Econ 508-A

Matrix Algebra and Linear transformations

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NOTATION

$$A_{M\times N} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix}$$

$$=[\mathrm{a}_{ij}]$$

$$=[\mathrm{a}^1 \dots a^N], \text{ where } a^i \in \mathbb{R}^M,\ i = 1, ..., N,$$

BASIC OPERATIONS

1. Scalar multiplication:

$$\alpha A = [\alpha a_{ij}]$$

2. Addition:

$$A + B = [a_{ij} + b_{ij}]$$

3. Multiplication:

$$A_{M \times N} B_{N \times Q} = D_{M \times Q} = [d_{ij}]$$

$$\text{where } d_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$$

Law of matrix algebra

Theorem:

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$

TRANSPOSE

Definition: Transpose of $A_{M\times N} = [a_{ij}]$:

$$A'_{N\times M} = [a'_{ij}] = [a_{ji}]$$

Theorem:

1. $(A+B)' = A' + B'$
2. $(AB)' = B'A'$
3. $(\alpha A)' = \alpha A',\ \alpha \in \mathbb{R}$

SPECIAL CLASSES OF MATRICES

1. Square matrix:

   $A_{N\times N} = [a_{ij}]$;
   a_{ii} : diagonal elements

2. Diagonal matrix:

   $A_{N\times N} = [a_{ij}]$; $a_{ij} = 0$ for $i \neq j$

3. Identity matrix:

   $I_N = [a_{ij}]$; $a_{ii} = 1, a_{ij} = 0$ for $i \neq j$;
   $I_M A_{M\times N} = A_{M\times N} = A_{M\times N} I_N$

4. Symmetric matrix:

   $A_{N\times N} = [a_{ij}]$; $a_{ij} = a_{ji}$ for every i, j. $A' = A$

TRACE

Definition: given $A_{N\times N}$,

$$\mathrm{Tr}(A) = \sum_{i=1}^{N} a_{ii}$$

Theorem:

Given $A_{N\times N}, B_{N\times N}$,

1. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
2. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

DETERMINANTS

Given $A_{N\times N}$, $|A|$ denotes the determinant of A

Definition: (i,j)th **minor** of A :

A_{ij} : $(N-1)\times(N-1)$ matrix obtained by deleting row i and col j

Definition: (i,j)th **cofactor** of A:

$$c_{ij}(A) = (-1)^{(i+j)} \left|A_{ij}\right|$$

Definition:

$$|A| = \sum_{k=1}^{N} a_{ij} c_{ij}(A)$$

(expansion along ith row $(k=j)$ or jth col $(k=i)$)

PROPERTIES OF DETERMINANTS

Theorem:

1. $|A| = \left|A^T\right|$
2. $|AB| = |A|\,|B|$
3. $|I_N| = 1$
4. $|kA| = k^N\,|A|$
5. $|A| = 0$ if A has a row or col of 0's
6. If we multiply a row (col) of A by k to get $\hat{A}$,
   $\left|\hat{A}\right| = k\,|A|$
7. If we add a multiple of a row (col) to another row (col) to get $\hat{A}$, $\left|\hat{A}\right| = |A|$

INVERSE

Definition: given $A_{N \times N}$, its inverse, denoted A^{-1} is the $N \times N$ matrix such that

$$A^{-1}A = AA^{-1} = I_N$$

Theorem:

1. A^{-1} exists $\Leftrightarrow |A| \neq 0$
2. If A^{-1} exists, it is unique

We say that A is **nonsingular** if A^{-1} exists.

Properties of Inverses

Theorem:

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $\left|A^{-1}\right| = \dfrac{1}{|A|}$
4. $(A')^{-1} = (A^{-1})'$

COMPUTATION OF THE INVERSE

Definition: the Adjoint of an $N \times N$ matrix A is given by

$$\text{Adj}A = [C(A)]^T,$$

$$\text{where } \; C(A) = [c_{ij}(A)]$$

The inverse of A, A^{-1} is given by:

$$A^{-1} = \frac{1}{|A|}\text{Adj}A$$

LINEAR DEPENDENCE AND INDEPENDENCE

Definition: nonzero vectors $\{x^1, \ldots, x^K\} \in R^N$ are **linearly dependent** if $\exists$ scalars $\alpha_1, \ldots, \alpha_K$ not all 0 such that:

$$0_N = \sum_{i=1}^{K} \alpha_i x^i$$

where 0_N is the N-dimensional zero vector.

Definition: $\{x^1, \ldots, x^K\} \in \mathbb{R}^N$ are **linearly independent** if

$$0_N = \sum_{i=1}^{K} \alpha_i x^i \;\Rightarrow\; \alpha_k = 0 \;\; \forall k = 1, \ldots K$$

RANK

Definition: the column (row) rank of $A_{M \times N}$ is the max number of linearly independent cols (rows).

$$\text{col rank}(A_{M \times N}) \leq N$$
$$\text{row rank}(A_{M \times N}) \leq M$$

Theorem: col rank$(A) =$ row rank$(A) =$ rank(A)

Lemma: rank$(A) \leq$ min $\{N, M\}$

Corollary: rank$(A') =$rank(A)

Definition: $A_{M \times N}$, $M \leq N$ is of **full rank** iff

$$\text{rank}(A) = M$$

RANK OF SQUARE MATRICES

Theorem: $|A| \neq 0 \Leftrightarrow \text{rank}(A_{N \times N}) = N \Leftrightarrow A^{-1}$ exists

SIMULTANEOUS LINEAR EQUATIONS

A linear equation:

$$a_1 x_1 + a_2 x_2 + \ldots + a_N x_N = b$$

A system of M simultaneous linear equations:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1N} x_N &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2N} x_N &= b_2 \\ &\vdots \\ a_{M1} x_1 + a_{M2} x_2 + \ldots + a_{MN} x_N &= b_M \end{aligned}$$

Can be represented in matrix terms:

$$A_{M \times N} x_{N \times 1} = b_{M \times 1}$$

INTERPRETATION OF AX

A linear combination of the columns of A:

$$
\begin{aligned}
A\boldsymbol{x} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N \\
&\quad a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N \\
&\quad\quad \vdots \\
&\quad a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_N \\
&= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{M1} \end{bmatrix} \mathrm{x}_1 + \ldots + \begin{bmatrix} a_{1N} \\ a_{2N} \\ \vdots \\ a_{MN} \end{bmatrix} x_N \\
&= \mathrm{a}^1 x_1 \; + \; \ldots \; + \; a^N x_N
\end{aligned}
$$

EXAMPLE

Consider the system of equations of the form $Ax = b$ given by

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

For which values of $b \in \mathbb{R}^3$ is the system solvable?

Re-write the system: $$\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} x_2 \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} x_3 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The system is solvable iff b can be written as a linear combination of the columns of A.

EXAMPLE (CONTINUED)

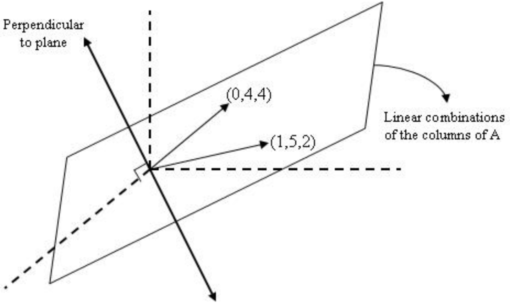
What if $b = 0$?

- The trivial solution is always $x = 0$.
- Can there be other solutions?

Notice that

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

GEOMETRIC INTUITION



$Ax = b$ can be solved iff b lies in the plane that is spanned by the column vectors of A.

The plane is called the **column space** of A.

The line perpendicualr to the plane is called the **null space** of A.

COLUMN SPACE AND NULL SPACE (HANDWAVY)

The **column space** of a matrix $A_{M \times N}$ is the set of vectors that can be written as a linear combination of the columns of A.

The dimension of the column space is the rank of A.

The **null space or kernel** of A is the set of vectors x such that $Ax = 0$.

The dimension of the null space of A is equal to $N-$ Rank(A).

SUMMARY

Given A, a $N \times N$ matrix:

Nonsingular	Singular
A is invertible	A is not invertible
Cols are independent	Cols are dependent
Rows are independent	Rows are dependent
$Ax = 0$ has one solution $x = 0$	$Ax = 0$ has infinitely many solutions
$Ax = b$ has one solution $x = A^{-1}b$	$Ax = b$ has no solution or infinitely many
A has rank N	A has rank $R < N$
Col space of A is $\mathbb{R}^N$	Col space of A has dimension $R < N$
Null space of A has dimension 0	Null space of A has dimension $N - R$

References:
Section 1.3 Sundaram
Chapters 8 and 9 S&B
Turkington Ch 1, 2
Strang Ch 2