ECON 508-A

Convexity in $\mathbb{R}^N$

Juan Ignacio Vizcaino

Washington University in St. Louis

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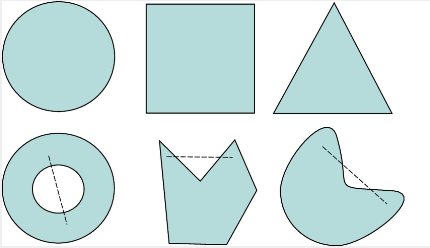
The material here is based on the slides on Convexity prepared by Carmen Aston-Figari.

All errors are mine.

Definition 1

A set $A \subseteq \mathbb{R}^N$ is convex iff for any $a, b \in A$ and any $\theta \in [0, 1]$ the point $\theta a + (1 - \theta)b$ is also in A.

Geometrically, *A* is convex iff *A* contains the line segment joining any two points in *A*.



Definition 2

Let $A, B \subseteq \mathbb{R}^N$. Let $\theta \subseteq \mathbb{R}$.

$A + B = \left\{x \in \mathbb{R}^N : \exists\, a \in A, b \in B \text{ such that } x = a + b\right\}$

$\theta A = \left\{x \in \mathbb{R}^N : \exists\, a \in A \text{ such that } x = \theta a\right\}$

Theorem 1

Let $A, B \subseteq \mathbb{R}^N$ be convex and let $\theta \subseteq \mathbb{R}$.

1. *$A + B$ is convex.*
2. *θA is convex.*
3. *$A \cap B$ is convex.*

Theorem 2

For any $x \in \mathbb{R}^N$ and any $\varepsilon > 0$, $N_\varepsilon(x)$ is convex.

Convex Hull

Definition 3

Given a set $X \subseteq \mathbb{R}^N$, the Convex Hull of X, denoted ConvX is the smallest convex set that contains X.

Alternatively ConvX can be defined as:

1. The set containing all possible convex combinations of points in X:

$$\text{ConvX} = \Big\{ y : y = \sum_{i=1}^{m} \alpha^i x^i \text{ for some} m, \text{ with } x^i \in X \\ \alpha^i \in [0,1] \text{ and } \sum_{i=1}^{m} \alpha^i = 1 \}$$

2. The intersection of all the convex sets that contain X.

Definition 4

$K \subseteq \mathbb{R}^N$ is a Convex Cone if for any $x, x' \in K$ and $\alpha, \beta > 0, \alpha x + \beta x' \in K$

Example 1: the non-negative orthant of $\mathbb{R}^N$:

$$\mathbb{R}^N_+ = \{x = (x_1, x_2, \ldots, x_N) \mid x_i \geq 0 \text{ for } i = 1, \ldots N\}$$

Example 2: Production Sets under Additivity and Non-Increasing returns to scale

$$Y \subset \mathbb{R}^L: \text{ production set}$$

$$\text{For } (y_1, \ldots, y_L) \in Y$$

$$y_\ell > 0 \Rightarrow \ell: \text{ output}$$

$$y_\ell < 0 \Rightarrow \ell: \text{ input}$$

Non-increasing returns to scale:

$$\text{If } y \in Y, \text{ then } \alpha y \in Y \text{ for all } \alpha \in [0, 1].$$

Additivity:

$$Y + Y \subset Y$$

Proposition 1

Y is additive and exhibits non-increasing returns to scale if and only if it is a convex cone.

Separation 1

Definition 5

$A, B \subseteq \mathbb{R}^N$ can be strictly separated iff there exists a $v \in \mathbb{R}^N$ and an $r \in \mathbb{R}$ such that $v \cdot a > r > v \cdot b$ for every $a \in A, b \in B$.

- Any $N - 1$ dimensional plane in $\mathbb{R}^N$ can be represented as a set
$$\left\{x \in \mathbb{R}^N : v \cdot x = r\right\}$$
where $v \in \mathbb{R}^N, v \neq 0$ and $r \in \mathbb{R}$.
- The vector v gives the tilt of the plane and the number r determines the position of the plane.
- If $r = 0$, then the plane passes through the origin:
$$T_v = \left\{x \in \mathbb{R}^N : v \cdot x = 0\right\}$$
- Thus, the plane T_v is the set of points orthogonal to v.

Separation 2

- For $r \neq 0$, the plane is a parallel copy of T_v, shifted away from the origin.
- Strict separation means that there is a plane such that the set A lies strictly on one side of the plane and B lies strictly on the other side.

Theorem 3 (Supporting Hyperplane Theorem)

Let $A \subseteq \mathbb{R}^N$ be non-empty, closed, and convex. If x^* is not interior to A, then A is supported at x^*.

Proof: Omitted. □

Theorem 4 (Separating Hyperplane Theorem)

Suppose that $A, B \subseteq \mathbb{R}^N$ are non-empty, closed, convex, and disjoint. If at least one is compact, then A and B can be strictly separated.

Proof: Omitted. $\square$