Econ 508-A

Definite Matrices

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August 1, 2011

QUADRATIC FORMS

DEFINITION: A **quadratic form** on $\mathbb{R}^N$ is a real valued function of the form

$$Q(x_1, ..., x_N) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} x_i x_j$$

where each term is a monomial of degree two.

Matrix representation of a quadratic form

Let $\boldsymbol{x} = (x_1, ..., x_N)$.

Then $Q(\boldsymbol{x})$ can also be represented in matrix form:

$$Q(\boldsymbol{x}) = \boldsymbol{x}' A \boldsymbol{x}$$

where A is a symmetric $N \times N$ matrix.

EXAMPLE 1

The general quadratic form in $\mathbb{R}^2$:

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

can be written in matrix terms:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

EXAMPLE 2

The general quadratic form in $\mathbb{R}^3$:

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

can be written in matrix terms:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

And so forth

DEFINITE MATRICES

DEFINITION: Let A be a **symmetric** $N \times N$ matrix. Then A is:

1. **Positive Definite** (PD) iff $Q(\boldsymbol{x}) = \boldsymbol{x}' A\boldsymbol{x} > 0 \quad \forall x \in \mathbb{R}^N \setminus \{0\}$

2. **Positive Semidefinite** (PSD) iff $Q(\boldsymbol{x}) = \boldsymbol{x}' A\boldsymbol{x} \geq 0 \quad \forall x \in \mathbb{R}^N \setminus \{0\}$

3. **Negative Definite** (ND) iff $Q(\boldsymbol{x}) = \boldsymbol{x}' A\boldsymbol{x} < 0 \quad \forall x \in \mathbb{R}^N \setminus \{0\}$

4. **Negative Semidefinite** (NSD) iff $Q(\boldsymbol{x}) = \boldsymbol{x}' A\boldsymbol{x} \leq 0 \quad \forall x \in \mathbb{R}^N \setminus \{0\}$

DEFINITE MATRICES (CONTINUED)

5. **Indefinite** if $Q(\boldsymbol{x}) = \boldsymbol{x}'A\boldsymbol{x} > 0$ for some $\boldsymbol{x} \in \mathbb{R}^N$, and $Q(\boldsymbol{x}) = \boldsymbol{x}'A\boldsymbol{x} < 0$ for some other $\boldsymbol{x} \in \mathbb{R}^N$

REMARK: A matrix that is PD (ND) is automatically PSD (NSD). Otherwise, every symmetric matrix falls into one of the five mentioned categories.

EXAMPLE 1

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

   Pick any $x \in \mathbb{R}^2 \backslash \{0\}$

   $$Q(\boldsymbol{x}) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 > 0$$

   So A is PD

EXAMPLE 3

3 $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Pick any $x \in \mathbb{R}^2 \backslash \{0\}$

$Q(\boldsymbol{x}) = \boldsymbol{x}' C\boldsymbol{x} = 2x_1x_2$

$$\begin{aligned} \boldsymbol{x}^T C\boldsymbol{x} &= 2 > 0 \quad \text{for } \boldsymbol{x} = (1,1) \\ \boldsymbol{x}^T C\boldsymbol{x} &= -2 < 0 \quad \text{for } \boldsymbol{x} = (-1,1) \end{aligned}$$

So C is Indefinite

PICTURES

(i) $Q(\boldsymbol{x}) = x_1^2 + x_2^2$

(ii) $Q(\boldsymbol{x}) = -x_1^2 - x_2^2$

(iii) $Q(\boldsymbol{x}) = x_1^2 - x_2^2$

(iv) $Q(\boldsymbol{x}) = x_1^2 + 2x_1x_2 + x_2^2$

(v) $Q(\boldsymbol{x}) = -x_1^2 - 2x_1x_2 - x_2^2$

Identifying definiteness and semidefiniteness

1. Eigenvalues

2. Principal minors

EIGENVALUES

Let A be a $N \times N$ square matrix.

The following equation is called **characteristic equation:**

$$\det[A - \lambda I] = 0$$

The solutions to the characteristic equation are called **characteristic roots** or **eigenvalues**.

Example

$$A = \begin{bmatrix} 2 & -6 \\ -6 & -7 \end{bmatrix}$$

The characteristic equation is:

$$\det[A - \lambda I] = 0$$

$$\left| \begin{bmatrix} 2 & -6 \\ -6 & -7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 2 - \lambda & -6 \\ -6 & -7 - \lambda \end{bmatrix} \right| = 0$$

Example 1 (continued)

$$(2-\lambda)(-7-\lambda)-(-6)(-6)=0$$

$$\lambda^2+5\lambda-50=0$$

$$(\lambda-5)(\lambda+10)=0$$

$$\lambda_1 = 5$$

$$\lambda_2 = -10$$

IDENTIFYING DEFINITENESS USING EIGENVALUES

THEOREM: Let A be an $N \times N$ **symmetric** matrix. Then

1. A is **PD** iff all its eigenvalues are **positive**.
2. A is **PSD** iff all its eigenvalues are **nonnegative**.
3. A is **ND** iff all its eigenvalues are **negative**.
4. A is **NSD** iff all its eigenvalues are **nonpositive**.
5. A is **indefinite** iff it has at least one positive eigenvalue and at least one negative eigenvalue.

Example 1

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   The characteristic equation is

   $$|I - \lambda I| = \left| \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \right| = (1-\lambda)^2$$

   $$\lambda_1 = \lambda_2 = 1$$

   So A is PD

EXAMPLE 2

2. $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

The characteristic equation is

$$\left| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \lambda I \right| = \left| \begin{bmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{bmatrix} \right| = (1-\lambda)(-\lambda) = 0$$

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$

So B is PSD

Example 3

3. $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The characteristic equation is

$$\left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda I \right| = \left| \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right| = (-\lambda)^2 - 1 = 0$$

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

So C is Indefinite

PRINCIPAL MINORS

DEFINITION: Let A be an $N \times N$ square matrix.

1. The $K \times K$ submatrix obtained from A by deleting **any** $(N-K)$ columns of A and the corresponding $(N-K)$ rows of A is called ***K*-order principal submatrix of *A***.

2. The determinant of a K-order principal submatrix of A is called a ***K*-order principal minor** (principal minor (PM) of order K).

3. The K-order principal submatrix of A obtained by deleting the **last** $(N-K)$ columns of A and the **last** $(N-K)$ rows of A is called the ***K*-order leading principal submatrix of *A***, denoted A_K.
   (An $N \times N$ matrix has N leading principal submatrices.)

4. The determinant of A_K is called the ***K*-order leading principal minor** (LPM), denoted $|A_K|$.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Leading principal submatrices and leading principal minors:

$$A_1 = \begin{bmatrix} a_{11} \end{bmatrix} \quad ; \quad |A_1| = a_{11}$$

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad ; \quad |A_2| = a_{11}a_{22} - a_{12}a_{21}$$

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad ; \quad |A|$$

Example (continued)

Principal submatrices and principal minors:

Of order 1:

$$\begin{bmatrix} a_{22} \end{bmatrix} \qquad a_{22}$$

$$\begin{bmatrix} a_{33} \end{bmatrix} \qquad a_{33}$$

Of order 2:

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \qquad \left| \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \right| = a_{11}a_{33} - a_{13}a_{31}$$

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad \left| \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \right| = a_{22}a_{33} - a_{23}a_{32}$$

Identifying definiteness using principal minors

Theorem: Let A be an $N \times N$ **symmetric** matrix. Then

1. A is **PD** iff all its **LPM**'s are **positive**.

2. A is **PSD** iff all its **PM**'s are **nonnegative**.

3. A is **ND** iff its **LPM**'s alternate in signs with $(-1)^K |A_K| > 0$:
   every **LPM** of *odd* order is **negative**
   every **LPM** of *even* order is **positive**

4. A is **NSD** iff every **PM** of *odd* order is **nonpositive**
   every **PM** of *even* order is **nonnegative**.

5. A is **indefinite** iff some **LPM**'s of order k are nonzero but do not fit into (1) or (3).

Example 1

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   Look at the A_K's and LPM's first:

   $A_1 = \begin{bmatrix} 1 \end{bmatrix}$, $\quad |A_1| = 1$

   $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad |A_2| = 1$

   So A is PD

EXAMPLE 2

2 $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Look at the B_K's and LPM's first:

$B_1 = \begin{bmatrix} 1 \end{bmatrix}$, $\quad |B_1| = 1$

$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad |B_2| = 0$

So we know that B is not PD. Is it PSD?

Check all the remaining principal submatrices and PM's

Of order 1:

$\begin{bmatrix} 0 \end{bmatrix}$ $\quad \left| \begin{bmatrix} 0 \end{bmatrix} \right| = 0$

So B is PSD

EXAMPLE 3

3. $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Look at the C_K's and LPM's first:

$C_1 = \begin{bmatrix} 0 \end{bmatrix}, \qquad |C_1| = 0$

$C_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad |C_2| = -1$

Since the second order LPM is negative, C doesn't fit into any category

So C is Indefinite

DEFINITENESS ON SUBSPACES

Let B be a $N \times M$ matrix, A be $N \times N$, x be $N \times 1$.

The set

$$T = \{x \in \mathbb{R}^N : B'x = 0\}$$

consists of all vectors that are orthogonal to the columns of B.

T is a vector subspace in R^N.

If the columns of B are linearly independent, T is $N - M$ dimensional.

DEFINITE QUADRATIC FORMS

THEOREM: Let A be symmetric, B of rank M. The quadratic form $x' Ax > 0 \;\; \forall\, x \in T$ (is PD on T) iff

$$\mathbf{(-1)}^{\boldsymbol{M}} \begin{vmatrix} A_{RR} & B_{RM} \\ B'_{MR} & 0 \end{vmatrix} > 0 \text{ for } R = M+1, ..., N.$$

That is, the border preserving LPM's of orders $M+1, ..., N$ **have the same sign as** $\mathbf{(-1)}^{\boldsymbol{M}}$.

DEFINITE QUADRATIC FORMS (CONTINUED)

THEOREM: Let A be symmetric, B of rank M. The quadratic form $x'Ax < 0 \;\; \forall\, x \in T$ (is ND on T) iff

$$(\mathbf{-1})^{\boldsymbol{R}} \begin{vmatrix} A_{RR} & B_{RM} \\ B'_{MR} & 0 \end{vmatrix} > 0 \text{ for } R = M+1, ..., N.$$

In this case, the border preserving LPM's of orders $M+1, ..., N$ **have the same sign as** $(\mathbf{-1})^{\boldsymbol{R}}$ for $R = M+1, ..., N$. That is, they alternate signs.

Example

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$

So $N = 4$, $M = 2$, $R = 3, 4$

The bordered matrix is $\hat{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_{11} & b_{12} \\ a_{21} & a_{22} & a_{23} & a_{24} & b_{21} & b_{22} \\ a_{31} & a_{32} & a_{33} & a_{34} & b_{31} & b_{32} \\ a_{41} & a_{42} & a_{43} & a_{44} & b_{41} & b_{42} \\ b_{11} & b_{21} & b_{31} & b_{41} & 0 & 0 \\ b_{12} & b_{22} & b_{23} & b_{24} & 0 & 0 \end{bmatrix}$

The border preserving leading principal submatrix of order R is obtained by deleting the rows and columns of $\hat{A}$ corresponding to th last $N - R$ rows and columns of the original matrix.

EXAMPLE (CONTINUED)

The border preserving leading principal submatrix of order 3 is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_{11} & b_{12} \\ a_{21} & a_{22} & a_{23} & b_{21} & b_{22} \\ a_{31} & a_{32} & a_{33} & b_{31} & b_{32} \\ b_{11} & b_{21} & b_{31} & 0 & 0 \\ b_{12} & b_{22} & b_{23} & 0 & 0 \end{bmatrix}$$

For semidefiniteness, the respective requierments extend to all the border preserving principal minors. Also, the inequality regarding the sign is weak.