Econ 508-A

Convex Sets and Concave Functions

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CONVEX SETS IN $\mathbb{R}^N$

DEFINITION: **A set** $X \subseteq \mathbb{R}^N$ is **convex** iff given any two points $\boldsymbol{x}', \boldsymbol{x}'' \in X$, the point

$$\boldsymbol{x}_{\boldsymbol{\alpha}} = \alpha \boldsymbol{x}' + (1 - \alpha)\boldsymbol{x}'' \in X \text{ for every } \alpha \in [0, 1].$$

$\boldsymbol{x}_{\boldsymbol{\alpha}}$ is called a **convex combination** of $\boldsymbol{x}'$ and $\boldsymbol{x}''$.

Convex sets in $\mathbb{R}^N$ (2)

Another ways of writing down the convex combination of two points is:

$$\boldsymbol{x_\alpha} = \boldsymbol{x''} + \alpha(\boldsymbol{x'} - \boldsymbol{x''})$$

The set of all convex combinations between any two points is the line segment that joins them.

Graphically, a set is convex if given any two points, the line that joins them is in the set.

[picture]

CONVEX SETS IN $\mathbb{R}^N$ (3)

A set is **strictly convex** if the line joining any two points lies in the interior of the set:

For any $\boldsymbol{x}', \boldsymbol{x}'' \in X$, and $\alpha \in [0, 1]$,

$$\boldsymbol{x}_{\boldsymbol{\alpha}} = \alpha \boldsymbol{x}' + (1 - \alpha)\boldsymbol{x}'' \in \text{int}(X)$$

[picture]

PROPERTIES OF CONVEX SETS

THEOREM: Let $X, Y \subseteq \mathbb{R}^N$ be convex sets and $r \in \mathbb{R}$. Then the sets:

1. $rX = \{\boldsymbol{z} \in \mathbb{R}^N : \exists\, \boldsymbol{x} \in X \text{ s.t. } \boldsymbol{z} = r\boldsymbol{x}\}$
2. $X + Y = \{\boldsymbol{z} \in \mathbb{R}^N : \exists\, \boldsymbol{x} \in X \text{ and } \boldsymbol{y} \in Y \text{ s.t. } \boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}\}$
3. $X \cap Y$

Are convex sets.

PROOF: HW

CONVEX COMBINATION

DEFINITION: $\boldsymbol{x} \in \mathbb{R}^N$ is a **convex combination** of the vectors $\boldsymbol{x^1}, \ldots, \boldsymbol{x^m} \in \mathbb{R}^N$ if it can be written as

$$\boldsymbol{x} = \sum_{i=1}^m \alpha^i \boldsymbol{x^i} = \begin{bmatrix} \sum_{i=1}^m \alpha^i x_1^i \\ \vdots \\ \sum_{i=1}^m \alpha^i x_N^i \end{bmatrix}$$

with $\alpha^i \in [0, 1]$ for all i and $\sum_{i=1}^m \alpha^i = 1$

CONVEX COMBINATION (2)

THEOREM: A set X is convex iff every convex combination of points in X lies in X.

PROOF: Rockafellar.

CONVEX HULL

DEFINITION: Given a set $X \subseteq \mathbb{R}^N$, the **convex hull** of X, denoted $\text{Conv}X$ is the smallest convex set that contains X.

[picture]

Alternative characterizations of $\text{Conv}X$

1. The set containing all possible convex combinations of points in X:
   $\text{Conv}X = \{\boldsymbol{y} : \boldsymbol{y} = \sum_{i=1}^{m} \alpha^i \boldsymbol{x^i}$ for some m, with $\boldsymbol{x^i} \in X$, $\alpha^i \in [0,1]$ and $\sum_{i=1}^{m} \alpha^i = 1\}$.

2. The intersection of all the convex sets that contain X.

Convex cone

Definition: $K \subseteq \mathbb{R}^N$ is a **convex cone** if for any $\boldsymbol{x}, \boldsymbol{x}' \in K$ and $\alpha, \beta > 0$, $\alpha\boldsymbol{x} + \beta\boldsymbol{x}' \in K$.

Example: the nonnegative orthant of $\mathbb{R}^N$:

$$\mathbb{R}^N_+ = \{\boldsymbol{x} = (x_1, x_2, ..., x_N) \mid x_i \geq 0 \text{ for } i = 1, ...N\}$$

[picture]

Separation Theorems

Definition: Given $\boldsymbol{p} \in \mathbb{R}^N$, $\boldsymbol{p} \neq \mathbf{0}$, $c \in \mathbb{R}$, the **hyperplane** generated by $\boldsymbol{p}$ and c is the $(N-1)$ dimensional plane

$$H_{\boldsymbol{p},c} = \{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{p} \cdot \boldsymbol{x} = c\}$$

The sets $\{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{p} \cdot \boldsymbol{x} \geq c\}$ and $\{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{p} \cdot \boldsymbol{x} \leq c\}$ are called the **half-space above** and **the half-space below** the hyperplane $H_{\boldsymbol{p},c}$.

[picture]

Hyperplanes and half spaces are convex sets.

Separation Theorems (2)

Definition: Consider two nonempty sets $X, Y \subseteq \mathbb{R}^N$.

- X and Y **are separated by a hyperplane** $H_{\boldsymbol{p},c}$ iff $\exists\, \boldsymbol{p} \neq \mathbf{0} \in \mathbb{R}^N$ and $c \in \mathbb{R}$ such that

$$\boldsymbol{p} \cdot \boldsymbol{x} \geq c \geq \boldsymbol{p} \cdot \boldsymbol{y} \;\; \forall\;\; \boldsymbol{x} \in X, \;\; \boldsymbol{y} \in Y.$$

- X and Y are **strictly separated by a hyperplane** $H_{\boldsymbol{p},c}$ iff there exists $\boldsymbol{p} \neq \mathbf{0} \in \mathbb{R}^N$ and $c \in \mathbb{R}$ such that

$$\boldsymbol{p} \cdot \boldsymbol{x} > c > \boldsymbol{p} \cdot \boldsymbol{y} \;\; \forall\;\; \boldsymbol{x} \in X, \;\; \boldsymbol{y} \in Y.$$

[picture]

Separation Theorems (3)

Theorem: **separating Hyperplane Theorem**: Let X, $Y \subseteq \mathbb{R}^N$ be nonempty.

1. If X and Y are convex, and $X \cap Y = \emptyset$, then X and Y are separated by a hyperplane.

2. If X and Y are convex and closed, if $X \cap Y = \emptyset$, and if at least one of them is compact, then X and Y are strictly separated by a hyperplane.

Proof: omitted.

SEPARATION THEOREMS (4)

DEFINITION: Let $X \subseteq \mathbb{R}^N$ be nonempty.

- X **is supported at** $\boldsymbol{x}^*$ iff $\exists\, \boldsymbol{p} \neq \mathbf{0} \in \mathbb{R}^N$ such that $\boldsymbol{p} \cdot \boldsymbol{x} \geq \boldsymbol{p} \cdot \boldsymbol{x}^*$ for all $\boldsymbol{x} \in X$.
- X **is strictly supported at** $\boldsymbol{x}^*$ iff there exists $\boldsymbol{p} \neq \mathbf{0} \in \mathbb{R}^N$ such that $\boldsymbol{p} \cdot \boldsymbol{x} > \boldsymbol{p} \cdot \boldsymbol{x}^*$ for all $\boldsymbol{x} \in X$, $\boldsymbol{x} \neq \boldsymbol{x}^*$.

[picture]

Separation Theorems (5)

Theorem: **Supporting Hyperplane Theorem**. Suppose that $X \subseteq \mathbb{R}^N$ is nonempty, closed and convex and that $\boldsymbol{x}^* \notin \text{Int}(X)$. Then X is supported by a hyperplane at $\boldsymbol{x}^*$.

[picture]

Proof: omitted.

DEFINITIONS

From now on, let $C \subseteq \mathbb{R}^N$ be a convex set, and $f : C \to \mathbb{R}$

The **hypograph** of f (hyp f). is the set of points lying on and below the graph of f:

$$\text{hyp} f = \{(\boldsymbol{x}, y) \in \mathbb{R}^{N+1} : \boldsymbol{x} \in \mathbb{R}^N \text{ and } y \leq f(\boldsymbol{x})\}$$

Similarly, the **epigraph** of f (epi f). is the set of points lying on and above the graph of f:

$$\text{epi} f = \{(\boldsymbol{x}, y) \in \mathbb{R}^{N+1} : \boldsymbol{x} \in \mathbb{R}^N \text{ and } y \geq f(\boldsymbol{x})\}$$

[pictures]

Concave functions

Definition: Let $\boldsymbol{x}', \boldsymbol{x}'' \in C$, and define $\boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha \boldsymbol{x}' + (1-\alpha)\boldsymbol{x}''$, where $\alpha \in [0,1]$. f **is concave** iff

$$f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq \alpha f(\boldsymbol{x}') + (1-\alpha) f(\boldsymbol{x}'')$$

[picture]

Theorem: f is concave iff the set hyp f is convex.

PROOF

(i) f concave $\Rightarrow$ hyp f convex

Pick two arbitrary points in hyp f : $(\boldsymbol{x}', y'), (\boldsymbol{x}'', y'')$, and let $(\boldsymbol{x}^{\boldsymbol{\alpha}}, y^{\alpha}) = \alpha(\boldsymbol{x}', y') + (1-\alpha)(\boldsymbol{x}'', y'')$.

Want to show that $(\boldsymbol{x}^{\boldsymbol{\alpha}}, y^{\alpha}) \in$ hyp f.

First, we know that $y' \leq f(\boldsymbol{x}')$ and $y'' \leq f(\boldsymbol{x}'')$.

So $\alpha y' \leq \alpha f(\boldsymbol{x}')$ and $(1-\alpha)y'' \leq (1-\alpha)f(\boldsymbol{x}'')$.

Therefore, $y^{\alpha} = \alpha y' + (1-\alpha)y'' \leq \alpha f(\boldsymbol{x}') + (1-\alpha)f(\boldsymbol{x}'')$

By concavity of f, $\alpha f(\boldsymbol{x}') + (1-\alpha)f(\boldsymbol{x}'') \leq f(\boldsymbol{x}^{\boldsymbol{\alpha}})$

Then $y^{\alpha} \leq f(\boldsymbol{x}^{\boldsymbol{\alpha}})$ and, hence, $(\boldsymbol{x}^{\boldsymbol{\alpha}}, y^{\alpha}) \in$ hyp f

PROOF (2)

(ii) f concave $\Leftarrow$ hyp f convex

Given any two points $\boldsymbol{x}', \boldsymbol{x}''$ in the domain of f, $(\boldsymbol{x}', f(\boldsymbol{x}'))$ and $(\boldsymbol{x}'', f(\boldsymbol{x}''))$ are in hyp f.

Let $\boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha \boldsymbol{x}' + (1-\alpha)\boldsymbol{x}''$ and $f^{\alpha} = \alpha f(\boldsymbol{x}') + (1-\alpha) f(\boldsymbol{x}'')$

Want to show that $f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq f^{\alpha}$

By convexity of hyp f , the point $(\boldsymbol{x}^{\boldsymbol{\alpha}}, f^{\alpha})$ is also in hyp f

Which means that $f^{\alpha} \leq f(\boldsymbol{x}^{\boldsymbol{\alpha}})$

So $\alpha f(\boldsymbol{x}') + (1-\alpha) f(\boldsymbol{x}'') \leq f(\boldsymbol{x}^{\boldsymbol{\alpha}})$, and, hence, f is concave. ■

CONVEX FUNCTIONS

DEFINITION: f is **convex** iff $-f$ is concave.

THEOREM: the following are equivalent:

1. f is convex.
2. epi f is a convex set.
3. For any $\boldsymbol{x}', \boldsymbol{x}'' \in C$, $\boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha\boldsymbol{x}' + (1-\alpha)\boldsymbol{x}''$,

$$f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \leq \alpha f(\boldsymbol{x}') + (1-\alpha) f(\boldsymbol{x}'').$$

PROOF: HW.

Strict concavity and convexity

Definition:

1. f is **strictly concave** iff for any $\boldsymbol{x}', \boldsymbol{x}'' \in C$, $\boldsymbol{x}' \neq \boldsymbol{x}''$, and any $\alpha \in (0,1)$

$$f(\boldsymbol{x}^{\boldsymbol{\alpha}}) > \alpha f(\boldsymbol{x}') + (1-\alpha) f(\boldsymbol{x}'')$$

2. f is **strictly convex** iff $-f$ is strictly concave.

Some useful theorems

THEOREM: Let $f_1, ...f_k$ be concave functions, each defined on C. Let $a_1, ...a_k \in \mathbb{R}_+$. Then $a_1 f_1 + ... + a_k f_k$ is a concave function on C.

PROOF: HW.

THEOREM: Let $f : C \to \mathbb{R}$ where C is open and convex. If f is concave on C, it is continuous on C. More generally, f is continuous on the interior of C.

Other useful properties

a. If f is concave and F is concave and increasing, then $U(\boldsymbol{x}) = F(f(\boldsymbol{x}))$ is concave.

b. If f is convex and F is convex and increasing, then $U(\boldsymbol{x}) = F(f(\boldsymbol{x}))$ is convex.

c. Let f and g be concave functions. Then $h(\boldsymbol{x}) = \min\{f(\boldsymbol{x}), g(\boldsymbol{x})\}$ is concave.

d. Let f and g be convex functions. Then $h(\boldsymbol{x}) = \max\{f(\boldsymbol{x}), g(\boldsymbol{x})\}$ is convex.

QUASICONCAVE FUNCTIONS

DEFINITION:

1. f is **quasiconcave** iff
   $\forall\ \boldsymbol{x}', \boldsymbol{x}'' \in C,\ \boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha \boldsymbol{x}' + (1-\alpha)\boldsymbol{x}'',\ \alpha \in [0,1].$

$$f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq \min\{f(\boldsymbol{x}'), f(\boldsymbol{x}'')\}.$$

2. f is **strictly quasiconcave** iff
   $\forall\ \boldsymbol{x}', \boldsymbol{x}'' \in C,\ \boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha \boldsymbol{x}' + (1-\alpha)\boldsymbol{x}'',\ \alpha \in [0,1]$

$$f(\boldsymbol{x}^{\boldsymbol{\alpha}}) > \min\{f(\boldsymbol{x}'), f(\boldsymbol{x}'')\}$$

The function f is quasiconvex iff $-f$ is quasiconcave

strictly quasiconvex iff $-f$ is strictly quasiconcave.

QUASICONCAVITY AND UPPER CONTOUR SETS

Given a point $a \in \mathbb{R}$, the upper contour set of f is the set

$U_a = \{\boldsymbol{x} \in C : f(\boldsymbol{x}) \geq a\}$

THEOREM: f is quasiconcave iff the upper contour sets are convex.

PROOF

(i) f is quasiconcave $\Rightarrow$ U_y convex.

If $U_a = \varnothing$, it is convex trivially.

If $U_a \neq \varnothing$, take any two points $\boldsymbol{x}', \boldsymbol{x}'' \in U_a$.

By definition of U_a, both $f(\boldsymbol{x}') \geq a$ and $f(\boldsymbol{x}'') \geq a$.

By quasiconcavity of f, $f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq \min\{f(\boldsymbol{x}'), f(\boldsymbol{x}'')\} \geq a$

So $\boldsymbol{x}^{\boldsymbol{\alpha}} \in U_a$, and, hence, U_a is convex.

(ii) f is quasiconcave $\Leftarrow$ U_a convex: HW.

[picture: examples $\mathbb{R}$, $\mathbb{R}^2$]

Similarly, f is quasiconvex iff the lower contour sets, $L_a = \{\boldsymbol{x} \in C : f(\boldsymbol{x}) \leq a\}$, are convex.

Useful Theorems

Theorem: let f be quasiconcave, and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a weakly increasing function defined on an interval I that contains $f(C)$. Then the composite function $g \circ f(\boldsymbol{x})$ is quasiconcave in C.

Proof:

Pick any two points $x', x'' \in C$. WLOG, assume that $f(x'') \geq f(x')$

By quasiconcavity of f, $f(x^{\alpha}) \geq f(x')$.

Since g is nondecreasing, $g(f(x'')) \geq g(f(x'))$ and $g(f(x^{\alpha})) \geq g(f(x'))$

Hence, $g \circ f(x^{\alpha}) \geq \min\{g(f(x'')), g(f(x'))\}$. ■

Remark: nondecreasing transformations preserve quasiconcavity, NOT concavity.

Useful Theorems (2)

Theorem: if f is concave, then f is quasiconcave.

Proof:

Let f be concave, with $f(\boldsymbol{x}'') \geq f(\boldsymbol{x}')$.

By definition of concavity, $f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq \alpha f(\boldsymbol{x}'') + (1-\alpha)f(\boldsymbol{x}')$

Since $f(\boldsymbol{x}'') \geq f(\boldsymbol{x}')$,
$\alpha f(\boldsymbol{x}'') + (1-\alpha)f(\boldsymbol{x}') \geq \alpha f(\boldsymbol{x}') + (1-\alpha)f(\boldsymbol{x}') = f(\boldsymbol{x}')$.

Then $f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq f(\boldsymbol{x}') = \min\{f(\boldsymbol{x}''), f(\boldsymbol{x}')\}$.

So f is quasiconcave. ■

The converse is not true.

Example 1

1. $f(x) = e^x$:

   Quasiconcave?

   For any $y > 0$: $U_y = [\ln y, \infty)$, which is a convex set.
   For $y \leq 0$: $U_y = \mathbb{R}$, which is also convex.
   So $f(x) = e^x$ is quasiconcave

   Concave?

   No, convex. [picture]

EXAMPLE 2

2. $f(x) = e^{-x^2}$ (similar to the standard normal density function)

Quasiconcave?

For any $y \in (0,1]$, $U_y = \left[-\sqrt{\ln(1/y)}, \sqrt{\ln(1/y)}\right]$, which is a convex set.
For any $y > 1$, $U_y = \varnothing$, which is also convex
For any $y \leq 0$, $U_y = \mathbb{R}$, which is also convex

Concave?

No. [picture]

EXAMPLE 3

3. Consider the following Cobb Douglas utility function:

$$u : \mathbb{R}_{++} \to \mathbb{R},\ \ u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$

It's both concave and quasiconcave

But $\hat{u} = [u(x_1, x_2)]^3 = x_1, x_2$ is not concave anymore.

EXERCISE: (HW)

Show that the Cobb Douglas function $f : C \to \mathbb{R},\ \ C \subseteq \mathbb{R}^N$

$$f(\boldsymbol{x}) = A \prod_{i=1}^{N} x_i^{\alpha_i} \qquad \text{where } \alpha_i > 0 \ \forall i$$

is quasiconcave for all $\boldsymbol{x} > \boldsymbol{0}$.