ECONOMICS 508

Homework 2

Exercise 1: Assume that $f: \mathbb{R}^N \to \mathbb{R}$ and $g: \mathbb{R}^N \to \mathbb{R}$ are concave functions. Let $\alpha_1, \alpha_2 > 0$ be positive real numbers. Show that the function $h(x) = \alpha_1 f(x) + \alpha_2 g(x)$ is also concave.

Exercise 2: Assume $f: (0,1) \to \mathbb{R}$ is convex. Show that f must be a continuous function.

Exercise 3: Let $f: \mathbb{R}^N \to \mathbb{R}$ and $g: \mathbb{R}^N \to \mathbb{R}$ be convex functions. Show that the function $s(x) = \max\{f(x), g(x)\}$.

Exercise 4: (Exercise 63, page 73, Sundaram) Find the Hessians $D^2 f$ of each of the following functions. Evaluate the Hessians at the specified points and determine whether the resulting matrix is positive definite, negative definite, PSD, NSD or indefinite.

a) $f(x_1, x_2) = x_1^2 + \sqrt{x_2}$ at $x^* = (1,1)$
b) $f(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}$ at $x^* = (2,2,2)$
c) $f(x_1, x_2, x_3) = \sqrt{x_1 x_2 x_3}$ at $x^* = (2,2,2)$

Exercise 5: (Exercise 46, page 71, Sundaram) Assume $f: \mathbb{R}^N \to \mathbb{R}$ is a continuous function. Show that the set $L = \{x \in \mathbb{R}^N : f(x) = 0\}$ is closed.

Exercise 6: (Exercise 45, page 71, Sundaram) Assume the function $f: \mathbb{R}^N \to \mathbb{R}$ is continuous at the point $p \in \mathbb{R}^N$ and $f(p) > 0$. Show that there exists an open ball $B \subseteq \mathbb{R}^N$ such that $p \in B$, and for all $x \in B$, we have $f(x) > 0$.

Exercise 7: Assume that $f: \mathbb{R}^N \to \mathbb{R}$ and $g: \mathbb{R}^N \to \mathbb{R}$ are continuous functions. Show that $h = f + g$ and $s = f \cdot g$ are also continuous.

Exercise 8: (Exercise 57, page 73, Sundaram) Assume $f: \mathbb{R}^2 \to \mathbb{R}$ satisfies $f(0,0) = 0$ and at all other points (x, y)

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

Show that the cross-partials $\frac{\partial^2 f}{\partial x \partial y}(x, y)$ and $\frac{\partial^2 f}{\partial y \partial x}(x, y)$ exist at all $(x, y) \in \mathbb{R}^2$, but that these partials fail to be continuous at $(0,0)$. Also, show that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$$

Homework is due on Tuesday, August 12.