Econ 508-A

Concavity, Quasiconcavity and Derivatives

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August 8, 2010

CONCAVITY AND Df

THEOREM: let $C \subseteq \mathbb{R}^N$ be convex. Let $f : C \to \mathbb{R}$ be at least C^1. Then f is concave iff for every $\boldsymbol{x}'$, $\boldsymbol{x}'' \in C$, it is the case that

$$f(\boldsymbol{x}'') \leq f(\boldsymbol{x}') + Df(\boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{x}')$$

[picture]

REMARK: f is strictly concave iff

$$f(\boldsymbol{x}'') < f(\boldsymbol{x}') + Df(\boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{x}') \text{ for } \boldsymbol{x}' \neq \boldsymbol{x}'' \in C.$$

PROOF

(i) f concave $\Rightarrow f(\boldsymbol{x}'') \leq f(\boldsymbol{x}') + Df(\boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{x}')$

Suppose f is concave.

Pick any $\boldsymbol{x}'$, $\boldsymbol{x}'' \in C$, $\alpha \in (0,1)$. Define $\boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha \boldsymbol{x}'' + (1-\alpha)\boldsymbol{x}'$

By concavity of f, $f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq \alpha f(\boldsymbol{x}'') + (1-\alpha) f(\boldsymbol{x}')$

Rearranging terms, $f(\boldsymbol{x}' + \alpha(\boldsymbol{x}'' - \boldsymbol{x}')) \geq \alpha\left(f(\boldsymbol{x}'') - f(\boldsymbol{x}')\right) + f(\boldsymbol{x}')$

Hence, $\dfrac{f(\boldsymbol{x}' + \alpha(\boldsymbol{x}'' - \boldsymbol{x}')) - f(\boldsymbol{x}')}{\alpha} \geq f(\boldsymbol{x}'') - f(\boldsymbol{x}')$

PROOF

(i) f concave $\Rightarrow f(\boldsymbol{x}'') \leq f(\boldsymbol{x}') + Df(\boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{x}')$ (continued)

Taking limits as $\alpha \to 0$, the above expression is the one-sided directional derivative of f at $\boldsymbol{x}''$ in the direction $(\boldsymbol{x}' - \boldsymbol{x}'')$

Since f is C^1,

$$\lim_{\alpha \to 0} \frac{f(\boldsymbol{x}' + \alpha(\boldsymbol{x}'' - \boldsymbol{x}')) - f(\boldsymbol{x}')}{\alpha} = D_{(\boldsymbol{x}'' - \boldsymbol{x}')} f(\boldsymbol{x}') = Df(\boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{x}')$$

So $Df(\boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{x}') \geq f(\boldsymbol{x}'') - f(\boldsymbol{x}')$,

and $f(\boldsymbol{x}'') \leq f(\boldsymbol{x}') + Df(\boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{x}')$

PROOF

(ii) f concave $\Leftarrow$ $f(\boldsymbol{x}'') \leq f(\boldsymbol{x}') + Df(\boldsymbol{x}')(\boldsymbol{x}'' - \boldsymbol{x}')$

Reverse the inequality for convexity and strict convexity.

CONCAVITY AND D^2f

THEOREM: Let $C \subseteq \mathbb{R}^N$ be convex and $f : C \to \mathbb{R}$ be C^2. Then

1. $D^2f(\boldsymbol{x})$ $\forall \boldsymbol{x} \in C$ is **NSD** iff f is **concave**.
2. $D^2f(\boldsymbol{x})$ $\forall \boldsymbol{x} \in C$ is **PSD** iff f is **convex**.
3. $D^2f(\boldsymbol{x})$ $\forall \boldsymbol{x} \in C$ is **ND** then f is **strictly concave**.
4. $D^2f(\boldsymbol{x})$ $\forall \boldsymbol{x} \in C$ is **PD** then f is **strictly convex**.

REMARK: in (3) and (4), negative and positive definiteness are only sufficient.

Example

$g(x) = -x^4$ [picture]

The hypograph of g is convex, so we know that g is concave.

However, $D^2g(x) = -12x^2$

For $x = 0$, $D^2g(x) = 0$

which makes $D^2g(x)$ a NSD matrix

(because $v'[D^2g(0)]v = 0$. for $v \neq 0$)

LOCAL CONCAVITY

THEOREM: Let $C \subseteq \mathbb{R}^N$ be convex, $f : C \to \mathbb{R}$ be C^2, and let $\boldsymbol{x}^* \in C$.

1. If $D^2f(\boldsymbol{x}^*)$ is ND then there is an $\varepsilon > 0$ such that f is strictly concave on $N_\varepsilon(\boldsymbol{x}^*)$.

2. If $D^2f(\boldsymbol{x}^*)$ is PD then there is an $\varepsilon > 0$ such that f is strictly convex on $N_\varepsilon(\boldsymbol{x}^*)$.

Note that if $N_\varepsilon(\boldsymbol{x}^*)$ contains points outside of C, the statement that f is concave on $N_\varepsilon(\boldsymbol{x}^*)$ means that it's concave on $N_\varepsilon(\boldsymbol{x}^*) \cap C$.

DIFFERENTIABLE STRICT CONCAVITY

DEFINITION: let, $f : \mathbb{R}^N \to \mathbb{R}$ be C^2. f is **differentiably strictly concave** iff $D^2 f(\boldsymbol{x})$ is ND $\forall\, \boldsymbol{x}$.

In some cases checking for negative definiteness of $D^2 f(\boldsymbol{x})$ can be easier, like in the next example.

EXAMPLE

Suppose f is additively separable: $f(\boldsymbol{x}) = f_1(x_1) + \ldots + f_N(x_N)$

where $f : \mathbb{R}^N \rightarrow \mathbb{R}$ and $f_n : \mathbb{R} \rightarrow \mathbb{R}$

Then $$D^2 f(\boldsymbol{x}) = \begin{bmatrix} D^2 f_1(\boldsymbol{x}) & & O \\ & \ddots & \\ O & & D^2 f_N(\boldsymbol{x}) \end{bmatrix}$$

EXAMPLE (CONTINUED)

$$f : \mathbb{R}_{++} \to \mathbb{R}\, f(\boldsymbol{x}) = a_1 \ln x_1 + a_2 \ln x_2 + \ldots + a_N ln(x_N)$$

$$D^2 f(x) = \begin{bmatrix} \frac{-a_1}{x_1^2} & & O \\ & \ddots & \\ O & & \frac{-a_N}{x_N^2} \end{bmatrix}$$

USEFUL THEOREMS

THEOREM: if f is separable, f is differentiably strictly concave if the f_n are differentiably strictly concave.

PROOF: Trivial.

THEOREM: Let $C \subseteq \mathbb{R}^N$ be convex, $f : C \to \mathbb{R}$ be concave. Let D be any interval containing $f(C)$ and let $g : D \to \mathbb{R}$ be concave and weakly increasing. Then $h : C \to \mathbb{R}$ defined by $h(\boldsymbol{x}) = g(f(\boldsymbol{x}))$ is concave. Moreover, if f is strictly concave and g is strictly increasing, then h is strictly concave. Analogous claims hold for f convex (with g increasing).

Proof

Consider any $\boldsymbol{x}', \boldsymbol{x}'' \in C$ and any $\alpha \in [0,1]$.

Let $\boldsymbol{x}_{\boldsymbol{\alpha}} = \alpha \boldsymbol{x}' + (1-\alpha)\boldsymbol{x}''$.

Since f is concave, $f(\boldsymbol{x}_{\boldsymbol{\alpha}}) \geq \alpha f(\boldsymbol{x}') + (1-\alpha) f(\boldsymbol{x}'')$

Since g is weakly increasing and concave, this implies that

$$
\begin{aligned}
h(\boldsymbol{x}_{\boldsymbol{\alpha}}) &= g(f(\boldsymbol{x}_{\boldsymbol{\alpha}})) \\
&\geq g(\alpha f(\boldsymbol{x}') + (1-\alpha) f(\boldsymbol{x}'')) \\
&\geq \alpha g(f(\boldsymbol{x}')) + (1-\alpha) g(f(\boldsymbol{x}'')) \\
&= \alpha h(\boldsymbol{x}') + (1-\alpha) h(\boldsymbol{x}'')
\end{aligned}
$$

Replace weak inequality with strict inequality for strictly concave, and reverse the inequality for convex. ■

EXAMPLE

Let the domain be $\mathbb{R}_{++}$.

Consider $h(x) = e^{1/x}$

Let $f(x) = \frac{1}{x}$ and $g(y) = e^y$

Then $h(x) = g(f(x))$

f is strictly convex and g is (strictly) convex and strictly increasing.

Therefore, by the previous theorem, h is strictly convex.

MORE USEFUL THEOREMS

THEOREM: Let $C \subseteq \mathbb{R}^N$ be an interval and let , $f : C \rightarrow \mathbb{R}$ be concave and strictly positive for all $\boldsymbol{x} \in C$. Then $h : C \rightarrow \mathbb{R}$ defined by $h(\boldsymbol{x}) = 1/f(\boldsymbol{x})$ is convex.

PROOF: Since f is concave, then $-f$ is convex. Since f is strictly positive, $-f$ is strictly negative. The function $g(y) = -1/y$ is convex and increasing on $\mathbb{R}_-$, so $h(x) = g(-f(x))$ is convex. ■

THEOREM: Let $C \subseteq \mathbb{R}^N$ be convex and let , $f : C \rightarrow \mathbb{R}$ be invertible. Then f is (strictly) concave iff f^{-1} is (strictly) convex.

PROOF: Omitted. Can be shown using the fact that hyp$f$ and epif^{-1} are identical.

QUASICONCAVITY AND Df

THEOREM: Let $f : C \to \mathbb{R}$ be C^1. Then f is quasiconcave iff $\forall\, \boldsymbol{x}', \boldsymbol{x}'' \in C$,

$$f(\boldsymbol{x}'') \geq f(\boldsymbol{x}') \;\Rightarrow\; Df(\boldsymbol{x}')(\boldsymbol{x}' - \boldsymbol{x}'') \geq 0$$

Remember that $Df(\boldsymbol{x}'')(\boldsymbol{x}' - \boldsymbol{x}'') = \nabla f(\boldsymbol{x}'') \cdot (\boldsymbol{x}' - \boldsymbol{x}'')$

[picture: inner product of two vectors]

INTUITION

The characterization for quasiconcavity in terms of D^2f is similar to that of concavity.

However, quasiconcavity only requires that $\boldsymbol{v}' D^2 f(\boldsymbol{x}^*)\boldsymbol{v}$ be negative for all $\boldsymbol{v}$ such that $\nabla f(\boldsymbol{x}^*) \cdot \boldsymbol{v} = 0$.

What this means geometrically is that $D^2 f(\boldsymbol{x}^*)$ needs to be ND on the tangent to the level set of f through $\boldsymbol{x}^*$. That is, we need to check the curvature of the function along the tangent to the level set.

[picture]

Notice that when $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$, the tangent is all of $\mathbb{R}^N$, so the condition is equivalent to checking for concavity.

QUASICONCAVITY AND D^2f

Let $T_{\nabla f(\boldsymbol{x})} = \{\boldsymbol{v} \in \mathbb{R}^N : \nabla f(\boldsymbol{x}) \cdot \boldsymbol{v} = 0\}$

THEOREM: Let $C \subseteq \mathbb{R}^N$ be convex and $f : C \to \mathbb{R}$.

1. Let f be C^2 Fix $\boldsymbol{x}^* \in C$,. If $D^2 f(\boldsymbol{x}^*)$ is ND on $T_{\nabla f(\boldsymbol{x}^*)}$, then f is locally strictly quasiconcave.
2. For every $\boldsymbol{x} \in C$, if $D^2 f(\boldsymbol{x})$ is NSD on $T_{\nabla f(\boldsymbol{x})}$, then f is quasiconcave.
3. If f is quasiconcave and C^2, then $\forall\, \boldsymbol{x} \in C$, $D^2 f(\boldsymbol{x})$ is NSD on $T_{\nabla f(\boldsymbol{x})}$.

PROOF: omitted. ■

Differentiable Strict Quasiconcavity (DSQC)

Definition: If $D^2f(\boldsymbol{x}^*)$ is ND on $T_{\nabla f(\boldsymbol{x}^*)}$ for all $\boldsymbol{x} \in C$, then f is **differentiably strictly quasiconcave (DSQC)**.

If $\nabla f(\boldsymbol{x}^*) \neq \mathbf{0}$, we can check for differential strict quasiconcavity by checking the border preserving principal minors of the bordered Hessian:

$$\begin{bmatrix} D^2f(\boldsymbol{x}^*) & \nabla f(\boldsymbol{x}^*) \\ Df(\boldsymbol{x}^*) & O \end{bmatrix}$$

DSQC CONTINUED

f is DSQC iff the border preserving principal minors of the bordered Hessian alternate in signs.

If the number of columns in $D^2f(\boldsymbol{x}^*)$ is odd, then the determinant of the smallest submatrix should be negative.

If the number of columns in $D^2f(\boldsymbol{x}^*)$ is even, then the determinant of the smallest submatrix should be positive.

Example

Let $f : \mathbb{R}^2_{++} \to \mathbb{R}$ be given by $f(x, y) = x^{3/4}y^{2/3}$

Then $Df(x) = \left[\frac{3}{4}x^{-1/4}y^{2/3} \quad \frac{2}{3}x^{3/4}y^{-1/3}\right]$

And $D^2f(x) = \begin{bmatrix} -\frac{3}{16}x^{-5/4}y^{2/3} & \frac{1}{2}x^{-5/4}y^{-1/3} \\ \frac{1}{2}x^{-5/4}y^{-1/3} & -\frac{2}{3}x^{3/4}y^{-4/3} \end{bmatrix}$

f is NOT concave.

$$BH = \begin{bmatrix} -\frac{3}{16}x^{-5/4}y^{2/3} & \frac{1}{2}x^{-5/4}y^{-1/3} & \frac{3}{4}x^{-1/4}y^{2/3} \\ \frac{1}{2}x^{-5/4}y^{-1/3} & -\frac{2}{3}x^{3/4}y^{-4/3} & \frac{2}{3}x^{3/4}y^{-1/3} \\ \frac{3}{4}x^{-1/4}y^{2/3} & \frac{2}{3}x^{3/4}y^{-1/3} & 0 \end{bmatrix}$$

Only need to check whether $|BH| > 0$

$$|BH| = \frac{1}{2}x^{5/4}y \begin{vmatrix} -\frac{3}{8}x^{-1} & y^{-1}] & \frac{3}{2} \\ \frac{1}{2}x^{-1} & -\frac{2}{9}y^{-1} & \frac{2}{3} \\ \frac{3}{4}x^{-1} & \frac{2}{3}y^{-1} & 0 \end{vmatrix}$$

$$|BH| = \frac{1}{2}x^{5/4}y\left[(1)(\tfrac{3}{2})[\tfrac{1}{3}x^{-1}y^{-1}] - (-1)(\tfrac{2}{3}[-x^{-1}y^{-1}]\right]$$

$$|BH| = \frac{17}{24}x^{-1/4} > 0$$

So f is quasiconcave on $\mathbb{R}^2_{++}$

USEFUL THEOREM

THEOREM: if f is DSQC, and $\nabla f(\boldsymbol{x}^*) \neq \mathbf{0}$ then

$$\begin{vmatrix} D^2 f(\boldsymbol{x}^*) & \nabla f(\boldsymbol{x}^*) \\ Df(\boldsymbol{x}^*) & O \end{vmatrix} \neq 0$$

PROOF: omitted. ■