Econ 508-A

KT Sufficient Conditions

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Introduction

Given the problem

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g_k(\boldsymbol{x}) \leq 0 \end{aligned}$$

The KT conditions are also sufficient for a maximum when:

- The objective function f is concave.
- The constraint set X is convex (the constraints g_k are convex).

We'll develop weaker conditions in the following sections.

Interior Local Optima

Theorem: Let $f : X \to \mathbb{R}$ be differentiable, where $X \subseteq \mathbb{R}^N$. Given $\boldsymbol{x}^* \in X$, suppose that X is locally convex (there is ε s.t. $N_\varepsilon(\boldsymbol{x}^*) \cap X$ is convex). Suppose that the following conditions hold:

1. $Df(\boldsymbol{x}^*) = 0$
2. f is locally concave at $\boldsymbol{x}^*$.

Then $\boldsymbol{x}^*$ is a local maximum.

Moreover, if f is locally strictly concave, then $\boldsymbol{x}^*$ is a local strict maximum.

Proof

By contradiction:

Suppose that f is locally concave at $\boldsymbol{x}^*$ but $\boldsymbol{x}^*$ is not a local maximum.

Choose any $\varepsilon > 0$ s.t. f is concave on $N_\varepsilon(\boldsymbol{x}^*)$.

Since $\boldsymbol{x}^*$ is not a local maximum, $\exists\, \hat{\boldsymbol{x}} \in N_\varepsilon(\boldsymbol{x}^*)$ s.t. $f(\hat{\boldsymbol{x}}) > f(\boldsymbol{x}^*)$.

Take any $\alpha \in (0,1)$ and let $\boldsymbol{x}^\alpha = \alpha\hat{\boldsymbol{x}} + (1-\alpha)\boldsymbol{x}^*$.

By concavity of $f(\boldsymbol{x})$,

$$f(\boldsymbol{x}^\alpha) \geq \alpha f(\hat{\boldsymbol{x}}) + (1-\alpha)f(\boldsymbol{x}^*) = f(\boldsymbol{x}^*) + \alpha(f(\hat{\boldsymbol{x}}) - f(\boldsymbol{x}^*))$$

Proof (continued)

Rewriting $\boldsymbol{x}^{\alpha} = \boldsymbol{x}^{*} + \alpha(\hat{\boldsymbol{x}} - \boldsymbol{x}^{*})$ and dividing by α,

$$\frac{f(\boldsymbol{x}^{*} + \alpha(\hat{\boldsymbol{x}} - \boldsymbol{x}^{*})) - f(\boldsymbol{x}^{*})}{\alpha} \geq \frac{\alpha(f(\hat{\boldsymbol{x}}) - f(\boldsymbol{x}^{*}))}{\alpha} = f(\hat{\boldsymbol{x}}) - f(\boldsymbol{x}^{*}) > 0$$

Taking limits as $\alpha \to 0$, this implies that

$$D_{(\hat{\boldsymbol{x}} - \boldsymbol{x}^{*})} f(\boldsymbol{x}^{*}) = Df(\boldsymbol{x}^{*})(\hat{\boldsymbol{x}} - \boldsymbol{x}^{*}) > 0, \text{ so } Df(\boldsymbol{x}^{*}) \neq 0 \quad \blacksquare$$

For strict, do the same with strict inequality.

COROLLARY OF PREVIOUS THEOREM

Let $f : X \to \mathbb{R}$ be C^2, where $X \subseteq \mathbb{R}^N$. Given $\boldsymbol{x}^* \in X$, suppose that X is locally convex. Suppose that the following conditions hold:

1. $Df(\boldsymbol{x}^*) = 0$ (sufficiency FOC)

2. $D^2 f(\boldsymbol{x}^*)$ is ND (sufficiency SOC)

Then $\boldsymbol{x}^*$ is a strict local maximum.

Notice that $D^2 f(\boldsymbol{x}^*)$ being ND is sufficient for strict concavity of f at $\boldsymbol{x}^*$.

Necessary conditions

Theorem: Let $f : X \to \mathbb{R}$ be C^2, where $X \subseteq \mathbb{R}^N$. If $\boldsymbol{x}^*$ is interior and a local maximum, then the following conditions hold:

1. $Df(\boldsymbol{x}^*) = 0$ (Necessity FOC)

2. $D^2 f(\boldsymbol{x}^*)$ is NSD (Necessity SOC)

Proof

If $\boldsymbol{x}^*$ is an interior local maximum, then there is $\varepsilon > 0$ such that for $N_\varepsilon(\boldsymbol{x}^*) \subseteq X$ and $\forall\, \boldsymbol{x} \in N_\varepsilon(\boldsymbol{x}^*),\ f(\boldsymbol{x}^*) \geq f(\boldsymbol{x})$. Note that $N_\varepsilon(\boldsymbol{x}^*)$ is convex.

1. Already proved.
2. Suppose $Df(\boldsymbol{x}^*) = 0$ but $D^2 f(\boldsymbol{x}^*)$ is not negative semidefinite.

   Then for some $\boldsymbol{v} \in \mathbb{R}^N,\ \boldsymbol{v} \neq \mathbf{0},\ \boldsymbol{v}' D^2 f(\boldsymbol{x}^*) \boldsymbol{v} > 0$.

   Thus, the function is locally strictly convex on the line through $\boldsymbol{x}^*$ in the direction $\boldsymbol{v}$.

   So along this line $\boldsymbol{x}^*$ is a local strict minimum, and, hence, cannot be a local strict maximum.

■

EXAMPLE

Note that there is a discrepancy between the necessity and sufficiency SOC. We need strict negative definiteness at $\boldsymbol{x}^*$ to guarantee negative definiteness in $N_\varepsilon(\boldsymbol{x}^*)$.

Consider the following example:

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^4$.

$Df(0) = 0$ and $D^2f(0) = 0$, so the necessary conditions for a maximum are satisfied.

However, $x = 0$ is a minimum, since f is strictly convex.

CONSTRAINED LOCAL OPTIMA

THEOREM: Let $f : X \to \mathbb{R}$ be C^2, where $X \subseteq \mathbb{R}^N$. Given $\boldsymbol{x}^* \in X$, suppose that X is locally convex. Assuming the following:

1. The KT conditions hold.
2. f is locally concave at $\boldsymbol{x}^*$

Then $\boldsymbol{x}^*$ is a local maximum.

Moreover, if f is locally strictly concave, then $\boldsymbol{x}^*$ is a local strict maximum.

PROOF: omitted (similar to the proof of the first theorem: there would exist some feasible direction in which f increases and the g_k are not increasing).

NECESSARY CONDITIONS

THEOREM: Let f and g be C^2. Fix $\boldsymbol{x}^*$ and let

$$T = \{\boldsymbol{v} \in \mathbb{R}^N : \nabla g_k(\boldsymbol{x}^*) \cdot \boldsymbol{v} = 0 \; \forall k \text{ binding } \}$$

1. Suppose that constraint qualification holds. Suppose that $\boldsymbol{x}^*$ is a local maximum. Then there are unique numbers $\lambda_k \geq 0$ such that the following conditions hold:

   (a) i. $\nabla f(\boldsymbol{x}^*) = \sum_k \lambda_k \nabla g_k(\boldsymbol{x}^*)$

   ii. $\lambda_k g_k(\boldsymbol{x}^*) = 0$ for all k.

   (b) $D^2 f(\boldsymbol{x}^*) - \sum_k \lambda_k D^2 g_k(\boldsymbol{x}^*)$ is negative semidefinite on T.

Sufficient Conditions

2. Suppose that at a feasible point $\boldsymbol{x}^*$ there exist numbers $\lambda_k \geq 0$ such that the following conditions hold:

   (a) i. $\nabla f(\boldsymbol{x}^*) = \sum_k \lambda_k \nabla g_k(\boldsymbol{x}^*)$

   ii. $\lambda_k > 0$ iff $g_k(\boldsymbol{x}^*) = 0 \; \forall \, k$.

   (b) $D^2 f(\boldsymbol{x}^*) - \sum_k \lambda_k D^2 g_k(\boldsymbol{x}^*)$ is negative definite on T.

   Then $\boldsymbol{x}^*$ is a strict local maximum.

PROOF: omitted. ■

Note that condition 2(a)ii is stronger than KT(2), since it requires that active constraints be binding.

GLOBAL OPTIMIZATION

A local maximum is a global maximum if f is concave.

THEOREM: Let $X \subseteq \mathbb{R}^N$ be convex. Let $f : X \to \mathbb{R}$.

1. Let $\boldsymbol{x}^*$ be a local maximum.
   (a) If f is concave on X, then $\boldsymbol{x}^*$ is a global maximum.
   (b) If f is strictly quasiconcave on X, then $\boldsymbol{x}^*$ is a global strict maximum.
2. Let $\boldsymbol{x}^*$ be a local strict maximum. If f is quasiconcave on X, then $\boldsymbol{x}^*$ is a global strict maximum.

PROOF

1. (a) Suppose $\boldsymbol{x}^*$ is not a global maximum.

   Then there is $\boldsymbol{x} \in X$ such that $f(\boldsymbol{x}) > f(\boldsymbol{x}^*)$.

   For $\alpha \in (0, 1)$, let $\boldsymbol{x}^\alpha = \alpha\boldsymbol{x} + (1 - \alpha)\boldsymbol{x}^*$.

   By concavity, $f(\boldsymbol{x}^\alpha) \geq \alpha f(\boldsymbol{x}) + (1 - \alpha)f(\boldsymbol{x}^*) > f(\boldsymbol{x}^*)$.

   Since α is arbitratry, this implies that $\boldsymbol{x}^*$ is not a local maximum.

PROOF (CONTINUED)

1. (b) Suppose $\boldsymbol{x}^*$ is not a global strict maximum.

   Then there is $\boldsymbol{x} \in X$ such that $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*)$.

   For $\alpha \in (0, 1)$, let $\boldsymbol{x}^\alpha = \alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{x}^*$.

   By strict quasiconcavity, $f(\boldsymbol{x}^\alpha) > f(\boldsymbol{x}^*)$.

   Since α is arbitratry, this implies that $\boldsymbol{x}^*$ is not a local maximum.

Proof (continued)

2. Suppose $\boldsymbol{x}^*$ is not a global strict maximum.

   Then there is $\boldsymbol{x} \in X$ such that $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*)$.

   For $\alpha \in (0, 1)$, let $\boldsymbol{x}^\alpha = \alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{x}^*$.

   By quasiconcavity, $f(\boldsymbol{x}^\alpha) \geq f(\boldsymbol{x}^*)$.

   Since α is arbitratry, this implies that $\boldsymbol{x}^*$ is not a local maximum.