

Continuity properties and sensitivity analysis of parameterized fixed points and approximate fixed points

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Abstract

In this paper we consider continuity of the set of fixed points and approximate fixed points for parameterized set-valued mappings. Continuity properties are provided for the fixed points of general multivalued mappings, with additional results shown for contraction mappings. Further analysis is provided for the continuity of the approximate fixed points of set-valued functions. Additional results are provided on sensitivity of the fixed points via set-valued derivatives related to tangent cones.

Key words: fixed point problems; approximate fixed point problems; set-valued continuity; data dependence; generalized differentiation

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1 Introduction

Fixed points are utilized in many applications. Often the parameters of such models are estimated from data. As such it is important to understand how the set of fixed points

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changes with respect to the parameters. We motivate our general approach by consider applications from economics and finance. In particular, the solution set of a parameterized game (i.e., Nash equilibria) fall under this setting. Thus if the parameters of the individuals playing the game are not perfectly known, sensitivity analysis can be done in a general way even without unique equilibrium. For the author, the immediate motivation was from financial systemic risk models such as those in [10, 8, 4, 12] where methodology to estimate system parameters are studied in, e.g., [15]. Additionally, the systemic risk measures in [13] utilize continuity of fixed points for continuity and closedness properties of the risk measures.

The continuity of fixed points has been studied in the literature before. For single-valued mappings we refer to, e.g., [7, 21, 2, 16, 32, 33, 38, 17], for set-valued or multivalued mappings we refer to, e.g., [35, 36, 37, 44, 34, 31, 43, 11, 14, 42, 39]. In this paper we will give general theorems that extend upon the work on data dependence of fixed point sets, particularly in [31, 39], to consider cases in general topological spaces and provide additional results on continuity under the Vietoris topology. In this work, we go beyond simple continuity to also find conditions on the set-valued derivatives (via tangent cones) for fixed points. This is related to the work of [17] which proves the continuous differentiability of fixed points for single-valued contraction mappings. We additionally study the continuity of the set of approximate fixed points (see, e.g., [45, 5, 9, 6] for discussion of approximate fixed points).

Notably, by deducing continuity properties of the set of fixed points, we provide results that would allow for well-behaved selectors and optimal fixed points. For some results on selection theorems, we refer to the books [28, 40]. In the case of optimization over fixed point sets, we focus primarily on the (semi)continuity of the value function of such an optimization problem. We refer to [27, 26] for fixed point optimization algorithms and brief discussions of such a problem. We refer to [25, 24] for a few applications from the literature that utilize this optimization problem.

In Section 2, we present some definitions and results from the literature on continuity of set-valued mappings to be used in this paper. Section 3 presents general results on continuity

of fixed points for set-valued maps. Section 4 considers the special case when studying contraction mappings. We conclude in Section 5 by providing results on the continuity of the set of approximate fixed points.

2 Background

In this section we will present definitions of continuity for set-valued or multivalued mappings, i.e., functions mapping into the power set of some space. We will additionally provide a brief overview of some results on these forms of continuity from the literature. Throughout this paper we will let \mathbb{X} and \mathbb{Y} be Hausdorff spaces. Additionally we will denote the power set of \mathbb{Y} by $\mathcal{P}(\mathbb{Y}) := \{Y \subseteq \mathbb{Y}\}$.

Definition 2.1. *A set-valued mapping $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ is called (upper, lower) continuous if it is continuous with respect to the (resp. upper, lower) Vietoris topology.*

Remark 2.2. The mapping $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ is continuous if, and only if, it is upper and lower continuous. If F is single-valued, i.e., $F(x) = \{f(x)\}$ for some function $f : \mathbb{X} \rightarrow \mathbb{Y}$, then F is upper continuous if, and only if, F is lower continuous if, and only if, f is continuous.

Remark 2.3. Upper (lower) continuity is often referred to as upper (resp. lower) hemi-continuity (in, e.g., [1]), upper (resp. lower) semicontinuity (in, e.g., [3, 23, 28, 40]), and inner (resp. outer) continuity (in, e.g., [41]) in the literature. We use the terminology from [30, 19, 22] to emphasize that a single-valued function is upper continuous if, and only if, it is lower continuous if, and only if, it is continuous. This has the additional advantage of avoiding the need to distinguish single-valued semicontinuity from set-valued continuity concepts.

The following equivalent representations for upper and lower continuity are standard in the literature (see [23, Propositions 1.2.6 and 1.2.7] and [1, Lemmas 17.4 and 17.5]).

Proposition 2.4. *For a set-valued mapping $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ the following are equivalent:*

(i) F is upper continuous;

(ii) $F^+[V] := \{x \in \mathbb{X} \mid F(x) \subseteq V\}$ is open in \mathbb{X} for any $V \subseteq \mathbb{Y}$ open;

(iii) $F^-[\bar{V}] := \{x \in \mathbb{X} \mid F(x) \cap \bar{V} \neq \emptyset\}$ is closed in \mathbb{X} for any $\bar{V} \subseteq \mathbb{Y}$ closed.

Proposition 2.5. *For a set-valued mapping $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ the following are equivalent:*

(i) F is lower continuous;

(ii) $F^-[V] := \{x \in \mathbb{X} \mid F(x) \cap V \neq \emptyset\}$ is open in \mathbb{X} for any $V \subseteq \mathbb{Y}$ open;

(iii) $F^+[\bar{V}] := \{x \in \mathbb{X} \mid F(x) \subseteq \bar{V}\}$ is closed in \mathbb{X} for any $\bar{V} \subseteq \mathbb{Y}$ closed.

We additionally wish to provide simple conditions for upper and lower continuity via the graphs of our multivalued functions.

Theorem 2.6. *[1, Theorem 17.11] Let \mathbb{Y} be compact. A set-valued mapping $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ has a closed graph, i.e.,*

$$\text{graph } F := \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid y \in F(x)\}$$

is closed in the product topology, if, and only if, F is upper continuous and closed-valued.

Lemma 2.7. *[1, Lemma 17.12] For set-valued mappings $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ we have that F is lower continuous if F has open fibers, i.e.,*

$$F^-(\{y\}) := \{x \in \mathbb{X} \mid y \in F(x)\}$$

is open for any $y \in \mathbb{Y}$. And F has open fibers if the graph of F is open in the product topology.

Often we are interested not only in a set-valued mapping but in a parameterized set of optimization problems. From the literature we can derive results on the value function (and in the case of the Berge maximum theorem, the optimizers) of such an optimization problem.

Theorem 2.8. *Let $f : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ be some desired objective function and $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$. And let $v : \mathbb{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be defined by*

$$v(x) := \sup\{f(x, y) \mid y \in F(x)\} \quad (2.1)$$

for any $x \in \mathbb{X}$.

- (i) *The value function v is upper semicontinuous if f is upper semicontinuous and F is upper continuous with nonempty and compact images.*
- (ii) *The value function v is lower semicontinuous if f is lower semicontinuous and F is lower continuous.*
- (iii) *The value function v is continuous if f is continuous and F is continuous with nonempty and compact images. Further, the set of maximizers $V : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$, defined by*

$$V(x) := \arg \max\{f(x, y) \mid y \in F(x)\} = \{y \in F(x) \mid f(x, y) = v(x)\} \quad (2.2)$$

for every $x \in \mathbb{X}$, is upper continuous with compact images.

Proof. These are trivial consequences of [1, Lemmas 17.29 and 17.30] and the Berge maximum theorem (see, e.g., [1, Theorem 17.31]). □

We wish to give one more definition of continuity from the literature for multivalued mappings. For fixed points, contraction mappings play a prominent role, for example in the Banach and Nadler fixed point theorems. The definition of such a mapping is given below.

Definition 2.9. *Let \mathbb{X} and \mathbb{Y} be metric spaces with metrics $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ respectively. Let $D_{\mathbb{Y}}$ denote the Hausdorff metric and $D_{\mathbb{Y}}^*$ denote the excess in $\mathcal{P}(\mathbb{Y})$, i.e.,*

$$D_{\mathbb{Y}}(A, B) := \max \{D_{\mathbb{Y}}^*(A, B), D_{\mathbb{Y}}^*(B, A)\}$$

$$D_{\mathbb{Y}}^*(A, B) := \sup_{a \in A} d_{\mathbb{Y}}(a, B),$$

for any $A, B \subseteq \mathbb{Y}$ where the distance between a point and a set is given by

$$d_{\mathbb{Y}}(a, B) := \inf_{b \in B} d_{\mathbb{Y}}(a, b) \quad \forall a \in \mathbb{Y} \forall B \subseteq \mathbb{Y}.$$

Let $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ be a set-valued mapping. F is called (L, α) -Hölder continuous if

$$D_{\mathbb{Y}}(F(x_1), F(x_2)) \leq L d_{\mathbb{X}}(x_1, x_2)^\alpha$$

for every $x_1, x_2 \in \mathbb{X}$. F is called L -Lipschitz continuous if it is $(L, 1)$ -Hölder continuous. F is called a contraction mapping if it is L -Lipschitz continuous for some $L < 1$.

We complete this section by providing two definitions of derivatives for set-valued mappings. These definitions can be used for sensitivity analysis of multivalued functions. Other notions of derivative of set-valued derivatives exist in the literature as well. We refer the reader to [23] for some of these results.

Definition 2.10. [23, Definition 6.6.19] Let \mathbb{X} and \mathbb{Y} be normed spaces. Let $F : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ and assume $(x_0, y_0) \in \text{graph } F$.

- (i) The contingent derivative $DF(x_0, y_0)$ of F at (x_0, y_0) is the multifunction defined such that the graph of $DF(x_0, y_0)$ is given by the contingent tangent cone of the graph of F at (x_0, y_0) , i.e., $y \in DF(x_0, y_0)(x)$ if there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \searrow 0$ and sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow (x, y)$ such that $y_0 + t_n y_n \in F(x_0 + t_n x_n)$ for every $n \in \mathbb{N}$.
- (ii) The adjacent derivative $D_n F(x_0, y_0)$ of F at (x_0, y_0) is the multifunction defined such that the graph of $D_n F(x_0, y_0)$ is given by the adjacent tangent cone of the graph of F at (x_0, y_0) , i.e., $y \in D_n F(x_0, y_0)(x)$ if for every sequence $\{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \searrow 0$ there exists a sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow (x, y)$ such that $y_0 + t_n y_n \in F(x_0 + t_n x_n)$ for every $n \in \mathbb{N}$.

Proposition 2.11. [23, Corollary 6.6.25] If $f : \mathbb{X} \rightarrow \mathbb{Y}$ is single-valued, then

(i) if f is Gâteaux differentiable at $x_0 \in \mathbb{X}$, then $f'(x_0) = Df(x_0, f(x_0))$,

(ii) if f is Fréchet differentiable at $x_0 \in \mathbb{X}$, then $f'(x_0) = D_n f(x_0, f(x_0))$.

3 Continuity of parameterized fixed points

In this section we will provide continuity results and simple sensitivity analysis for the collection of fixed points for set-valued mappings. These results follow from the definitions of continuity given in the prior section for the set-valued mapping. Throughout let us consider the fixed points of the mapping $H : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y})$. We will denote the parameterized fixed points of H by the function $h : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$, i.e.,

$$h(x) := \text{FIX}_{y \in \mathbb{Y}} H(x, y) = \{y \in \mathbb{Y} \mid y \in H(x, y)\} \quad \forall x \in \mathbb{X}.$$

Lemma 3.1. (i) If $\text{graph } H \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Y}$ is closed in the product topology then $\text{graph } h \subseteq \mathbb{X} \times \mathbb{Y}$ is closed in the product topology.

(ii) If the properties of (i) are satisfied and \mathbb{Y} is a compact Hausdorff space then h is an upper continuous multivalued map with closed and compact images.

Proof. (i) Recall that the graph of h is given by

$$\text{graph } h := \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid y \in h(x)\}.$$

Let $\{(x_i, y_i)\}_{i \in I} \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow (x, y)$ such that $(x_i, y_i) \in \text{graph } h$ for every $i \in I$. By definition of the mapping h it is immediate that $(x_i, y_i, y_i) \in \text{graph } H$ for every $i \in I$. By convergence in the product topology and closedness of the graph of H it immediately follows that $y \in H(x, y)$, i.e., $y \in h(x)$.

(ii) If we additionally assume that \mathbb{Y} is compact then we can apply the closed graph theorem (Theorem 2.6) to recover that h is upper continuous and closed-valued. Since a closed

subset of a compact set is compact, we recover that h is additionally compact-valued. \square

Remark 3.2. If the properties of Lemma 3.1(ii) are satisfied, \mathbb{Y} is a locally convex space and convex, and H has nonempty convex images then h has nonempty images by the Kakutani fixed point theorem (see, e.g., [1, Corollary 17.55])

Lemma 3.3. (i) *If graph $H \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Y}$ is open in the product topology then graph $h \subseteq \mathbb{X} \times \mathbb{Y}$ is open in the product topology.*

(ii) *If H has open fibers (i.e., $H^{-}(\{\bar{y}\}) := \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid \bar{y} \in H(x, y)\}$ is open for every $\bar{y} \in \mathbb{Y}$) then h has open fibers.*

Proof. (i) Let $\{(x_i, y_i)\}_{i \in I} \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow (x, y)$ such that $(x_i, y_i) \notin \text{graph } h$ for every $i \in I$. By definition of the mapping h it is immediate that $(x_i, y_i, y_i) \notin \text{graph } H$ for every $i \in I$. By convergence in the product topology and openness of the graph of H it immediately follows that $y \notin H(x, y)$, i.e., $y \notin h(x)$.

(ii) Fix $\bar{y} \in \mathbb{Y}$. Let $\{x_i\}_{i \in I} \subseteq \mathbb{X} \rightarrow x$ such that $x_i \notin h^{-}(\{\bar{y}\})$ for every $i \in I$. By definition of the mapping h it is immediate that $(x_i, \bar{y}) \notin H^{-}(\{\bar{y}\})$ for every $i \in I$. By convergence in the product topology and openness of the fibers of H it immediately follows that $(x, \bar{y}) \notin H^{-}(\{\bar{y}\})$, i.e., $\bar{y} \notin h(x)$. \square

Remark 3.4. The condition of Lemma 3.3(i) and (ii) imply h is lower continuous, in fact the condition of Lemma 3.3(i) implies (ii) (see, e.g., Lemma 2.7).

Example 3.5. Even if H is continuous, h need not be lower continuous. Consider $H(x, y) =$

$\{\min(xy, 1)\}$ for every $x \in \mathbb{X} := [0, 2]$ and $y \in \mathbb{Y} := [0, 1]$. Immediately it is clear that

$$h(x) := \begin{cases} \{0\} & \text{if } x < 1 \\ [0, 1] & \text{if } x = 1 \\ \{0, 1\} & \text{if } x > 1 \end{cases}$$

If $x = 1$ and $y \in (0, 1)$ then for any $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{X} \setminus \{1\} \rightarrow 1$ and any $\{y_n\}_{n \in \mathbb{N}} \subseteq \mathbb{Y} \rightarrow y$ there exists $N \geq 0$ such that $y_n \notin h(x_n)$ for every $n \geq N$.

Remark 3.6. Theorem 2.8(i) can be applied under the setting of Lemma 3.1(ii) such that $h(x) \neq \emptyset$ for every $x \in \mathbb{X}$ (see, e.g., Remark 3.2) to determine that the value function v defined in (2.1) is upper semicontinuous if the objective function f is upper semicontinuous. Similarly, Theorem 2.8(ii) can be applied under the setting of Lemma 3.3(ii) to determine that the value function v is lower semicontinuous if the objective function f is lower semicontinuous.

We will finish this section on general results for fixed point mappings by considering the derivatives of the fixed points. As fixed point mappings are inherently set-valued functions we will consider the two types of set-valued derivatives discussed in the prior section.

Theorem 3.7. *Let \mathbb{X} and \mathbb{Y} be normed spaces. The contingent (adjacent) derivative of the fixed points $h : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ is included in the fixed points of the contingent (resp. adjacent) derivative of H , i.e.,*

$$Dh(x_0, y_0)(x) \subseteq \text{FIX}_{y \in \mathbb{Y}} DH(x_0, y_0, y_0)(x, y)$$

(resp. with $D_n h$ and $D_n H$) for every $(x_0, y_0) \in \text{graph } h$ and $x \in \mathbb{X}$.

Proof. Let $(x_0, y_0) \in \text{graph } h$, $x \in \mathbb{X}$, and $y \in Dh(x_0, y_0)(x)$. Then there exist sequences $\{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \searrow 0$ and $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow (x, y)$ such that $y_0 + t_n y_n \in h(x_0 + t_n x_n)$ for every $n \in \mathbb{N}$. By the definition of h this implies (under the same sequences

as before) $y_0 + t_n y_n \in H(x_0 + t_n x_n, y_0 + t_n y_n)$. Thus we can immediately conclude that $y \in DH(x_0, y_0, y_0)(x, y)$. The case for the adjacent derivative follows similarly. \square

4 Continuity of fixed points of contraction mappings

In this section we will consider additional, stronger, results that can be deduced when we consider the mapping $y \mapsto H(x, y)$ to be a contraction mapping (see Definition 2.9). We will assume for this section that \mathbb{Y} is a complete metric space (with metric $d_{\mathbb{Y}}$). As in Definition 2.9, we will let $D_{\mathbb{Y}} : \mathcal{P}(\mathbb{Y}) \times \mathcal{P}(\mathbb{Y}) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ denote the Hausdorff metric in $\mathcal{P}(\mathbb{Y})$. Lemma 4.1 and Corollary 4.2 are similar to results found in, e.g., [31, 39]; we include these results for sake of completeness.

Lemma 4.1. *Let $H : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ with closed and $d_{\mathbb{Y}}$ -bounded values. Let $y \mapsto H(x, y)$ be a contraction mapping with Lipschitz constant $L_{\mathbb{Y}} < 1$ for every $x \in \mathbb{X}$. Then the parameterized set of fixed points $h : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ has nonempty and closed images and:*

(i) *If $\psi_x^{H(\cdot, y)} : \mathbb{X} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined by*

$$\psi_x^{H(\cdot, y)}(\bar{x}) := D_{\mathbb{Y}}^*(H(\bar{x}, y), H(x, y)) = \sup_{\bar{y} \in H(\bar{x}, y)} \inf_{y^* \in H(x, y)} d_{\mathbb{Y}}(\bar{y}, y^*) \quad \forall \bar{x} \in \mathbb{X}$$

is continuous at x for every $(x, y) \in \mathbb{X} \times \mathbb{Y}$, then $\text{graph } h$ is closed in the product topology.

(ii) *If H has compact images and $x \mapsto H(x, y)$ is a lower continuous mapping for every $y \in \mathbb{Y}$, then h is lower continuous with compact images.*

(iii) *If \mathbb{X} is a metric space and $x \mapsto H(x, y)$ is a $(L_{\mathbb{X}}, \alpha)$ -Hölder continuous mapping for every $y \in \mathbb{Y}$ with constant parameters $L_{\mathbb{X}}$ and α , then h is $(\frac{L_{\mathbb{X}}}{1-L_{\mathbb{Y}}}, \alpha)$ -Hölder continuous.*

Proof. First, by the Nadler fixed point theorem (see, e.g., [29, Theorem 5]), we have that $h(x) \neq \emptyset$ for every $x \in \mathbb{X}$. To show that h has closed images, let $x \in \mathbb{X}$ and $\{y_n\}_{n \in \mathbb{N}} \subseteq \mathbb{Y} \rightarrow y$

such that $y_n \in h(x)$ for every $n \in \mathbb{N}$. By H closed-valued, $y \in h(x)$ if, and only if, $d_{\mathbb{Y}}(y, H(x, y)) := \inf_{y' \in H(x, y)} d_{\mathbb{Y}}(y, y') = 0$. And immediately by the contraction property we see

$$\begin{aligned} d_{\mathbb{Y}}(y, H(x, y)) &\leq \limsup_{n \rightarrow \infty} [d_{\mathbb{Y}}(y, H(x, y_n)) + D_{\mathbb{Y}}(H(x, y_n), H(x, y))] \\ &\leq (1 + L_{\mathbb{Y}}) \limsup_{n \rightarrow \infty} d_{\mathbb{Y}}(y, y_n) = 0. \end{aligned}$$

- (i) The result follows directly from Lemma 3.1(i) so long as H has a closed graph. Let $\{(x_i, \bar{y}_i, y_i)\}_{i \in I} \subseteq \text{graph } H \rightarrow (x, \bar{y}, y)$:

$$\begin{aligned} d_{\mathbb{Y}}(y, H(x, \bar{y})) &\leq \limsup_{i \in I} [d_{\mathbb{Y}}(y, H(x_i, \bar{y})) + D_{\mathbb{Y}}^*(H(x_i, \bar{y}), H(x, \bar{y}))] \\ &\leq \limsup_{i \in I} [d_{\mathbb{Y}}(y, H(x_i, \bar{y}_i)) + D_{\mathbb{Y}}^*(H(x_i, \bar{y}_i), H(x_i, \bar{y})) \\ &\quad + \psi_x^{H(\cdot, \bar{y})}(x_i)] \\ &\leq \limsup_{i \in I} [d_{\mathbb{Y}}(y, y_i) + L_{\mathbb{Y}} d_{\mathbb{Y}}(\bar{y}_i, \bar{y}) + \psi_x^{H(\cdot, \bar{y})}(x_i)] \\ &\leq \limsup_{i \in I} d_{\mathbb{Y}}(y, y_i) + L_{\mathbb{Y}} \limsup_{i \in I} d_{\mathbb{Y}}(\bar{y}_i, \bar{y}) \\ &\quad + \limsup_{i \in I} \psi_x^{H(\cdot, \bar{y})}(x_i) = 0. \end{aligned}$$

That is, $y \in H(x, \bar{y})$ by H having closed images.

- (ii) First note that since H has compact images, so does h (see, e.g., [39, Theorem 3.25.1]). To prove lower continuity of the fixed point mapping h , we will utilize [23, Theorem 1.2.68(b)] that h is lower continuous if, and only if, $\phi_x^h : \mathbb{X} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, defined by

$$\phi_x^h(\bar{x}) := D_{\mathbb{Y}}^*(h(x), h(\bar{x})) = \sup_{y \in h(x)} \inf_{\bar{y} \in h(\bar{x})} d_{\mathbb{Y}}(y, \bar{y}) \quad \forall \bar{x} \in \mathbb{X},$$

is continuous at x for every $x \in \mathbb{X}$. Fix $x \in \mathbb{X}$ and let $\{x_i\}_{i \in I} \subseteq \mathbb{X} \rightarrow x$ be a convergent net. Since $\phi_x^h(x) = 0$ by definition, we wish to show that $\lim_{i \in I} \phi_x^h(x_i) = 0$. Utilizing

the proof idea of [18, Lemma 15.1] we can conclude

$$\phi_x^h(\bar{x}) \leq \frac{1}{1 - L_{\mathbb{Y}}} \sup_{y \in h(x)} D_{\mathbb{Y}}^*(H(x, y), H(\bar{x}, y))$$

for any $x, \bar{x} \in \mathbb{X}$. Thus, by the same logic as in [18, Theorem 15.2], h is lower continuous if $\lim_{i \in I} \sup_{y \in h(x)} D_{\mathbb{Y}}^*(H(x, y), H(x_i, y)) = 0$. Since h is compact-valued, for any $i \in I$ there exists some $y_i \in h(x)$ such that $D_{\mathbb{Y}}^*(H(x, y_i), H(x_i, y_i)) = \sup_{y \in h(x)} D_{\mathbb{Y}}^*(H(x, y), H(x_i, y))$.

$$\begin{aligned} 0 &\leq \liminf_{i \in I} \sup_{y \in h(x)} D_{\mathbb{Y}}^*(H(x, y), H(x_i, y)) \\ &\leq \limsup_{i \in I} \sup_{y \in h(x)} D_{\mathbb{Y}}^*(H(x, y), H(x_i, y)) \\ &= \limsup_{i \in I} D_{\mathbb{Y}}^*(H(x, y_i), H(x_i, y_i)) = \lim_{j \in J} D_{\mathbb{Y}}^*(H(x, y_j), H(x_j, y_j)) \\ &= \lim_{k \in K} D_{\mathbb{Y}}^*(H(x, y_k), H(x_k, y_k)) \\ &\leq \limsup_{k \in K} [D_{\mathbb{Y}}^*(H(x, y_k), H(x, y^*)) + D_{\mathbb{Y}}^*(H(x, y^*), H(x_k, y^*)) \\ &\quad + D_{\mathbb{Y}}^*(H(x_k, y^*), H(x_k, y_k))] \\ &\leq \limsup_{k \in K} [D_{\mathbb{Y}}(H(x, y_k), H(x, y^*)) + D_{\mathbb{Y}}^*(H(x, y^*), H(x_k, y^*)) \\ &\quad + D_{\mathbb{Y}}(H(x_k, y^*), H(x_k, y_k))] \\ &\leq \limsup_{k \in K} [2L_{\mathbb{Y}}d_{\mathbb{Y}}(y_k, y^*) + D_{\mathbb{Y}}^*(H(x, y^*), H(x_k, y^*))] \\ &\leq 2L_{\mathbb{Y}} \limsup_{k \in K} d_{\mathbb{Y}}(y_k, y^*) + \limsup_{k \in K} D_{\mathbb{Y}}^*(H(x, y^*), H(x_k, y^*)) = 0. \end{aligned}$$

Where the subnet $\{D_{\mathbb{Y}}^*(H(x, y_j), H(x_j, y_j))\}_{j \in J}$ for $J \subseteq I$ is chosen so that it converges to the limit supremum of the full net and the subnet $\{y_k\}_{k \in K} \rightarrow y^* \in h(x)$ for $K \subseteq J$ is chosen as a convergent subnet in the compact set $h(x)$. The final equality to 0 is due to the convergence of $\{y_k\}_{k \in K}$ to y^* and the continuity of $\bar{x} \mapsto \phi_x^{H(\cdot, y^*)}(\bar{x}) := D_{\mathbb{Y}}^*(H(x, y^*), H(\bar{x}, y^*))$ at x (from [23, Theorem 1.2.68]).

(iii) By [18, Lemma 15.1] and Hölder continuity of $x \mapsto H(x, y)$ for every $y \in \mathbb{Y}$, we

immediately have for any $x_1, x_2 \in \mathbb{X}$

$$\begin{aligned} D_{\mathbb{Y}}(h(x_1), h(x_2)) &\leq \frac{1}{1 - L_{\mathbb{Y}}} \sup_{y \in \mathbb{Y}} D_{\mathbb{Y}}(H(x_1, y), H(x_2, y)) \\ &\leq \frac{L_{\mathbb{X}}}{1 - L_{\mathbb{Y}}} d_{\mathbb{X}}(x_1, x_2)^{\alpha}. \end{aligned}$$

□

Corollary 4.2. *Let \mathbb{Y} be a compact metric space. Let $H : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ with closed and $d_{\mathbb{Y}}$ -bounded values. Let $y \mapsto H(x, y)$ be a contraction mapping with Lipschitz constant $L_{\mathbb{Y}} < 1$ for every $x \in \mathbb{X}$. Then the parameterized set of fixed points $h : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ has closed and compact images and is (upper, lower) continuous if $x \mapsto H(x, y)$ is (resp. upper, lower) continuous for every $y \in \mathbb{Y}$.*

Proof. First note that since H has closed images, H has compact images as well by \mathbb{Y} a compact space. Thus h has compact images by it having closed images (see Lemma 4.1). Upper continuity follows directly from Lemma 3.1(ii) so long as H has a closed graph, but this follows directly from [23, Theorem 1.2.68(a)] and the proof of Lemma 4.1(i). Lower continuity follows immediately from (ii). Finally, continuity of the fixed points follow from upper and lower continuity. □

Remark 4.3. (i) By Lemma 4.1(i), h has a closed graph if $x \mapsto H(x, y)$ is $D_{\mathbb{Y}}$ -continuous for every $y \in \mathbb{Y}$. Additionally, by [23, Proposition 1.2.61] and Lemma 4.1(i), h has a closed graph if $x \mapsto H(x, y)$ is upper continuous for every $y \in \mathbb{Y}$. In fact, the proof of upper continuity for Corollary 4.2 only relies on compactness for the closed graph theorem (Theorem 2.6).

(ii) In the case of H single-valued, it is immediate that H has compact images. Therefore, from Lemma 4.1(ii), continuity of the fixed points follows from continuity of $x \mapsto H(x, y)$ for every $y \in \mathbb{Y}$ (see Remark 2.2).

- (iii) The results of Lemma 4.1(iii) can easily be generalized to a local version.
- (iv) The results of Lemma 4.1 are standard in the case that H is single-valued. We refer the reader to, e.g., [20, Theorems I.1.3.2 and I.1.6.A.4].
- (v) The results of Corollary 4.2 can be deduced directly from [31, Proposition 3.1 and Corollary 3.3].
- (vi) In the case of continuity (i.e., Corollary 4.2), we can directly apply the Berge maximum theorem (Theorem 2.8(iii)) to deduce continuity of the value function $v : \mathbb{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ and upper continuity of the set of maximizers $V : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ for any continuous objection function $f : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$.

We now wish to consider the derivatives of the fixed points of contraction mappings. It was shown in [17, Theorem 4.7] that for single-valued contraction mappings, continuous differentiability of H implies continuous differentiability of h . The following result allows us to consider the case when H need not be single-valued or *continuously* differentiable.

Corollary 4.4. *Let \mathbb{X} be a normed space and \mathbb{Y} be a Banach space. Let $H : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ have closed and $d_{\mathbb{Y}}$ -bounded values. Let $y \mapsto H(x, y)$ be a contraction mapping with Lipschitz constant $L_{\mathbb{Y}} < 1$ for every $x \in \mathbb{X}$. The contingent (adjacent) derivative of $h : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ is defined by the set of fixed points of the contingent (resp. adjacent) derivative of H , i.e.,*

$$Dh(x_0, y_0)(x) = \text{FIX}_{y \in \mathbb{Y}} DH(x_0, y_0, y_0)(x, y)$$

(resp. with $D_n h$ and $D_n H$) for every $(x_0, y_0) \in \text{graph } h$ and $x \in \mathbb{X}$.

Proof. As above, by the Nadler fixed point theorem, we have that $h(x) \neq \emptyset$ for every $x \in \mathbb{X}$. We will prove the result for the contingent derivatives only, the proof of equivalence for adjacent derivatives follows from the same logic. By Theorem 3.7 we have that $Dh(x_0, y_0)(x) \subseteq \text{FIX}_{y \in \mathbb{Y}} DH(x_0, y_0, y_0)(x, y)$ for any $(x_0, y_0) \in \text{graph } h$ and $x \in \mathbb{X}$. Let $(x_0, y_0) \in \text{graph } h$, $x \in \mathbb{X}$, and $y \in \text{FIX}_{y \in \mathbb{Y}} DH(x_0, y_0, y_0)(x, y)$. By definition there exists

a sequence $\{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \searrow 0$ and $\{(x_n, \bar{y}_n, y_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Y} \rightarrow (x, y, y)$ such that $y_0 + t_n y_n \in H(x_0 + t_n x_n, y_0 + t_n \bar{y}_n)$ for every $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ we can conclude

$$\begin{aligned}
d_{\mathbb{Y}}\left(y, \frac{1}{t_n}[h(x_0 + t_n x_n) - y_0]\right) &\leq \|y - y_n\|_{\mathbb{Y}} + \frac{1}{t_n} d_{\mathbb{Y}}(y_0 + t_n y_n, h(x_0 + t_n x_n)) \\
&\leq \|y - y_n\|_{\mathbb{Y}} + \frac{1}{t_n} \frac{L_{\mathbb{Y}}}{1 - L_{\mathbb{Y}}} \|(y_0 + t_n y_n) - (y_0 + t_n \bar{y}_n)\|_{\mathbb{Y}} \\
&= \|y - y_n\|_{\mathbb{Y}} + \frac{L_{\mathbb{Y}}}{1 - L_{\mathbb{Y}}} \|y_n - \bar{y}_n\|_{\mathbb{Y}} \\
&\leq \frac{1}{1 - L_{\mathbb{Y}}} [\|y - y_n\|_{\mathbb{Y}} + L_{\mathbb{Y}} \|y - \bar{y}_n\|_{\mathbb{Y}}]
\end{aligned} \tag{4.1}$$

Inequality (4.1) follows from the definition of y_n and Proposition 4.5 below. By this sequence of inequalities we recover

$$\lim_{n \rightarrow \infty} d_{\mathbb{Y}}\left(y, \frac{1}{t_n}[h(x_0 + t_n x_n) - y_0]\right) = 0$$

and, as an immediate consequence, there exists a sequence $\{y_n^*\}_{n \in \mathbb{N}} \subseteq \mathbb{Y} \rightarrow y$ such that $y_0 + t_n y_n^* \in h(x_0 + t_n x_n)$ for every $n \in \mathbb{N}$. That is, $y \in Dh(x_0, y_0)(x)$. \square

The following proposition is utilized in the proof of Corollary 4.4. It gives an upper bound to the distance between a point and the nearest fixed point.

Proposition 4.5. *Let $H : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ have nonempty, closed, and $d_{\mathbb{Y}}$ -bounded values. If $y \mapsto H(x, y)$ is a contraction mapping with Lipschitz constant $L_{\mathbb{Y}} < 1$ for $x \in \mathbb{X}$ and if $y_0 \in H(x, \bar{y})$ for some $\bar{y} \in \mathbb{Y}$, then*

$$d_{\mathbb{Y}}(y_0, h(x)) \leq \frac{L_{\mathbb{Y}}}{1 - L_{\mathbb{Y}}} d_{\mathbb{Y}}(y_0, \bar{y}).$$

Proof. This proof follows from that of [18, Lemma 15.1]. Define $T_1 : \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y})$ by $T_1 y := \{y_0\}$ for every $y \in \mathbb{Y}$. Since T_1 is a constant function, it is Lipschitz continuous for any choice of Lipschitz constant, in particular this is true with constant $L_{\mathbb{Y}}$. Additionally, the set of fixed points of T_1 is given by y_0 by definition. By utilizing only the first half of the

proof of [18, Lemma 15.1] we recover

$$\begin{aligned} d_{\mathbb{Y}}(y_0, h(x)) &= D_{\mathbb{Y}}^*(y_0, h(x)) \\ &\leq \frac{1}{1 - L_{\mathbb{Y}}} D_{\mathbb{Y}}^*(T_1 y_0, H(x, y_0)) = \frac{1}{1 - L_{\mathbb{Y}}} D_{\mathbb{Y}}^*(y_0, H(x, y_0)). \end{aligned}$$

Thus by the choice of $y_0 \in H(x, \bar{y})$ and Lipschitz continuity of $H(x, \cdot)$ we can immediately conclude

$$d_{\mathbb{Y}}(y_0, h(x)) \leq \frac{1}{1 - L_{\mathbb{Y}}} D_{\mathbb{Y}}(H(x, y_0), H(x, \bar{y})) \leq \frac{L_{\mathbb{Y}}}{1 - L_{\mathbb{Y}}} d_{\mathbb{Y}}(y_0, \bar{y}).$$

□

5 Continuity of parameterized approximate fixed points

As in Section 4 above, throughout this section we will consider \mathbb{Y} to be a metric space with metric $d_{\mathbb{Y}}$. We additionally introduce the notion of the open and closed ball around $A \subseteq \mathbb{Y}$ defined respectively by

$$\begin{aligned} B_{\epsilon}(A) &:= \{y \in \mathbb{Y} \mid d_{\mathbb{Y}}(y, A) := \inf_{a \in A} d_{\mathbb{Y}}(y, a) < \epsilon\} \\ \bar{B}_{\epsilon}(A) &:= \{y \in \mathbb{Y} \mid d_{\mathbb{Y}}(y, A) \leq \epsilon\} \end{aligned}$$

for any $\epsilon > 0$. Approximate fixed points have been studied in, e.g., [45, 5, 9, 6].

Corollary 5.1. *Let \mathbb{Y} be a metric space and $H : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ have nonempty images.*

- (i) *Let \mathbb{Y} additionally be a compact space and let H be upper continuous with closed images (equivalently H has a closed graph). The closed approximate fixed point mapping $\bar{h}_{\epsilon} : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ defined by $\bar{h}_{\epsilon}(x) := \text{FIX}_{y \in \mathbb{Y}} \bar{B}_{\epsilon(x)}(H(x, y))$ is closed and compact-valued and upper continuous (equivalently H has a closed graph) if $\epsilon : \mathbb{X} \rightarrow \mathbb{R}_{++}$ is upper semicontinuous.*

(ii) Let H be lower continuous. The open approximate fixed point mapping $h_\epsilon : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ defined by $h_\epsilon(x) := \text{FIX}_{y \in \mathbb{Y}} B_{\epsilon(x)}(H(x, y))$ is lower continuous with an open graph if $\epsilon : \mathbb{X} \rightarrow \mathbb{R}_{++}$ is lower semicontinuous.

Proof. (i) Let ϵ be an upper semicontinuous function. Define $\bar{H}_\epsilon : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ by $\bar{H}_\epsilon(x, y) = \bar{B}_{\epsilon(x)}(H(x, y)) \cap \mathbb{Y}$. Then by definition $\bar{h}_\epsilon(x) = \text{FIX}_{y \in \mathbb{Y}} \bar{H}_\epsilon(x, y)$. Therefore, by Lemma 3.1(i), if \bar{H}_ϵ has a closed graph then \bar{h}_ϵ is upper continuous with closed and compact images. Let $\{(x_i, y_i, \tilde{y}_i)\}_{i \in I} \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Y} \rightarrow (x, y, \tilde{y})$ such that $\tilde{y}_i \in \bar{H}_\epsilon(x_i, y_i)$, i.e., $d_{\mathbb{Y}}(\tilde{y}_i, H(x_i, y_i)) \leq \epsilon(x_i)$, for every $i \in I$. Since H is nonempty and compact-valued, there exists some $y_i^* \in H(x_i, y_i) \subseteq \mathbb{Y}$ such that $d_{\mathbb{Y}}(\tilde{y}_i, y_i^*) = d_{\mathbb{Y}}(\tilde{y}_i, H(x_i, y_i))$ for every $i \in I$. By \mathbb{Y} compact, there exists a convergent subnet $\{y_j^*\}_{j \in J} \rightarrow y^* \in \mathbb{Y}$ where $J \subseteq I$. Immediately we can conclude

$$\begin{aligned} d_{\mathbb{Y}}(\tilde{y}, H(x, y)) &\leq d_{\mathbb{Y}}(\tilde{y}, y^*) = \lim_{j \in J} d_{\mathbb{Y}}(\tilde{y}_j, y_j^*) \\ &= \lim_{j \in J} d_{\mathbb{Y}}(\tilde{y}_j, H(x_j, y_j)) \leq \limsup_{j \in J} \epsilon(x_j) \leq \epsilon(x). \end{aligned}$$

That is, $\tilde{y} \in \bar{H}_\epsilon(x, y)$.

(ii) Let ϵ be a lower semicontinuous function. Define $H_\epsilon : \mathbb{X} \times \mathbb{Y} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ by $H_\epsilon(x, y) = B_{\epsilon(x)}(H(x, y)) \cap \mathbb{Y}$. Then by definition $h_\epsilon(x) = \text{FIX}_{y \in \mathbb{Y}} H_\epsilon(x, y)$. Therefore, by Lemma 3.3(i), if H_ϵ has an open graph then h_ϵ is lower continuous with an open graph. We will show that H_ϵ is open by proving its compliment is closed. Let $\{(x_i, y_i, \tilde{y}_i)\}_{i \in I} \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Y} \rightarrow (x, y, \tilde{y})$ such that $\tilde{y}_i \notin H_\epsilon(x_i, y_i)$, i.e., $d_{\mathbb{Y}}(\tilde{y}_i, H(x_i, y_i)) \geq \epsilon(x_i)$, for every $i \in I$. Since H is nonempty-valued, for every $\delta > 0$ there exists some $y^\delta \in H(x, y)$ such that $d_{\mathbb{Y}}(\tilde{y}, y^\delta) < d_{\mathbb{Y}}(\tilde{y}, H(x, y)) + \delta$. By H lower continuous, there exists a net $\{y_j^\delta\}_{j \in J} \subseteq \mathbb{Y} \rightarrow y^\delta$ such that $y_j^\delta \in H(x_j, y_j)$ for every $j \in J \subseteq I$. Immediately

we can conclude that $(x, y, \tilde{y}) \in (\text{graph } H_\epsilon)^c$ since for any $\delta > 0$

$$\begin{aligned} d_{\mathbb{Y}}(\tilde{y}, H(x, y)) + \delta &> d_{\mathbb{Y}}(\tilde{y}, y^\delta) = \lim_{j \in J} d_{\mathbb{Y}}(\tilde{y}_j, y_j^\delta) \\ &\geq \liminf_{j \in J} d_{\mathbb{Y}}(\tilde{y}_j, H(x_j, y_j)) \geq \liminf_{j \in J} \epsilon(x_j) \geq \epsilon(x). \end{aligned}$$

□

Remark 5.2. (i) Corollary 5.1 is presented with functional approximation error $\epsilon : \mathbb{X} \rightarrow \mathbb{R}_{++}$. A typical choice would be given by a constant, i.e., $\epsilon(x) := \bar{\epsilon} > 0$ for every $x \in \mathbb{X}$.

(ii) Consider the setting of Corollary 5.1 and fix $\epsilon : \mathbb{X} \rightarrow \mathbb{R}_{++}$. We can immediately conclude if $h(x) \neq \emptyset$ then $\bar{h}_\epsilon(x) \supseteq h_\epsilon(x) \supseteq h(x) \neq \emptyset$. In particular if the Kakutani fixed point theorem (see Remark 3.2) or the Nadler fixed point theorem hold, then h_ϵ and \bar{h}_ϵ have nonempty values. The literature on approximate fixed points provide additional results in this direction.

(iii) If H is single-valued and continuous then the open approximate fixed point mapping h_ϵ has an open graph for any lower semicontinuous $\epsilon : \mathbb{X} \rightarrow \mathbb{R}_{++}$ by Corollary 5.1(ii). If \mathbb{Y} is additionally a compact Hausdorff space then the closed approximate fixed point mapping \bar{h}_ϵ has a closed graph for any upper semicontinuous ϵ by Corollary 5.1(i).

Corollary 5.3. *Consider the settings of Corollary 5.1(i) and (ii) both satisfied with $\epsilon : \mathbb{X} \rightarrow \mathbb{R}_{++}$ continuous and assume the set of fixed points $h(x)$ is nonempty for every $x \in \mathbb{X}$. Let the value function $v : \mathbb{X} \rightarrow \mathbb{R}$ be defined as in Theorem 2.8 for some continuous objective function $f : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$.*

(i) $\bar{V}_\epsilon : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y}) \setminus \{\emptyset\}$ defined by

$$\bar{V}_\epsilon(x) := \{y \in \bar{h}_\epsilon(x) \mid f(x, y) \geq v_\epsilon(x)\} \quad \forall x \in \mathbb{X}$$

is upper continuous with closed and compact images where

$$v_\epsilon(x) := \sup\{f(x, y) \mid y \in h_\epsilon(x)\} \quad \forall x \in \mathbb{X}.$$

(ii) $V_{\epsilon, \delta} : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ defined by

$$V_{\epsilon, \delta}(x) := \{y \in h_\epsilon(x) \mid f(x, y) > v_\epsilon(x) - \delta(x)\} \quad \forall x \in \mathbb{X}$$

is lower continuous with an open graph for any $\delta : \mathbb{X} \rightarrow \mathbb{R}_+$ lower semicontinuous.

Additionally $V_{\epsilon, \delta}(x) \neq \emptyset$ if $\delta(x) > 0$.

Proof. (i) Immediately by Theorem 2.8(ii) and Corollary 5.1(ii), we can conclude $v_\epsilon : \mathbb{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous. To prove \bar{V}_ϵ is upper continuous with closed images (and therefore compact images) we will prove that $\text{graph } \bar{V}_\epsilon$ is closed. Notice that $\text{graph } \bar{V}_\epsilon = \text{graph } \bar{h}_\epsilon \cap \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid f(x, y) \geq v_\epsilon(x)\}$. By Corollary 5.1(i) we know that \bar{h}_ϵ has a closed graph, therefore it remains to show that $\{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid f(x, y) \geq v_\epsilon(x)\}$ is closed. Let $\{(x_i, y_i)\}_{i \in I} \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow (x, y)$ such that $f(x_i, y_i) \geq v_\epsilon(x_i)$ for every $i \in I$. Immediately we recover

$$f(x, y) = \lim_{i \in I} f(x_i, y_i) \geq \liminf_{i \in I} v_\epsilon(x_i) \geq v_\epsilon(x).$$

Finally, \bar{V}_ϵ has nonempty values since for every $x \in \mathbb{X}$ there exists some element $y^* \in \text{cl } h_\epsilon(x) \subseteq \bar{h}_\epsilon(x)$ such that $f(x, y^*) = v_\epsilon(x)$ (by $\text{cl } h_\epsilon(x)$ compact), i.e., $y^* \in \bar{V}_\epsilon(x)$.

(ii) Let $\delta : \mathbb{X} \rightarrow \mathbb{R}_+$ be a lower semicontinuous function. By the definition of $V_{\epsilon, \delta}$ we immediately find that $\text{graph } V_{\epsilon, \delta} = \text{graph } h_\epsilon \cap \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid f(x, y) > v_\epsilon(x) - \delta(x)\}$. Therefore $V_{\epsilon, \delta}$ has an open graph (and as a consequence $V_{\epsilon, \delta}$ is lower continuous via Lemma 2.7) if $\{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid f(x, y) \leq v_\epsilon(x) - \delta(x)\}$ is closed. Let $\{(x_i, y_i)\}_{i \in I} \subseteq$

$\mathbb{X} \times \mathbb{Y} \rightarrow (x, y)$ where $f(x_i, y_i) \leq v(x_i) - \delta(x_i)$ for every $i \in I$. Immediately we recover

$$f(x, y) + \delta(x) \leq \liminf_{i \in I} [f(x_i, y_i) + \delta(x_i)] \leq \limsup_{i \in I} v(x_i) \leq v(x)$$

by v upper semicontinuous (see Theorem 2.8(i) and Lemma 3.1(ii)). Now assume $\delta(x) > 0$ for some $x \in \mathbb{X}$. By h nonempty and the definition of v , for any $\bar{\delta} > 0$ there exists some element $y_{\bar{\delta}}^* \in h(x) \subseteq h_\epsilon(x)$ such that $f(x, y_{\bar{\delta}}^*) > v(x) - \bar{\delta}$. In particular this implies $f(x, y_{\delta(x)}^*) > v(x) - \delta(x)$, i.e., $y_{\delta(x)}^* \in V_{\epsilon, \delta}(x)$.

□

Remark 5.4. The assumption that $\delta : \mathbb{X} \rightarrow \mathbb{R}_+$ be lower semicontinuous in Corollary 5.3(ii) can be relaxed so that

$$\delta(x) \leq \limsup_{\bar{x} \rightarrow x} \delta(\bar{x})$$

for every $x \in \mathbb{X}$. Similarly to Remark 5.2(i), a typical choice for δ is given by a constant, i.e., $\delta(x) := \bar{\delta} > 0$ for every $x \in \mathbb{X}$.

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