Efficiency and Competitive Equilibrium

1 Preliminaries.

I focus here on exchange economies (no production). The analysis can be extended to production.

There are \( I < \infty \) consumers and \( N < \infty \) commodities. Consumer \( i \)'s consumption of commodity \( n \) is denoted \( x^i_n \). I assume that \( x^i_n \geq 0 \) for each \( n \): I do not allow negative consumption. The vector giving consumer \( i \)'s consumption of each good is called a consumption bundle and is denoted \( x^i \): \( x^i = (x^i_1, \ldots, x^i_N) \). The set of of possible (in principle) consumption bundles, called the consumption space, is \( \mathbb{R}_+^N \).

An allocation is a vector \( x = (x^1, \ldots, x^I) \in \mathbb{R}_+^{IN} \), giving a consumption bundle for each consumer. I denote aggregate consumption by a bar: aggregate consumption of good \( n \) is

\[
\bar{x}_n = \sum_{i} x^i_n.
\]

The aggregate consumption vector is \( \bar{x} \in \mathbb{R}_+^N \).

The aggregate endowment of good \( n \) is \( \bar{e}_n \) (\( e \) for “endowment”; not \( e \approx 2.718 \)). To avoid triviality, I assume that \( \bar{e}_n > 0 \). The aggregate endowment is denoted \( \bar{e} \in \mathbb{R}^{N+} \). An allocation \( x \) is feasible iff for each \( n \)

\[
\bar{x}_n \leq \bar{e}_n.
\]

Implicitly, feasibility assumes free disposal: it is possible to consume less than the endowment.

For some of the discussion, I assume that each \( i \) has a personal endowment \( e^i \in \mathbb{R}_+^N \). The endowment allocation is \( e = (e^1, \ldots, e^N) \). An endowment allocation \( e \) is compatible with an aggregate endowment \( \bar{e} \) iff

\[
\bar{e} = \sum_i e^i.
\]

Although I assume \( \bar{e}_n > 0 \) for each \( n \), I do not necessarily assume that \( e^i_n > 0 \) for every \( i \) and every \( n \).

I assume that each consumer \( i \)'s choices can be rationalized by preferences \( W^i \) (with strict preference written \( S^i \)) that in turn can be represented by a utility function \( u^i : \mathbb{R}_+^N \to \mathbb{R} \). I always assume that the utility representation is continuous.
(see the Decision Theory notes). Recall also that any increasing transformation of \( u^i \) represents the same preferences and generates the same choices.

**Remark 1.** These assumptions rule out utility function representations such as

\[ u^i(x^i) = \sum \beta_n \log(x^i_n) \]

with \( \beta_n \geq 0, \sum \beta_n = 1 \), because this \( u^i \) is not defined on the boundaries of \( \mathbb{R}^+_N \).

Note, however, that I can represent the same preferences over consumption bundles by, for example,

\[ \hat{u}^i(x^i) = \prod_i (x^i_n)^{\beta_n} \]

which is defined over all \( \mathbb{R}^+_N \). \( \square \)

Note that I am restricting the domain of preferences to consumer \( i \)'s own consumption. In principle one could extend the domain to all of \( \mathbb{R}^{NI} \), and by doing so capture various forms of altruism and envy. But I am not doing this.

Preferences are **monotone** iff \( \hat{x}^i \ S^i x^i \) whenever \( \hat{x}^i \gg x^i \). In words, monotonicity says that more of every good makes the consumer strictly better off; monotonicity allows for the possibility that the consumer does not care one way or the other about some goods. Preferences are **strongly monotone** iff \( \hat{x}^i \ S^i x^i \) whenever \( x_i > x^* \). In words, strong monotonicity says that the consumer cares about every good and that more of any good makes the consumer strictly better off.

Preferences are **locally non-satiated** iff for any \( x^i \in \mathbb{R}^+_N \) and any \( \varepsilon > 0 \) there is a \( \hat{x}^i \in N_\varepsilon(x) \cap \mathbb{R}^+_N \) such that \( \hat{x}^i \ S^i x^i \). In words, local non-satiation says that for any \( x_i \) there is another bundle that is close to \( x^i \) and that consumer \( i \) likes strictly better than \( x^i \). Strong monotonicity implies local non-satiation but local non-satiation does not imply monotonicity, let alone strong monotonicity.

## 2 Efficiency

**Definition 1.** Fix two allocations \( x \) and \( x^* \).

1. \( x \) **Pareto dominates** \( x^* \) iff \( x^i \ W^i x^* \) with \( x^i S^i x^* \) for at least one \( i \).

2. \( x \) **strictly Pareto dominates** \( x^* \) iff \( x^i S^i x^* \) for every \( i \).

**Definition 2.** Fix a feasible allocation \( x^* \).

1. \( x^* \) is **efficient** (also called **Pareto efficient**, **Pareto optimal**) iff it is not Pareto dominated by any feasible allocation.

2. \( x^* \) is **weakly efficient** (weakly **Pareto efficient**, weakly **Pareto optimal**) iff it is not strictly Pareto dominated by any feasible allocation.

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\( \text{Recall that } x^i \gg x^* \text{ iff } x^i_n > x^*_n \text{ for every } n. \ x^i > x^* \text{ iff } x^i_n \geq x^*_n \text{ for every } n \text{ and } x^i_n > x^*_n \text{ for at least one } n. \)
In words, a feasible allocation is efficient iff it is not possible to make anyone better off without making someone else worse off. A feasible allocation is weakly efficient iff it is not possible to make everyone better off. Since \( x^* \) Pareto dominates \( x \) whenever \( x^* \) strictly Pareto dominates \( x \), efficiency implies weak efficiency. The converse is not true in general, but it is true whenever \( u^i \) has enough structure.

**Theorem 1.** If, for every \( i \), preferences are strongly monotone then efficiency and weak efficiency are equivalent.

**Proof.** Since efficiency implies weak efficiency, it remains to show that, under the given assumption, weak efficiency implies efficiency.

For each \( i \), let \( u^i \) be a continuous utility representation of \( i \)'s preferences. I argue by contraposition: if \( x^* \) is feasible but not efficient then it is not weakly efficient.

By monotonicity, \( u^i(x^*) \geq u^i(0) \) since \( x^* \geq 0 \). Therefore, since \( u^i(x^*) > u^i(x^{*i}) \), \( x^i > 0 \). Construct a new allocation \( \hat{x} \) by taking \( \epsilon x^i > 0 \) away from \( i \) and giving \( \epsilon x^i/(N-1) > 0 \) to everyone else. For \( \epsilon > 0 \) sufficiently small, this is feasible. In particular, \( \hat{x}^i = x^i - \epsilon x^i \geq 0 \), \( \hat{x}^j = x^j + \epsilon x^j/(N-1) > x^j \geq 0 \) for all other consumers \( j \).

By continuity, for \( \epsilon > 0 \) sufficiently small, \( u^i(x^i(\hat{x}^i)) > u^i(x^{*i}) \). And by strong monotonicity, for all other consumers \( j \), \( u^j(\hat{x}^j) > u^j(x^j) \geq u^j(x^{*j}) \). Thus \( \hat{x} \) is feasible and strictly Pareto dominates \( x^* \), which implies \( x^* \) was not weakly efficient. ■

Efficiency is a weak condition. For example, it may be efficient for consumer 1 to consume everything, and everyone else nothing. It may also be efficient for consumer 2 to consume everything, and everyone else nothing. And so on. Typically there is a large set of efficient allocations. One typically thinks of efficiency as necessary, but not sufficient, for an allocation to be “good”. An allocation that is not efficient can be unambiguously improved upon. But an allocation that is efficient may not satisfy some additional criterion of social justice (e.g., some form of fairness).

### 3 Efficiency and Social Welfare.

The main result of this section is that, subject to some qualifications, \( x^* \) is efficient if and only if it solves the problem of maximizing “social welfare”, which I define to be a weighted sum of individual utilities. This links efficiency with a particular form of optimization. Thus the terminology “Pareto optimal” (as an alternative to “efficient”) makes sense.

For the moment, fix utility representations \( u^i \). Given a vector of utility weights
\[ \alpha = (\alpha^1, \ldots, \alpha^I), \alpha > 0, \text{ let} \]

\[ S_\alpha(x) = \sum_i \alpha^i u^i(x^i). \]

“S” (not to be confused with strict preferences, S) is for “social.” \( S_\alpha(x) \) gives a measure of the social value (or social welfare) of an allocation \( x \). The \( S_\alpha \) maximization problem is to maximize \( S_\alpha \) over all feasible \( x \).

The Utility Possibility Set (UPS) is the subset of \( \mathbb{R}^I \), call it \( U \), such that \( c \in U \) iff there is a feasible \( x \) such that \( c^i = u^i(x^i) \) for every \( i \). Define \( C = U + \mathbb{R}^I_- \) (where \( \mathbb{R}^I_- = \{ c \in \mathbb{R}^I : c \leq 0 \} \)). \( C \) is the set of vectors \( c \in \mathbb{R}^I \) that are either in the UPS or are smaller than (in a vector sense) something in the UPS.

Since the set of feasible allocations is compact, \( U \) is compact if the utility representations \( u^i \) are continuous, which I always assume is the case. If \( U \) is compact then \( C \) is closed and bounded above. Finally, it is easy to verify that \( C \) is convex if the \( u^i \) are concave. I discuss concavity of the \( u^i \) further below.

**Theorem 2.** Fix an aggregate endowment.

1. (a) For any \( \alpha \in \mathbb{R}^I_{++} \), if \( x^* \) solves the \( S_\alpha \) maximization problem then \( x^* \) is efficient.

   (b) For any \( \alpha \in \mathbb{R}^I_+ \), \( \alpha \neq 0 \), if \( x^* \) solves the \( S_\alpha \) maximization problem then \( x^* \) is weakly efficient.

2. Suppose that \( U \) is compact and that \( C \) is convex. If \( x^* \) is weakly efficient then there is an \( \alpha \in \mathbb{R}^I_+, \alpha \neq 0 \) such that \( x^* \) solves the \( S_\alpha \) maximization problem.

**Proof.**

1. (a) This is almost immediate. If \( x^* \) is not efficient then there is a feasible \( x \) such that \( u^i(x^i) \geq u^i(x^{*i}) \) for all \( i \), with strict inequality for at least one \( i \); since all \( \alpha^i > 0 \), \( S_\alpha(x) > S_\alpha(x^*) \).

   (b) Again, this is almost immediate. If \( x^* \) is not weakly efficient then there is a feasible \( x \) such that \( u^i(x^i) > u^i(x^{*i}) \) for all \( i \). Then \( S_\alpha(x) > S_\alpha(x^*) \) even if some \( \alpha^i = 0 \).

2. If \( x^* \) is weakly efficient then \( c^* = (u^1(x^{*1}), \ldots, u^I(x^{*I})) \) is not interior to \( C \). In particular, if \( x^* \) is weakly efficient then there cannot be any \( c \in U \) such that \( c \gg c^* \), which implies that there cannot be any \( c \in C \) such that \( c \gg c^* \).

   By the Supporting Hyperplane theorem, there is an \( \alpha \in \mathbb{R}^I, \alpha \neq 0 \), such that for all \( c \in C \), and hence for all \( c \in U \),

\[ \sum_i \alpha^i c^i \leq \sum_i \alpha^i c^{*i}. \]
It remains to argue that $\alpha > 0$. Suppose that $\alpha^i < 0$ for some $i$. By construction of $C$, if $c$ is in $C$ then so is $\hat{c}$, defined to be the same as $c$ except for coordinate $i$, with $\hat{c}^i < c^i$. By taking $\hat{c}^i$ sufficiently negative, the above inequality cannot hold. This implies that $\alpha > 0$. 

\[\square\]

**Remark 2.** Theorem 2 assumes that $C$ is convex. As already noted, a sufficient condition for this is that the $u^i$ are concave. Concavity is typically considered to be strong. However, it is standard in economics to assume that preferences are convex (meaning that a convex combination of two consumption bundles is weakly preferred to either bundle), and concavity of $u^i$ implies convexity of the underlying preferences. In the converse direction, examples exist of convex preferences that do not admit any concave utility representation, but the examples tend to look pathological.

A related question is whether the analysis here depends on using some particular representation, concave or otherwise. If so, then this would be a very serious limitation. The answer, fortunately, is no. It will be enough, in particular, for most of the analysis, that preferences have some utility representation that is concave. I discuss this at greater length in Section 6.3. \[\square\]

**Theorem 3.** There exists an efficient allocation.

**Proof.** Fix any strictly positive vector of utility weights $\alpha$. By assumption, all preferences have continuous utility representations. For any such representations, $S_\alpha$ is continuous. Since the set of feasible allocations is non-empty and compact, a solution exists to the $S_\alpha$ maximization problem. By Theorem 2, this solution is efficient. 

\[\square\]

Theorem 2 provides a way to compute efficient allocations, at least in simple examples. For example, suppose $I = N = 2$. Then, assuming there exist concave utility representations, one can compute the set of efficient allocations by computing solutions to

$$\max_{x \in \mathbb{R}^4} \begin{array}{cc}
\alpha^1 u^1(x^1) + \alpha^2 u^2(x^2) \\
\sum_i x^i \leq \bar{e} \\
x \geq 0
\end{array}$$

for all $\alpha > 0$. Moreover, since a strictly increasing transformation of the objective function does not change the solution, one can assume $\alpha^1 + \alpha^2 = 1$, or $\alpha^2 = 1 - \alpha^1$, so that the problem is equivalent to solving

$$\max_{x \in \mathbb{R}^4} \begin{array}{cc}
\alpha^1 u^1(x^1) + (1 - \alpha^1)u^2(x^2) \\
\sum_i x^i \leq \bar{e} \\
x \geq 0
\end{array}$$
for every $\alpha_1 \in [0, 1]$.

This underscores that there are typically many efficient allocations, some more equitable than others. In particular, different efficient allocations correspond to different social weights. For example, the efficient allocation that gives everything to consumer $i$ corresponds to setting $\alpha_i = 1$ and all the other $\alpha^j = 0$.

Remark 3. The solution to the $S_\alpha$ maximization problem depends, of course, on $S_\alpha$, and hence on the utility representations and on $\alpha$. For example, if I represent the preferences of consumer $i$ using $2u^i$ instead of $u^i$, this typically changes the solution.

But, no matter what (concave) utility representations I use, I will compute the full set of efficiency allocations if I solve the $S_\alpha$ maximization problem for all vectors $\alpha$. $\square$

4 Competitive Equilibrium.

I focus on price vectors that are strictly positive: $p \in \mathbb{R}^N_{++}$. Note that if consumer $i$’s endowment $e^i$ is not zero, then, with strictly positive prices, the value of her endowment is positive: $p \cdot e^i > 0$. Let,

$$B^i(p) = \{x^i \in \mathbb{R}^N_+ : p \cdot x^i \leq p \cdot e^i\}.$$ 

$B^i(p)$ is the set of consumption bundles that consumer $i$ can afford.

**Definition 3.** Fix an endowment allocation $e^*$. $(p^*, x^*) \in \mathbb{R}^{NI}_+ \times \mathbb{R}^N_{++}$ is a competitive equilibrium (CE) iff the following hold.

1. For each $i$, $x^{*i}$ is preference maximal on $B^i(p^*)$.
2. $x^*$ is feasible.

$x^*$ is a competitive equilibrium allocation iff there exists a $p^* \in \mathbb{R}^N_{++}$ such that $(p^*, x^*)$ is a competitive equilibrium.

There is a literature exploring sufficient conditions for existence of competitive equilibrium. For these notes, I assume existence as needed.

5 The Welfare Theorems.

The next theorem establish that any CE is efficient.

**Theorem 4** (First Welfare Theorem (FWT)). Fix an endowment allocation $e^*$. If $x^*$ is a competitive equilibrium allocation and if preferences are locally non-satiated for every $i$, then $x^*$ is efficient.
Proof. By contraposition. Consider any $x \in \mathbb{R}_+^N$ and suppose that $x$ Pareto dominates $x^*$. I claim that $x$ is not feasible.

Let the endowment allocation be $e^*$. Since $x^*$ is a CE allocation, there is a price vector $p^* \in \mathbb{R}_+^N$ such that $(x^*, p^*)$ is a CE.

Since $x$ Pareto dominates $x^*$, there is an $i$ such that

$$x^i \not\triangleright^* x^{*i}.$$ 

Since $x^{*i}$ solves the preference maximization problem that is part of the definition of a CE, it must be that $x^i$ violates the budget constraint:

$$p \cdot x^i > p \cdot e^{*i}.$$  \hspace{1cm} (1)

Since $x$ Pareto dominates $x^*$, for any other consumer $j$,

$$x^j \not\triangleright^{*j} x^{*j}.$$ 

I claim that $p \cdot x^j \geq p \cdot e^j$. I argue by contraposition. Suppose $p \cdot x^j < p \cdot e^j$. Then by continuity of the budget constraint and local non-satiation, there is an $\tilde{x}^j$ such that $p \cdot \tilde{x}^j < p \cdot e^j$ and $\tilde{x}^j \not\triangleright^j x^j$. Since $x^j \not\triangleright^{*j} x^{*j}$, this implies that $\tilde{x}^j$ is budget feasible and is strictly preferred to $x^{*j}$, which contradicts the assumption that $x^{*j}$ solves the preference maximization problem. Thus, for all $j \neq i$

$$p \cdot x^j \geq p \cdot e^{*j}.$$  \hspace{1cm} (2)

Adding together (1) and (2)

$$\sum_i p \cdot x^i > \sum_i p \cdot e^{*i}$$

hence

$$p \cdot \bar{x} > p \cdot \bar{e}^*.$$ 

This implies $\bar{x} \not\leq^* \bar{e}^*$, hence $x$ is not feasible. By contraposition, if $x$ is feasible then it cannot Pareto dominate $x^*$.

The assumption that preferences are locally non-satiated is generally regarded as very weak. Hence, for efficiency to fail, one of two things must be true.

1. $x^*$ is not a CE allocation, either because of market power (some consumers are acting like monopolists) or because we are not at equilibrium.

2. The hidden assumption of complete markets fails. Complete markets says that every commodity that shows up as an argument in a utility function (or in a production function, if there is production) has a price and can be traded in a unified market (where by “unified” I mean that proceeds from the sale of one good can be used to make purchases of any other good). For example, if the music played by a neighbor enters your utility function, then complete markets requires that you be able to control that music, in volume, quantity, and variety, through a market purchase.
As a real world assumption, complete markets is clearly very strong. But it is satisfied in many formal models (typically, a failure of complete market failures has to be built into a model deliberately), which means that CE allocations in many models are efficient even if, as is sometimes the case, the allocations seem weird.

Whereas FWT establishes that a CE allocation is efficient, the next result, the Second Welfare Theorem (SWT), establishes that any efficient allocation can be interpreted as a CE allocation. SWT thus provides us with a reason to be interested in competitive equilibrium even if we believe competitive equilibrium to be unrealistic.

**Theorem 5** (Second Welfare Theorem (SWT).) *If \( x^* \) is efficient and if there is a competitive equilibrium \((p^*, x)\) for the economy with endowment \( e = x^* \) then \((p^*, x^*)\) is a competitive equilibrium of the \( e = x^* \) economy.*

**Proof.** Set \( e = x^* \). By assumption, there is a CE \((p^*, x)\) for this endowment allocation. For each \( i \), \( x^i \) satisfies the budget constraint (trivially, since wealth is \( p^* \cdot x^i \)), hence, since \( x^i \) is optimal, \( x^i W^i x^i \). On the other hand, since \( x \) is feasible (it is a CE allocation) and \( x^* \) is efficient, there cannot be any \( i \) such that \( x^i S^i x^* \). It follows that consumer \( i \) is indifferent between \( x^i \) and \( x^* \). This shows that \( x^* \) is also preference maximizing, hence \((p^*, x^*)\) is also a CE. ■

**Remark 4.** The version of SWT just given is the one that appears in Varian (1992). This is not the standard version. The standard version drops the assumption that a CE exists for the \( e = x^* \) economy and instead assumes that preferences are convex (any convex combination of two consumption bundles is weakly preferred to either of those bundles). The equilibrium assumption in Theorem 5 is undesirable because it is not an assumption about primitives of the model (i.e., not an assumption directly about preferences or endowments). But the standard version of SWT says only that an efficient allocation can be supported as something called a quasi-equilibrium; one needs additional assumptions about how the economy fits together to guarantee that a quasi-equilibrium is a CE. See the discussion in Mas-Colell, Whinston and Green (1995). □

6 The First-Order Approach to the Welfare Theorems.

I assume that any allocation \( x^* \) is strictly positive. This assumption is obnoxious (it does not make sense to assume that every consumer buys every brand of butter, for example). The assumption can be relaxed somewhat, but I will not do so. I also assume that the utility representations \( u^i \) are differentiable on the interior of \( \mathbb{R}_+^N \) and I assume that the KKT condition is both necessary and sufficient, which will be the case if the \( u^i \) are concave.

I present here a calculus-based approach to FWT and SWT that provides additional insights into the \( S_\alpha \) maximization problem and the interpretation of CE
prices. My treatment will be informal and in particular I will not state any results formally.

First, fix a vector \( \alpha \in \mathbb{R}^N_+ \) of utility weights and let \( x^* \) be a solution to the \( S_\alpha \) maximization problem. From the KKT condition for the \( S_\alpha \) problem, taking the derivative with respect to consumer \( i \)'s consumption of good \( n \),

\[
\alpha^i \frac{\partial u^i(x^*_i)}{\partial x^i_n} = \gamma^*_n, \tag{3}
\]

where \( \gamma^*_n \geq 0 \) is the KKT multiplier on the commodity \( n \) feasibility constraint (i.e., \( \bar{x}_n \leq \bar{e}_n \)). I assume that, in fact, \( \gamma^*_n > 0 \).

Next, suppose that \( x^{*i} \) solves the individual optimization problem at prices \( p^* \) and individual endowment \( e^{*i} \). Then,

\[
\frac{\partial u^i(x^{*i})}{\partial x^i_n} = \lambda^{*i} p^*_n, \tag{4}
\]

where \( \lambda^{*i} \geq 0 \) is the KKT multiplier on consumer \( i \)'s budget constraint. I assume that, in fact, \( \lambda^{*i} > 0 \).

The basic observation is that Equation (3) and Equation (4) look very similar. I use this observation as follows.

6.1 The Differential FWT.

Let \((p^*, x^*)\) be a CE. Then by feasibility in the definition of CE, \( x^* \) is feasible in a \( S_\alpha \) maximization problem, for any \( \alpha \). Moreover, since Equation (4) holds for every \( i \) and every \( n \), Equation (3) also holds, taking

- \( \gamma^*_n = p^*_n; \gamma^*_n > 0 \) since, by assumption, \( p^*_n > 0 \);
- \( \alpha^i = 1/\lambda^{*i}; \alpha^i > 0 \) since, by assumption, \( \lambda^{*i} > 0 \).

Since (by assumption), the KKT condition is sufficient as well as necessary in the \( S_\alpha \) maximization problem, this establishes that \( x^* \) solves the \( S_\alpha \) maximization problem and hence (by Theorem 2) is efficient. This is FWT.

Recall that, by the Envelope theorem, \( \gamma^*_n \) is the marginal affect on the value of the solution from relaxing the commodity \( n \) constraint: \( \gamma^*_n \) is the marginal social value of commodity \( n \), where social value is measured using utility weights \( \alpha \) (about which, more below).

This is a precise version of the idea that market prices are a measure of marginal social value. Recall the diamond/water paradox: the reason why diamonds command a higher market price than water, even though water is necessary for survival while diamonds are not, is that getting one more additional diamond is worth more to society than getting one more additional unit of water. It may well be the case
that the total value of diamonds to society is much lower than the total value of water, but market prices are not a measure of total value.

As for the utility weights, a rough intuition for why $\alpha^i = 1/\lambda^i$ makes sense is the following. If $\alpha^i$ is large then society is weighting consumer $i$ heavily. Recall also that, again by the Envelope Theorem, $\lambda^i$ is consumer $i$'s marginal utility of income: the amount of additional utility the consumer would get from relaxing the consumer’s budget constraint. If every consumer has the same concave utility function (it is at this point that the intuition gets a bit rough), then one can associate a high marginal utility of income with low income (the poor get more from an extra dollar than the rich do). So the math is saying that low $\alpha^i$ is associated with low wealth. Note that this is not a statement about the way things should be. It is a statement about how things are in competitive equilibrium: someone with little wealth (because the value of his endowment is low) is treated in equilibrium as though he is unimportant.

6.2 The Differential SWT.

Fix an efficient allocation $x^*$. By Theorem 2, if the $u^i$ are concave, there is a vector of utility weights $\alpha$ such that $x^*$ solves the $S_\alpha$ maximization problem. Then Equation (3) holds for every consumer $i$ and commodity $n$. Assume that $\alpha \gg 0$. Since $x^*$ solves the $S_\alpha$ maximization problem, it satisfies feasibility in the definition of CE. Moreover, if I set $e^{xi} = x^{*i}$ for each $i$, then, for any vector of prices, $x^{*i}$ is affordable. In addition, Equation (4) holds if I set,

- $p^*_n = \gamma^*_n$: since, by assumption $\gamma^*_n > 0$, $p^*_n > 0$;
- $\lambda^i = 1/\alpha^i$: since, by assumption $\alpha^i > 0$, $\lambda^i > 0$.

Since (by assumption), KKT is sufficient as well as necessary in the individual utility maximization problems, this establishes that $(p^*, x^*)$ is a CE. This is SWT. By the same reasoning as earlier, the interpretation is that any efficient allocation $x^*$ can be supported as a CE where the market price of good $n$ is the marginal social value of good $n$.

6.3 Theory of Value.

By a theory of value, I mean a vector $\gamma^* \in \mathbb{R}^N_+$ such that $\gamma^*$ is the vector of KKT multipliers for the solution, call it $x^*$, of a social welfare maximization problem. By the Envelope theorem, $\gamma^*_n$ is the marginal value to society of commodity $n$, starting at some reference efficient allocation $x^*$: $\gamma^*_n$ is the amount by which social welfare increases if the aggregate constraint on commodity $n$ were to be slightly relaxed. FWT says that any competitive equilibrium price vector is a theory of value. SWT says that any theory of value can be interpreted as a competitive equilibrium price vector.

This definition of “theory of value” wouldn’t make much sense if it depended delicately on a specific choice of utility representation. Fortunately, the dependence
is not very delicate. Suppose that I have gone through the above calculations for some $S_\alpha$ constructed from differentiable, concave utility representations. Let $(p^*, x^*)$ be the associated CE. Note that for any other utility representations, $(x^*, p^*)$ is still a CE and $x^*$ is still efficient. The question I am worried about is what happens to the vector of KKT multipliers $\gamma_n^*$ in the social welfare maximization problem if I change utility representations, because these multipliers are the theory of value.

For consumer $i$, consider an alternative representation $\hat{u}^i$ with $\hat{u}^i(x^i) = g^i(u^i(x^i))$, where $g^i$ has a strictly positive derivative. Define $c^i = u^i(x^*i)$. Then for any $n$, by the Chain Rule, and using the KKT condition for the original utility representation,

$$\frac{\partial \hat{u}^i(x^*i)}{\partial x^i_n} = Dg^i(c^i)\frac{\partial u^i(x^*i)}{\partial x^i_n} = Dg^i(c^i)\lambda^*i p^*_n.$$ 

In particular, the KKT condition holds for the new utility representation, but with the KKT multiplier for consumer $i$ being,

$$\hat{\lambda}^i = Dg^i(c^i)\lambda^*i.$$

It follows that, in the social welfare maximization problem, the commodity $n$, consumer $i$ KKT condition will continue to hold provided the utility weight on consumer $i$ is changed from $\alpha^i = 1/\lambda^*i$ to

$$\hat{\alpha}^i = \frac{1}{Dg^i(c^i)\lambda^*i}.$$ 

In particular, the KKT condition in the social welfare maximization problem holds for the same KKT multiplier $\gamma_n^*$: the change in utility representation has no affect on the measure of marginal social value.

Moreover this change in the utility weights is correct in the sense that a theory of value is always with respect to a particular efficient allocation, here $x^*$, and if we change the utility representations then we have to make a compensating change in utility weights in order to continue to get $x^*$ as the solution to the social welfare maximization problem.

Remark 5. Given an efficient allocation $x^*$, SWT says that there is a CE $(p^*, x^*)$ and we know that $p^*$ can be interpreted as a theory of value for appropriate choice of $\alpha$. This theory of value is almost unique. By Equation 4, the equilibrium price vector must be positively collinear with $\nabla u^i(x^*)$, for any $i$. On the other hand, any vector that is positively collinear with $\nabla u^i(x^*)$ is an equilibrium price vector: if $(x^*, p^*)$ is a CE then, for any $\beta > 0$, so is $(x^*, \beta p^*)$, since, for any consumer, the budget set $B^i$ has the property that $B^i(\beta p^*) = B^i(p^*)$. Thus, for a given reference efficient allocation $x^*$, the theory of value is determined uniquely up to multiplication by a scalar $\beta > 0$. □
6.4 The Labor Theory of Value.

As an alternative to a theory of value in the above sense, consider the Labor Theory of Value (LTV). LTV is often identified with Marx, who used it heavily, but LTV predates Marx. Under LTV, the value of a commodity $n$ is measured as the amount of labor that went into the manufacture of a unit of $n$ (this is for a production economy). A version of LTV is offered in this exchange from a scene early in the film *Treasure of Sierra Madre*, written and directed by John Huston (Walter Huston, John Huston’s father, plays the prospector).

[Prospector]: Say, answer me this one, will you? Why is gold worth some twenty bucks an ounce?

[Flophouse denison]: I don’t know. Because it’s scarce.

[Prospector]: A thousand men, say, go searchin’ for gold. After six months, one of them’s lucky: one out of a thousand. His find represents not only his own labor, but that of nine hundred and ninety-nine others to boot. That’s six thousand months, five hundred years, scramblin’ over a mountain, goin’ hungry and thirsty. An ounce of gold, mister, is worth what it is because of the human labor that went into the findin’ and the gettin’ of it.

[Flophouse denison]: I never thought of it just like that.

[Prospector] Well, there’s no other explanation, mister. Gold itself ain’t good for nothing except making jewelry with and gold teeth.

It can be shown that LTV produces a value vector $v$ that can be interpreted as a marginal theory of value if the production technology is constant returns to scale and if the only raw material is labor – no coal, no iron ore, wood, etc. – and there is only one type of labor – no skilled versus unskilled labor, for example. More generally, however, a marginal theory of value vector in a production economy will reflect preferences as well as production technologies.

References
