Exercise 3.2.2 Let $\mathbb{F}$ be the space of bounded continuous functions on $\mathbb{R}$. Show that $(\mathbb{F}, d_{sup})$ is a complete metric space.

Proof:

First, we show a Cauchy sequence $\{f_n\}$ does converge to a function f.

We pick an arbitrary x_0 and claim $\forall \epsilon > 0$, $\exists N$ $s.t.$ $n, m > N$ implies $|f_n(x_0) - f_m(x_0)| \leq \sup|f_n(x) - f_m(x)| < \epsilon$. Equivalently, we can say $\{f_n(x_0)\}$ converges to $y(x_0)$, which is in $\mathbb{R}$.

Since x_0 is arbitrarily chosen, it is true that $f_n(x)$ converges to its corresponding $y(x)$, for all x.

A real-valued function, $f(x) := y(x)$, where $y(x) = \lim_{n\to\infty} f_n(x)$ is what f_n converges to.

Next, we show function f is bounded.

For a large enough N, $n, m > N$ implies $|f_n - f_m|_{\text{sup}} < \epsilon$. So $\exists N\, and\, m > N\, s.t.$ $|f_N - f_m|_{\text{sup}} < 1$, which then implies $|f_N(x) - f_m(x)| \leq 1$ for all x and all $m > N$.

Hence $f_N(x) - 1 \leq f_m(x) \leq f_N(x) + 1$. Taking limit on both sides, we have $f_N(x) - 1 \leq f(x) \leq f_N(x) + 1$. For $\forall x \in \mathbb{R}$, $|f_N(x) - f(x)| \leq 1$.

$|f|_{\text{sup}} \leq |f - f_N|_{\text{sup}} + |f_N|_{\text{sup}} \leq 1+$ a finite number (b.c. f_N is bounded).

Last, we show function f is continuous.

Because f_n is Cauchy, for $\forall \frac{\epsilon}{3} > 0$ we can find N, s.t. $m > N$ implies:

$|f_N - f_m|_{\text{sup}} < \frac{\epsilon}{3}$, i.e. $f_N(x) - \frac{\epsilon}{3} \leq f_m(x) \leq f_N(x) + \frac{\epsilon}{3}$ for all x and all $m > N$. Taking limit on both sides, we have $f_N(x) - \frac{\epsilon}{3} \leq f(x) \leq f_N(x) + \frac{\epsilon}{3}$ for all x.

By triangle inequality, we have $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$.

From above, we can find N, s.t. $|f(x) - f_N(x)| \leq \frac{\epsilon}{3}$ and $|f_N(y) - f(y)| \leq \frac{\epsilon}{3}$ for all x.

Because f_N is continuous, we can find δ s.t. $|x - y| < \delta$ implies $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$.

Adding these together, we have shown that for $\forall \epsilon > 0$, we can find δ s.t. $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$.

Hence f is a bounded continuous function in the space $\mathbb{F}$.