Given a graph \( G \), let \( P_t \) be the transition probability matrix. We assign each node \( v \) an \( n \)-dimensional feature vector \( \mathbf{p}_v \), which describes the "structural role" of \( v \),

\[
\mathbf{p}_v = [P_t(v, i), P_t(v, i'), \ldots, P_t(v, i^{(n-1)})],
\]

where \( P_t(v, i) = \sum_j P_t(i, j) P_t(j, i) \), is the return probability of a \( t \)-step random walk starting from \( v \). Here, each graph is represented by a set of feature vectors in \( \mathbb{R}^n \) as \( \mathbf{P}_{G} = \{ \mathbf{p}_v \} \).

Properties of Return Probability Feature

(1) Isomorphism-invariant.
(2) Multi-resolution.
(3) Permutation map.
(4) Information.

Given two graphs \( G \) and \( H \) of \( n \) nodes, let \( x \) be the permutation map. Fix \( (\mathbf{p}_v, \mathbf{p}_w) \) be a pair of \( \mathbf{p}_v \) and \( \mathbf{p}_w \) respectively. \( x \) is a permutation map if \( P_t(x(i), j) = P_t(i, x(j)) \); then, the return probability feature is invariant under such a permutation map.

An illustrative example:

\[
\begin{align*}
G &= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
H &= \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\end{align*}
\]

Figure 2: (a) Toy Graph \( G \); (b) The \( 2 \)-step return probability of the nodes \( C_1 \) and \( C_2 \) in the toy graph, \( s = 2, \ldots, 200 \). The nested figure is a close-up view of the region of interest.

Observations: (i) Since \( C_1 \) and \( C_2 \) share the same neighborhoods at larger scales, their return probability values are close until the eighth step. Because \( C_1 \) plays a more different structural role from \( C_2 \) here, its return probability values deviate from those of \( C_1 \) and \( C_2 \) in early steps.
(ii) When the random walk steps approach infinity, the return probability \( P_t(i, i') \) will not change much and will converge to the stationary probability.