Connected Spaces.\(^1\)

1 Definition.

**Definition 1.** Let \((X, d)\) be a (non-empty) metric space. \(X\) is disconnected iff there are non-empty, disjoint, open sets \(A, B \subseteq X\) such that \(A \cup B = X\). \(X\) is connected iff it is not disconnected.

A subtlety is that if \(X\) is a subspace of a larger metric space \(Y\), then open means with respect to \(X\), not with respect to \(Y\). For example, consider \(X = \{0, 1\} \subseteq \mathbb{R}\). Then \(X\) is disconnected: take \(A = \{0\}\) and \(B = \{1\}\). Here \(A\) and \(B\) are open in \(X\) even though they are not open in \(\mathbb{R}\). The following result says that if \(X\) is a subset of a larger metric space \(Y\), then one can, in fact, also check connectedness by working with sets that are open in \(Y\). I use this fact in the proof of the main result of this section, Theorem 2.

**Theorem 1.** Let \((Y, d)\) be a (non-empty) metric space and let \(X \subseteq Y\). Then \(X\) is disconnected iff there are disjoint sets \(O_1, O_2 \subseteq Y\) that are open in \((Y, d)\) and such that \(O_1 \cap X \neq \emptyset, O_2 \cap X \neq \emptyset\) and \(X \subseteq O_1 \cup O_2\).

**Proof:** \(\Rightarrow.\) Since \(X\) is disconnected, there are non-empty, disjoint sets \(A, B \subseteq X\) that are open in \(X\) and such that \(A \cup B = X\). Since \(A\) is open in \(X\), for any \(a \in A\), there is an \(\varepsilon_a > 0\) such that \(\{x \in X : d(a, x) < \varepsilon_a\}\) has no intersection with \(B\). This implies that \(N_{\varepsilon_a}(a) = \{x \in Y : d(a, x) < \varepsilon_a\}\) also has no intersection with \(B\). Let

\[
O_1 = \bigcup_{a \in A} N_{\varepsilon_a/2}(a).
\]

Define \(O_2\) similarly for \(b \in B\). Note that, under this construction, for any \(a \in A, b \in B, \varepsilon_a, \varepsilon_b \leq d(a, b)\), hence \(\varepsilon_a/2, \varepsilon_b/2 \leq d(a, b)/2\).

All of the conditions for \(O_1\) and \(O_2\) are immediate except disjointness. Consider, then, any element \(x \in O_1\). Then there is an \(a \in A\) such that \(x \in N_{\varepsilon_a/2}(a)\). Take any \(b \in B\). Then, by the Triangle Inequality and the construction of \(\varepsilon_a\),

\[
d(x, b) \geq d(a, b) - d(x, a) > d(a, b) - \varepsilon_a/2 \geq d(a, b)/2 \geq \varepsilon_b/2.
\]

Therefor \(x \notin N_{\varepsilon_b/2}(b)\). Since \(b\) was arbitrary, \(x \notin O_2\).

\(\Leftarrow.\) Almost immediate: set \(A = O_1 \cap X\) and \(B = O_2 \cap X\). \(\blacksquare\)

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Thus, continuing the example from above, if \( X = \{0, 1\} \subseteq \mathbb{R} \) then \( X \) is disconnected since I can take \( O_1 = (-1/2, 1/2) \) and \( O_2 = (1/2, 3/2) \).

On the real line, connected sets must be of a certain form. Recall that a set \( X \subseteq \mathbb{R} \) is an interval iff for any \( a, b \in X \) with \( a \leq b \), if \( a < x < b \), then \( x \in X \).

**Theorem 2.** If \( X \subseteq \mathbb{R} \), then \( X \) is connected iff it is an interval.

**Proof.** By contraposition.

\( \Leftarrow \). Suppose that \( X \) is not connected. Then by Theorem 1, there are disjoint sets \( O_1, O_2 \) that are open in \( \mathbb{R} \) and such that \( O_1 \cap X \neq \emptyset \), \( O_2 \cap X \neq \emptyset \), and \( X \subseteq O_1 \cup O_2 \).

Choose any \( a \in O_1 \cap X, b \in O_2 \cap X \). Without loss of generality, suppose \( a < b \).

Let \( E = \{x \in O_1 : x < b\} \). \( E \) is bounded above (by \( b \)) and non-empty (since \( a \in E \)). Let \( x^* = \sup E \).

I claim that \( a < x^* < b \) and \( x^* \notin X \), hence \( X \) is not an interval.

1. Claim: \( a < x^* \). Since \( a \in O_1 \) and \( O_1 \) is open, there is an \( \varepsilon > 0 \) such that \( N_\varepsilon(a) \subseteq O_1 \). Since \( b \in O_2, b \notin O_1 \). Since \( a < b \), this implies \( N_\varepsilon(a) \subseteq E \). Since \( x^* \) is an upper bound for \( E \), this implies that \( a + \varepsilon \leq x^* \), hence \( a < x^* \).

2. Claim: \( x^* < b \). Since \( b \in O_2 \) and \( O_2 \) is open, there is an \( \varepsilon > 0 \) such that \( N_\varepsilon(b) \subseteq O_2 \). Therefore any element of \( N_\varepsilon(b) \) is an upper bound for \( E \), which implies \( x^* \leq b - \varepsilon \), hence \( x^* < b \).

3. Claim: \( x^* \notin X \). This follows from the following subclaims.

   - Claim: \( x^* \notin O_1 \). Take any \( x \in O_1 \) such that \( x < b \). As in the proof that \( a < x^* \), this implies \( x < x^* \). Thus, \( x^* < b \) but \( x^* \) is strictly greater than any \( x \in O_1 \) such that \( x < b \), hence \( x^* \notin O_1 \).
   - Claim: \( x^* \notin O_2 \). Take any \( x \in O_2 \) that is an upper bound of \( E \). As in the proof that \( x^* < b \), this implies \( x^* < x \). Thus, \( x^* \) is an upper bound of \( E \) but \( x^* \) is strictly less than any \( x \in O_2 \) that is an upper bound of \( E \), hence \( x^* \notin O_2 \).

\( \Rightarrow \). Suppose that \( X \) is not an interval. Then there are points \( a, b \in X \), with \( a < b \), and a number \( x^* \) such that \( a < x^* < b \) and \( x^* \notin X \). Take \( O_1 = (-\infty, x^*) \) and \( O_2 = (x^*, \infty) \) and apply Theorem 1. –

If I remove a point from the interior of an interval in \( \mathbb{R} \), I get a disconnected set: although \( [0, 3] \) is connected, \( [0, 1) \cup (2, 3] \) is disconnected. This is not true in higher dimensions. For example, \( S = \{ x \in \mathbb{R}^2 : ||x|| \leq 1 \} \), which is the unit disk in \( \mathbb{R}^2 \), is connected, but also so is \( S \setminus \{0\} \), which is the unit disk with the origin removed. This is an important topological difference between \( \mathbb{R} \) and Euclidean spaces of higher dimension. It is the fundamental reason why some useful results for \( \mathbb{R} \), notably the Mean Value Theorem, do not fully generalize, and why other useful results, such as the Intermediate Value Theorem, generalize only with great difficulty.
(here, I am interpreting the Brouwer Fixed Point Theorem as the generalization of the Intermediate Value Theorem).

Remark 1. If \( X = A \cup B \), where \( A \) and \( B \) are disjoint, then \( A \) is open iff \( B \) is closed, and vice versa. Hence \( X \) is disconnected iff \( X = A \cup B \) where \( A, B \) are non-empty, disjoint, closed sets. \( \square \)

Remark 2. Recall that in any metric space \( X \), \( \emptyset \) and \( X \) are both open and closed. The preceding remark establishes that \( X \) is disconnected iff there are sets other than \( \emptyset \) and \( X \), namely the sets \( A \) and \( B \) in the definition of disconnected, that are also both open and closed. For example, if \( X = \{0, 1\} \subseteq \mathbb{R} \), then \( \{0\} \) and \( \{1\} \) are both open and closed.

Stating the same idea differently, \( X \) is connected iff the only sets in \( X \) that are both open and closed are \( \emptyset \) and \( X \) itself. One can show that if \( X \) is a normed vector space, then it is connected and hence that there are no sets in such spaces, other than \( \emptyset \) and \( X \) itself, that are both open and closed. \( \square \)

Remark 3. Let \( (Y,d) \) be a metric space. \( A, B \subseteq Y \) are separated iff \( \overline{A} \cap B = A \cap \overline{B} = \emptyset \). \( A \) and \( B \) are separated iff \( X = A \cup B \) is disconnected. The argument is as follows.

\( \Rightarrow \). Suppose that \( A \) and \( B \) are separated. \( \overline{A} \cap B = \emptyset \) implies that \( A \) has no limit points in \( B \). Since \( A \cup B = X \), this implies that \( A \) is, in fact, closed in \( X \). A similar argument shows that \( B \) is closed in \( X \), which implies that \( X \) is disconnected (by Remark 1).

\( \Leftarrow \). Suppose that \( X \) is disconnected. Then there are non-empty, disjoint sets \( A, B \subseteq X \) that are open in \( X \) and such that \( X = A \cup B \). But then, since \( B \) is open in \( X \), its complement in \( X \), namely \( A \), is closed in \( X \), hence \( \overline{A} = A \), which implies \( \overline{A} \cap B = \emptyset \). Similarly, \( A \cap \overline{B} = \emptyset \). \( \square \)