Ker}(\lambda_k)$ and $T^{\Gamma_2} = \text{Ker}(\lambda)$. We have already shown that $T^{\Gamma_1} \subset T^{\Gamma_2}$. Thus, $\Gamma_2 \subset \Gamma_1$, and we have proven the claim. \square

Note that Proposition 3 implies that

$$\mathbb{C}[O] \subset \bigoplus_{\lambda} S(\lambda), \quad (5.3)$$

where λ runs through all integer combinations of $\lambda_1, \dots, \lambda_k$.

Proposition 5.3.6. *For each $\lambda \in \langle \lambda_1, \dots, \lambda_k \rangle$, $S(\lambda) \subset \mathbb{C}[X]$.*

Proof. We need to show that $S(\lambda_i) \subset \mathbb{C}[X]$ for all $i = 1, \dots, k$. Then the claim follows from the fact that $S(\lambda) \cdot S(\mu) = S(\lambda + \mu)$ for all $\lambda, \mu \in \mathfrak{X}^+(B)$. Take $f \in S(\lambda_i)$. Then $f(gb) = \lambda_i(b)f(g)$, for all $g \in G, b \in B$. Let v_i be a highest weight vector for $L(\lambda_i)$ as before. Then by (5.2), $f(g) = \langle f, gb \cdot v_i \rangle = \langle f, g \cdot v_i \rangle$. Let $O_i := G \cdot v_i$ for each $i = 1, \dots, k$, and define a map $O_i \rightarrow X$ by

$$g \cdot v_i \mapsto g \cdot (v_1, \dots, v_i, \dots, v_k),$$

for each i . We can think of these maps as inclusions of the O_i into X . In this way, we consider $g \cdot v_i$ to be an element of X . Then, by (5.2), f is a function on X , and $S(\lambda_i) \subset \mathbb{C}[X]$. \square

Note that Proposition 4 implies that

$$\bigoplus_{\lambda \in \langle \lambda_1, \dots, \lambda_r \rangle} S(\lambda) \subset \mathbb{C}[X]. \quad (5.4)$$

Then, by the chain of inclusions $\mathbb{C}[X] \hookrightarrow \mathbb{C}[O] \hookrightarrow \mathbb{C}[G]$ and the fact that λ cannot be a highest weight if the coefficients on any λ_i are negative, (5.3) and (5.4) imply the following theorem.

Theorem 5.3.7. *The homogeneous coordinate ring $\mathbb{C}[X]$ of the Kostant cone X given by the weights $\lambda_1, \dots, \lambda_k$ is*

$$\bigoplus_{\lambda \in \langle \lambda_1, \dots, \lambda_k \rangle} S(\lambda).$$

5.3.3 Multivariate Hilbert series on the Kostant cone

Note that Kostant's quadratic generation theorem gives a method of computing multivariate Hilbert series on the Kostant cone X . The multivariate Hilbert series of a variety whose ideal I has a quadratic generating set can be computed using methods from commutative algebra by looking at the graph of relations given by the leading quadratic monomials in I . However, this method rarely leads to simple direct computations of Hilbert series.

On the otherhand, the main theorem gives a direct way of computing the multivariate Hilbert series of X . Let $\mathbb{C}[X]$ be the homogeneous coordinate ring of X . By Theorem 3, we have

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \langle \lambda_1, \dots, \lambda_k \rangle} S(\lambda).$$

The representations $S(\lambda)$ are dual to the irreducible highest weight representations $L(\lambda)$. In particular, they have the same dimension. So we can use the formula for the main theorem to compute the multivariate Hilbert series on X given by the decomposition into weight spaces in Theorem 3. Let $HS(X)$ be the multivariate Hilbert series of X . Then, explicitly,

$$HS(X) = \prod_{\alpha \in \Phi^+} \left(1 + c_{\lambda_1}(\alpha)q_1 \frac{\partial}{\partial q_1} + \cdots + c_{\lambda_k}(\alpha)q_k \frac{\partial}{\partial q_k} \right) \prod_{i=1}^k \frac{1}{1 - q_i}.$$

This formula holds for any Kostant cone X , regardless of the number of dominant weights involved. Also, this formula is explicit, and can be computed for specific examples with the help of a computer algebra system.

5.4 Determinantal varieties

In this section, we use the Second Fundamental Theorems presented in §3.3 and Theorem 4.2.1 to compute multivariate Hilbert series on the three classical types of determinantal varieties. We may then use an appropriate specialization to find the standard single variable Hilbert series for these varieties. Explicit rational functions representing the single variable Hilbert series were computed by Thomas Enright, Markus Hunziker, and Andrew Pruet in [7] and [8] using BGG resolutions. We present the methods in this section as a simpler way of computing these Hilbert series directly, especially in low rank examples. In each case, there is a sense in which the multivariate series behaves ‘better’ than the single variate series. We have a recursive relationship for finding the multivariate series which does not exist when we restrict to the single variable case. Motivated by this fact, we compute the multivariate series recursively before restricting our variables to obtain the standard Hilbert series.

5.4.1 Symmetric determinantal varieties

As in §3.3, let $\mathcal{SD}_n^{\leq k}$ denote the variety of all rank at most k symmetric $n \times n$ complex matrices.

The Second Fundamental Theorem of Invariant Theory for $O(n)$ (cf., [10], p.561), states that the homogeneous coordinate ring $\mathbb{C}[\mathcal{SD}_n^{\leq k}]$ of $\mathcal{SD}_n^{\leq k}$ decomposes as an GL_n -module in the following way:

$$\mathbb{C}[\mathcal{SD}_n^{\leq k}] \cong \bigoplus_{\lambda} L(\lambda),$$

where λ runs over all even dominant integral weights of depth at most k . Here, an even weight of depth at most k is one that lies in the lattice cone $\langle 2\omega_1, \dots, 2\omega_k \rangle$, where $\omega_1, \dots, \omega_n$ are the fundamental dominant weights of GL_n , using the standard Borel subgroup of upper triangular matrices in GL_n . Note that GL_n is not a semisimple group. However, it is reductive. It can then be given a root system related to the root system of its semisimple part SL_n (cf. Chapter 3 in [10]). In this way, we can consider the decomposition in the Second Fundamental Theorem of Invariant Theory to hold as SL_n -representations, and the result of Theorem 4.2.1 holds. In a similar way, we will consider the Second Fundamental Theorems for GL_n and $Sp(n)$ to give decompositions of the coordinate rings of the anti-symmetric and standard determinantal varieties as SL_n -representations in the following two subsections.

We can then compute the series $HS_{\mathbf{q}}\langle 2\omega_1, \dots, 2\omega_k \rangle$ and specialize the variables in an appropriate way to recover the Hilbert series of the standard embedding of the symmetric determinantal variety. The standard Hilbert series on $\mathcal{SD}_n^{\leq k}$ is given by

$$\sum_{\lambda} \dim(L(\lambda))q^{|\lambda|},$$

where again, λ runs over all even dominant integral weights of depth at most k . After computing the series $HS_{\mathbf{q}}\langle 2\omega_1, \dots, 2\omega_k \rangle$, we specialize to the standard Hilbert series by making the substitution $q_i \mapsto q^i$ for $i = 1, \dots, k$.

We now compute some examples. We consider the variety $\mathcal{SD}_4^{\leq 2}$. We will compute the series $HS_{\mathbf{q}}\langle 2\omega_1, 2\omega_2 \rangle$, where ω_1 and ω_2 are the first two fundamental dominant weights of SL_4 . The main theorem gives us the following rational function for $HS_{\mathbf{q}}\langle 2\omega_1, 2\omega_2 \rangle$:

$$\prod_{1 \leq i < j \leq 4} \left(1 + 2c_{\omega_1}(\epsilon_i - \epsilon_j)q_1 \frac{\partial}{\partial q_2} + 2c_{\omega_2}(\epsilon_i - \epsilon_j)q_2 \frac{\partial}{\partial q_2} \right) \frac{1}{(1 - q_1)(1 - q_2)},$$

where $\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq 4\}$, and ϵ_i is the functional that gives the i th diagonal element of a matrix in $\mathfrak{g} = \mathfrak{sl}_4$. Then computing $c_{\omega_1}(\epsilon_i - \epsilon_j)$ and $c_{\omega_2}(\epsilon_i - \epsilon_j)$ for $1 \leq i < j \leq 4$ gives us

$$(1 + 2q_1 \frac{\partial}{\partial q_1})(1 + 2q_2 \frac{\partial}{\partial q_2})(1 + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2})(1 + q_2 \frac{\partial}{\partial q_2})(1 + \frac{2}{3}q_1 \frac{\partial}{\partial q_1} + \frac{2}{3}q_2 \frac{\partial}{\partial q_2}) \frac{1}{(1-q_1)(1-q_2)}.$$

Applying the differential operators then yields

$$\frac{1+6q_1+15q_2+q_1^2+16q_1q_2+15q_2^2+q_2^3-50q_1q_2^2-29q_1^2q_2-4q_1q_2^3-25q_1^2q_2^2+6q_1^3q_2+21q_1^2q_2^3+20q_1^3q_2^2+6q_1^3q_2^3}{(1-q_1)^4(1-q_2)^5}.$$

This formula seems unmanageable, but is easy to compute with Mathematica or Maple. We can also graph the coefficients the way we did for the coordinate ring of SL_3 to try and find a nice depiction of the numerator (see Figure 5.3). However, after we make the substitution $q_i \mapsto q^i$, we get

$$\frac{1 + 3q + 6q^2}{(1 - q)^7},$$

which is the Hilbert series for the standard embedding of $\mathcal{SD}_4^{\leq 2}$.

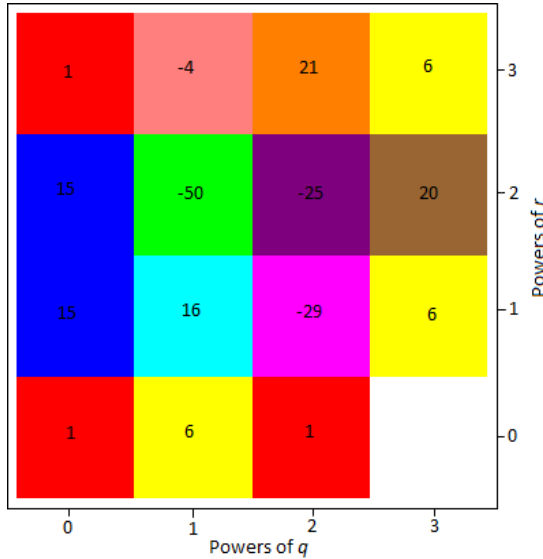


Figure 5.3: The coefficients of the numerator of $HS_{q,r}\langle 2\omega_1, 2\omega_2 \rangle$.

We can then increase the size of the matrices to recursively find the Hilbert series of $\mathcal{SD}_n^{\leq 2}$. Let $\{\alpha_1, \dots, \alpha_{n-1}\}$ be the simple roots of SL_n . The only positive roots of SL_n that contribute to the product in $HS_q\langle \omega_1, \omega_2 \rangle$, are those which can be written as a sum of consecutive simple roots $\sum \alpha_i$ beginning at either α_1 or α_2 . So, as we go from $n - 1$ to n , we add two differential operators, namely, those which correspond

to the positive roots $\alpha_2 + \cdots + \alpha_{n-1}$ and $\alpha_1 + \cdots + \alpha_{n-1}$. If we define $HS_{\mathbf{q}}^n \langle \omega_1, \omega_2 \rangle$ to be the series given by the first two fundamental dominant weights of SL_n , we have the following recursive formula. Note that

$$HS_{\mathbf{q}}^3 \langle 2\omega_1, 2\omega_2 \rangle = \frac{1 + 3q_1 + 3q_2 - 3q_1^2 q_2 - 3q_1 q_2^2 - q_1^2 q_2^2}{(1 - q_1)^3 (1 - q_2)^3}.$$

Lemma. For $n > 3$,

$$HS_{\mathbf{q}}^n \langle 2\omega_1, 2\omega_2 \rangle = \left(1 + \frac{2}{n-2} q_2 \frac{\partial}{\partial q_2}\right) \left(1 + \frac{2}{n-1} q_1 \frac{\partial}{\partial q_1} + \frac{2}{n-1} q_2 \frac{\partial}{\partial q_2}\right) HS_{\mathbf{q}}^{n-1} \langle 2\omega_1, 2\omega_2 \rangle.$$

We obtain the recursion by simply computing the coefficients for the two new weights. Note that this recursion is on the multivariate series, but it does not pass to a recursion on the single variable Hilbert series for the varieties $\mathcal{SD}_n^{\leq 2}$. The recursion is linear in the number of differential operators that contribute to the series when we go from SL_{n-1} to SL_n . In this way, the multivariate series behaves more nicely than the single variable Hilbert series. This multivariate series then allows us to more easily compute the Hilbert series of $\mathcal{SD}_n^{\leq k}$.

This method generalizes to the rank k symmetric determinantal variety. Again, we write $HS_{\mathbf{q}}^n \langle 2\omega_1, \dots, 2\omega_k \rangle$ when considering the weights as weights of SL_n . We have the following.

Proposition 5.4.1. *Let $n > k + 1$. Then*

$$HS_{\mathbf{q}}^n \langle 2\omega_1, \dots, 2\omega_k \rangle = \prod_{i=1}^k \left(1 + \frac{2}{n-i} \sum_{j=i}^k q_j \frac{\partial}{\partial q_j}\right) HS_{\mathbf{q}}^{n-1} \langle 2\omega_1, \dots, 2\omega_k \rangle.$$

Proof. We have a labeling of the fundamental dominant weights and the simple roots for SL_n as $\omega_1, \dots, \omega_{n-1}$ and $\alpha_1, \dots, \alpha_{n-1}$, resp., such that

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \leq i \leq j \leq n-1\},$$

and

$$c_{\omega_k}(\alpha_i + \cdots + \alpha_j) = \begin{cases} \frac{1}{j-i+1} & : i \leq k \leq j \\ 0 & : \text{otherwise} \end{cases}$$

independent of n . Under this labeling, the positive roots of SL_n which contribute to the product but did not contribute at the $(n-1)$ -st step are precisely those roots

whose sum begins with α_i for some $i \leq k$ and ends with α_{n-1} . There are k of these roots, namely $\alpha_1 + \cdots + \alpha_{n-1}, \alpha_2 + \cdots + \alpha_{n-1}, \dots, \alpha_k + \cdots + \alpha_{n-1}$. Then we have

$$\begin{aligned} 2c_{\omega_i}(\alpha_1 + \cdots + \alpha_{n-1}) &= \frac{2}{n-1}, \text{ for } 1 \leq i \leq k, \\ 2c_{\omega_i}(\alpha_2 + \cdots + \alpha_{n-1}) &= \frac{2}{n-2}, \text{ for } 2 \leq i \leq k, \\ &\dots \\ 2c_{\omega_k}(\alpha_k + \cdots + \alpha_{n-1}) &= \frac{2}{n-k}. \end{aligned}$$

This proves the Proposition. \square

We wish to emphasize the recursive nature of the multivariate series. This is due to the fact that we have a labeling of the fundamental dominant weights and positive roots of SL_n that is universal for all n . Then, by the product in Theorem 4.2.1, as we increase n , many of our differential operators can be reused, since $c_{\omega_i}(\alpha)$ does not depend on n . This situation is unique to the multivariate case: there is no analogue for the standard Hilbert series of $\mathcal{SD}_n^{\leq k}$. We will exploit this fact again in the next two subsections.

5.4.2 Antisymmetric determinantal varieties

As in §3.3, let $\mathcal{AD}_n^{\leq 2k}$ denote the variety of all rank at most $2k$ antisymmetric $n \times n$ complex matrices.

The Second Fundamental Theorem of Invariant Theory for $Sp(n)$ (cf., [10], p.562) states that the homogeneous coordinate ring $\mathbb{C}[\mathcal{AD}_n^{\leq 2k}]$ of $\mathcal{AD}_n^{\leq 2k}$ decomposes as an SL_n -module in the following way:

$$\mathbb{C}[\mathcal{AD}_n^{\leq 2k}] \cong \bigoplus_{\lambda} L(\lambda),$$

where λ runs over the lattice cone $\langle \omega_2, \dots, \omega_{2k} \rangle$, where as before, ω_i is the i th fundamental dominant weight of SL_n with respect to the standard Borel subgroup B of upper triangular matrices in SL_n .

We compute the series $HS_{\mathfrak{q}}\langle \omega_2, \dots, \omega_{2k} \rangle$ and specialize the variables in an appropriate way to recover the Hilbert series of the standard embedding of the an-

tisymmetric determinantal variety. The standard Hilbert series on $\mathcal{AD}_n^{\leq 2k}$ is given by

$$\sum_{\lambda} \dim(L(\lambda))q^{|\lambda|},$$

where again, λ runs over the lattice cone $\langle \omega_2, \dots, \omega_{2k} \rangle$. After computing the series $HS_{\mathbf{q}}\langle \omega_2, \dots, \omega_{2k} \rangle$, we specialize to the standard Hilbert series by making the substitution $q_i \mapsto q^i$ for $i = 1, \dots, k$.

As in the previous subsection, we start with the simplest case with a multigradation. Consider \mathcal{AD}_n^4 , where $n \geq 6$. In this case, we wish to compute $HS_{\mathbf{q}}^n\langle \omega_2, \omega_4 \rangle$, where again, the superscript refers to fact that we are considering ω_2 and ω_4 as fundamental dominant weights of the group SL_n . Let $n = 6$. There are twelve differential operators in the formula in Theorem 4.2.1 which are nonzero in this case. These differential operators correspond to positive root strings $\alpha_i + \dots + \alpha_j$, where α_2 and/or α_4 show up somewhere in the string. In this case, those roots and their values are given in the following figure.

| α | $c_{\omega_2}(\alpha)$ | $c_{\omega_4}(\alpha)$ |
|--|------------------------|------------------------|
| α_2 | 1 | 0 |
| α_4 | 0 | 1 |
| $\alpha_1 + \alpha_2$ | 1/2 | 0 |
| $\alpha_2 + \alpha_3$ | 1/2 | 0 |
| $\alpha_3 + \alpha_4$ | 0 | 1/2 |
| $\alpha_4 + \alpha_5$ | 0 | 1/2 |
| $\alpha_1 + \alpha_2 + \alpha_3$ | 1/3 | 0 |
| $\alpha_2 + \alpha_3 + \alpha_4$ | 1/3 | 1/3 |
| $\alpha_3 + \alpha_4 + \alpha_5$ | 0 | 1/3 |
| $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ | 1/4 | 1/4 |
| $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | 1/4 | 1/4 |
| $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | 1/5 | 1/5 |

Figure 5.4: Coefficients for the differential operators in $HS_{\mathbf{q}}^6\langle \omega_2, \omega_4 \rangle$

As we increase n , we add four new differential operators at each step. To see this, note that as n increases to $n + 1$, the new root strings which contribute at least

one nonzero differential operator are those which contain a copy of α_2 or α_4 and end in n . These new root strings are of the form $\alpha_i + \cdots + \alpha_n$ for $i = 1, 2, 3, 4$. There are obviously four such root strings. We then have the following recursive relation.

Lemma. Let $n > 6$. Then

$$HS_{\mathbf{q}}^n \langle \omega_2, \omega_4 \rangle = D \cdot HS_{\mathbf{q}}^{n-1} \langle \omega_2, \omega_4 \rangle,$$

where D is the partial differential operator

$$\left(1 + \frac{1}{n-1}q_1 \frac{\partial}{\partial q_1} + \frac{1}{n-1}q_2 \frac{\partial}{\partial q_2}\right) \left(1 + \frac{1}{n-2}q_1 \frac{\partial}{\partial q_1} + \frac{1}{n-2}q_2 \frac{\partial}{\partial q_2}\right) \left(1 + \frac{1}{n-3}q_2 \frac{\partial}{\partial q_2}\right) \left(1 + \frac{1}{n-4}q_2 \frac{\partial}{\partial q_2}\right).$$

This recursion is obtained by computing the coefficients of the four new differential operators obtained by increasing the size of the matrices from $(n-1) \times (n-1)$ to $n \times n$. Note that even though the recursion is significantly more complicated than in the symmetric case, we still add a fixed amount of differential operators at each step, and the actual computational complexity does not really increase. In a similar fashion, we can obtain a recursive relationship for the multivariate Hilbert series of $\mathcal{AD}_n^{\leq 2k}$.

5.4.3 The standard determinantal varieties

As in §3.3, let $\mathcal{D}_{n,m}^{\leq k}$ denote the variety of all $n \times m$ complex matrices of rank at most k .

The Second Fundamental Theorem of Invariant Theory for GL_n (cf., [10], p.559) states that the homogeneous coordinate ring $\mathbb{C}[\mathcal{D}_{n,m}^{\leq k}]$ decomposes as an $SL_n \times SL_m$ -module in the following way:

$$\mathbb{C}[\mathcal{D}_{n,m}^{\leq k}] \cong \bigoplus_{\lambda} L(\lambda)^* \otimes L(\lambda),$$

where the first representation in the tensor product is considered as an SL_n -representation, the second representation is considered as an SL_m -representation, and λ runs through all dominant integral weights such that $\text{depth}(\lambda) \leq \min(n, m, k)$. From here on out, we assume that $k \leq \min(n, m)$, and we assume that $\text{depth}(\lambda) \leq k$. Now, since SL_n is linear algebraic, we can replace the left action by inversion with the left action by taking the transpose. This yields the decomposition

$$\mathbb{C}[\mathcal{D}_{n,m}^{\leq k}] \cong \bigoplus_{\lambda} L(\lambda) \otimes L(\lambda),$$

where the terms in the product are as before. As an $SL_n \times SL_m$ -representation, the weights in the above decomposition run over the lattice cone $\langle (\omega_1, \omega_1), \dots, (\omega_k, \omega_k) \rangle$, where we use the notation (λ, μ) to denote a dominant integral weight of $G_1 \times G_2$ with $\lambda, \mu \in P_+(\mathfrak{g})$.

We again start by considering the simplest case that comes equipped with a multi-grading. Let $HS_{\mathfrak{q}}^{n,m} \langle (\lambda_1, \mu_1), \dots, (\lambda_k, \mu_k) \rangle$ denote the multivariate series, where the superscript denotes that we think of the λ_i as SL_n -representations and the μ_j as SL_m -representations. Consider the determinantal variety $\mathcal{D}_{3,3}^{\leq 2}$ of rank at most two 3×3 complex matrices. The decomposition of $\mathbb{C}[\mathcal{D}_{3,3}^{\leq 2}]$ into irreducible representations of $SL_3 \times SL_3$ has multivariate Hilbert series

$$HS_{\mathfrak{q}}^{3,3} \langle (\omega_1, \omega_1), (\omega_2, \omega_2) \rangle,$$

where ω_1, ω_2 are the first two fundamental dominant weights of SL_3 . Note that this is the multivariate Hilbert series of the coordinate ring of SL_3 considered in §2 of this chapter.

We can now begin increasing the size of n and m to obtain recursive relations on the multivariate Hilbert series. We first consider increasing m . As we increase $m - 1$ to m , we will add two new nonzero differential operators to the product, namely those corresponding to the positive roots $\alpha_1 + \dots + \alpha_m - 1$ and $\alpha_2 + \dots + \alpha_m - 1$. For these two positive roots, we have

$$\begin{aligned} c_{(\omega_1, \omega_1)}(\alpha_1 + \dots + \alpha_m) &= \frac{1}{m-1}, \\ c_{(\omega_2, \omega_2)}(\alpha_1 + \dots + \alpha_m) &= \frac{1}{m-1}, \\ c_{(\omega_2, \omega_2)}(\alpha_2 + \dots + \alpha_m) &= \frac{1}{m-2}. \end{aligned}$$

Thus, we have the following.

Lemma. Let $n, m > 3$. Then

$$\begin{aligned} &HS_{\mathfrak{q}}^{n,m} \langle (\omega_1, \omega_1), (\omega_2, \omega_2) \rangle = \\ &\left(1 + \frac{1}{m-2} q_2 \frac{\partial}{\partial q_2}\right) \left(1 + \frac{1}{m-1} q_1 \frac{\partial}{\partial q_1} + \frac{1}{m-1} q_2 \frac{\partial}{\partial q_2}\right) HS_{\mathfrak{q}}^{n,m-1} \langle (\omega_1, \omega_1), (\omega_2, \omega_2) \rangle \end{aligned}$$

Note that our original choice to increase m does not actually affect the formula in the recursion. The story is symmetric. If we had chosen to increase n instead of m , we would still have two new partial differential operators, with m replaced by n .

Another thing to note is that this recursion is strikingly similar to the recursion for the symmetric determinantal variety. This should come as no surprise, since $c_{n\lambda}(\alpha) = nc_\lambda(\alpha)$, and in either case, we are considering the same subset of the fundamental dominant weights of SL_n . For these reasons, we also get a similar recursive relationship to that in Proposition 5.4.1.

Proposition 5.4.2. *Let $n, m > k + 1$. Then*

$$HS_q^{n,m} \langle (\omega_1, \omega_1), \dots, (\omega_k, \omega_k) \rangle = \prod_{i=1}^k \left(1 + \frac{1}{m-i} \sum_{j=i}^k q_j \frac{\partial}{\partial q_j} \right) HS_q^{n,m-1} \langle (\omega_1, \omega_1), \dots, (\omega_k, \omega_k) \rangle.$$

The proof of Proposition 5.4.2 is identical to the proof of Proposition 5.4.1, and is again valid if we switch the roles of n and m . In a lot of ways, the multivariate series for $\mathcal{D}_{n,m}^{\leq k}$ behaves similarly to that of $\mathcal{SD}_n^{\leq k}$.

5.5 A nice lattice cone in $P_+(\mathfrak{sl}_n)$

Let $G = SL_n$. In this case, let $\omega_1, \dots, \omega_{n-1}$ be the fundamental dominant weights of G , and let $\alpha_1, \dots, \alpha_{n-1}$ be the set of simple roots for \mathfrak{g} . We would like to compute $HS_{q,r} \langle \omega_1, \omega_{n-1} \rangle$ for $n \geq 3$. Figure 5.5 describes the nonzero coefficients $c_{\omega_i}(\alpha)$ for $\alpha \in \Phi^+$ and $j = 1, \dots, n-1$.

Using these coefficients, we can compute that $HS_{q,r} \langle \omega_1, \omega_{n-1} \rangle$ is equal to

$$\prod_{i=1}^{n-2} \left(1 + \frac{q}{i} \frac{\partial}{\partial q} \right) \left(1 + \frac{r}{i} \frac{\partial}{\partial r} \right) \left(1 + \frac{q}{n-1} \frac{\partial}{\partial q} + \frac{r}{n-1} \frac{\partial}{\partial r} \right) \frac{1}{(1-q)(1-r)}.$$

We can rearrange the above product, since the operators commute, to get the following.

$$\left(1 + \frac{q}{n-1} \frac{\partial}{\partial q} + \frac{r}{n-1} \frac{\partial}{\partial r} \right) \left(\prod_{i=1}^{n-2} \left(1 + \frac{q}{i} \frac{\partial}{\partial q} \right) \frac{1}{1-q} \prod_{i=1}^{n-2} \left(1 + \frac{r}{i} \frac{\partial}{\partial r} \right) \frac{1}{1-r} \right)$$

| α | $c_{\omega_2}(\alpha)$ | $c_{\omega_4}(\alpha)$ |
|--|------------------------|------------------------|
| α_2 | 1 | 0 |
| α_4 | 0 | 1 |
| $\alpha_1 + \alpha_2$ | 1/2 | 0 |
| $\alpha_2 + \alpha_3$ | 1/2 | 0 |
| $\alpha_3 + \alpha_4$ | 0 | 1/2 |
| $\alpha_4 + \alpha_5$ | 0 | 1/2 |
| $\alpha_1 + \alpha_2 + \alpha_3$ | 1/3 | 0 |
| $\alpha_2 + \alpha_3 + \alpha_4$ | 1/3 | 1/3 |
| $\alpha_3 + \alpha_4 + \alpha_5$ | 0 | 1/3 |
| $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ | 1/4 | 1/4 |
| $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | 1/4 | 1/4 |
| $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ | 1/5 | 1/5 |

Figure 5.5: Coefficients of the differential operators in $HS_{q,r}\langle\omega_1, \omega_{n-1}\rangle$

Note that $\prod_{i=1}^{n-2} \left(1 + \frac{q}{i} \frac{\partial}{\partial q}\right) \frac{1}{1-q} = \frac{1}{(1-q)^{n-1}}$. Thus, the above simplifies to

$$\left(1 + \frac{q}{n-1} \frac{\partial}{\partial q} + \frac{r}{n-1} \frac{\partial}{\partial r}\right) \frac{1}{(1-q)^{n-1}(1-r)^{n-1}}.$$

We apply the above partial differential operator, and simplify to get

$$HS_{q,r}\langle\omega_1, \omega_{n-1}\rangle = \frac{1-qr}{(1-q)^n(1-r)^n}.$$

Note, that if we let $G = SL_2$, we could still compute $HS_{q,r}\langle\omega_1, \dots, \omega_{n-1}\rangle = HS_{q,r}\langle\omega, \omega\rangle$, where ω is the sole fundamental dominant weight of G . In this case, we compute that

$$HS_{q,r}\langle\omega, \omega\rangle = \frac{1-qr}{(1-q)^2(1-r)^2}.$$

In this way, the above discussion holds even in the case that $G = SL_2$ and the weights ω_1, ω_{n-1} are not independent.

There is a geometric interpretation of this example, using the First Fundamental Theorem for $GL(V)$. Note that $L(\omega_{n-1})$ is the dual space of $L(\omega_1)$. To put this

example in a more general context, consider the subset $X \in V \oplus V^*$, where V is a finite dimensional vector space with dual space V^* given by

$$X := \{(v, \lambda) \in V \oplus V^* \mid \lambda(v) = 0\}.$$

Then X is an example of a *null cone*. These spaces have been extensively studied in the context of Classical Invariant Theory. When $V = L(\omega_1)$, the null cone X has multivariate Hilbert series $HS_{\mathbf{q}}\langle\omega_1, \omega_n\rangle$.

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- AMS Travel Grant, Joint Mathematics Meetings, Winter 2015
- AMS Sectional Meeting Travel Grant, Fall 2014
- Summer Research Excellence Award, UWM, 2014
- Mark Lawrence Teply Award, UWM, 2014. *This award is designed to recognize mathematics graduate students who show remarkable potential in their research fields.*
- Morris and Miriam Marden Graduate Mathematics Award, UWM, 2014. *This award is designed to recognize a student who submits a mathematical paper of high quality with respect to both exposition and mathematical content. Paper submitted: A multi-variate generating function for the Weyl Dimension Formula.*
- Ernst Schwandt Teaching Award, UWM, 2014. *This award is designed to recognize demonstrated outstanding teaching performance for mathematics graduate teaching assistants.*
- AMS Student Membership, UWM, 2010-present.
- Graduate Assistant in An Area of National Need (GAANN) Fellowship, UWM, 2010-2014.

Research interests

Lie theory, representation theory of algebraic groups, computation algebraic geometry and commutative algebra, algebraic combinatorics

Conferences and workshops

- Joint Mathematics Meetings, San Antonio, Winter 2015
- AMS Central Sectional Meeting, University of Wisconsin, Eau Claire, Fall 2014
- Mathematical Sciences Research Institute (MSRI), Introductory Workshop: Geometric Representation Theory, Fall 2014
- Underrepresented Students in Topology and Algebra Research Symposium (USTARS), University of California, Berkeley, Spring 2014
- Midwest Algebraic Geometry Graduate Conference, University of Wisconsin, Madison, Fall 2012

Talks

- Joint Mathematics Meeting, San Antonio, TX, Winter 2015, Co-chair of the AMS Contributed Paper Session in Nonassociative Rings and Algebras, Category Theory, and Homological Algebra.
- AMS Central Sectional Meeting, University of Wisconsin, Eau Claire, Fall 2014, Special Session on Lie Algebras and Representation Theory, *invited*
- USTARS, University of California, Berkeley, Fall 2014, “A multi-variate generating function for the Weyl Dimension Formula”
- Beloit College Undergraduate Mathematics Colloquium, Spring 2011, *invited*
- UWM Algebra Seminar, 2010-2015, various talks