ECON 508-A

Envelope Theorem

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Section 1: Motivation

- Consider the problem of a firm that has a technology f available to produce output using a single input x.
- It acts competitively in both the input and the final output markets.
- Let p and w be the exogenous prices of final output and the production input, respectively.
- Let $\Pi(x, w, p)$ be the firm's profit when it uses x units of input and prices are w and p.

$$\Pi(x, w, p) \equiv p \cdot f(x) - w \cdot x$$

- Define $x^\star(w, p)$ to be the optimal input demand given prices w, p.

Motivation

Replacing $x^\star(w,p)$ into $\Pi(x,w,p)$:

$$\Pi^\star\Big(x^\star(w,p), w, p\Big) \equiv p \cdot f\left(x^\star(w,p)\right) - w \cdot x^\star(w,p)$$

In general, we call this object **Value Function**. In this particular case it is also called **Profit Function**.

We are interested in the behavior of the firm's profit under changes in exogenous variables (prices).

$$\left.\frac{\partial \Pi(x,w,p)}{\partial p}\right|_{x=x^\star(w,p)} = f\left(x^\star(w,p)\right)$$

$$\left.\frac{\partial \Pi(x,w,p)}{\partial w}\right|_{x=x^\star(w,p)} = -x^\star(w,p) \;\rightarrow\; \textbf{Hotelling's Lemma}$$

These results follow from:

$$\max_{\{x\}} \Pi\left(x, w, p\right) \equiv p \cdot f(x) - w \cdot x$$

F.O.C.

$$\frac{\partial \Pi\left(x, w, p\right)}{\partial x} = p \cdot f^{'}\left(x^{\star}\left(w, p\right)\right) - w = 0$$

$$\left.\frac{\partial \Pi\left(x, w, p\right)}{\partial p}\right|_{x=x^{\star}(w,p)} = f\left(x^{\star}\left(w, p\right)\right) + p \cdot f^{'}\left(x^{\star}\left(w, p\right)\right)\frac{\partial x^{\star}\left(w, p\right)}{\partial p} - w \cdot \frac{\partial x^{\star}\left(w, p\right)}{\partial p}$$

$$\left.\frac{\partial \Pi\left(x, w, p\right)}{\partial w}\right|_{x=x^{\star}(w,p)} = p \cdot f^{'}\left(x^{\star}\left(w, p\right)\right)\frac{\partial x^{\star}\left(w, p\right)}{\partial w} - x^{\star}\left(w, p\right) - w \cdot \frac{\partial x^{\star}\left(w, p\right)}{\partial w}$$

- Today we will see that this result is in fact a **theorem**.
- Start with The Envelope Theorem for Unconstrained Optimization Problems.
- Move towards a more general case: The Envelope Theorem for Constrained Optimization Problems.
- Study under what set of (*mild*) assumptions the theorem to hold.

The Envelope Theorem With No Binding Constraints

- Consider the following parameterized version of a differentiable maximization problem

$$\max_{\{x \in \mathbb{R}^N\}} f(x, \theta)$$

where $\theta \in \mathbb{R}^L$ is a vector of parameters.

- Let $x^\star(\theta)$ be the optimal x for a given θ.
- The value function is

$$f^V(\theta) = f(x^\star(\theta), \theta)$$

Section 2: The Envelope Theorem

Theorem 1

Fix a parametrized differentiable optimization problem. For each $\theta \in \mathbb{R}^L$, let $x^\star(\theta)$ be a solution to the problem. Let $f^v : \mathbb{R}^L \to R$ be defined by $f^v(\theta) = f(x^\star(\theta), \theta)$. Fix any $\hat{\theta} \in \mathbb{R}^L$. If $x^\star(\cdot)$ is differentiable at $\hat{\theta}$, then:

$$\frac{\partial f^v(\hat{\theta})}{\partial \theta} = \frac{\partial f\left(x^\star(\hat{\theta}), \hat{\theta}\right)}{\partial \theta}$$

Proof

By the Chain Rule

$$\frac{\partial f^{v}(\hat{\theta})}{\partial \theta} = \frac{\partial f\left(x^{\star}(\hat{\theta}), \hat{\theta}\right)}{\partial x^{\star}(\hat{\theta})} \cdot \frac{\partial x^{\star}(\hat{\theta})}{\partial \theta} + \frac{\partial f\left(x^{\star}(\hat{\theta}), \hat{\theta}\right)}{\partial \theta}$$

Since $x^{\star}(\theta)$ is optimal when $\theta = \hat{\theta}$, $\frac{\partial f\left(x^{\star}(\hat{\theta}),\hat{\theta}\right)}{\partial x^{\star}(\hat{\theta})} = 0$, and the result follows.

The Envelope Theorem With Binding Constraints

Consider the following parameterized version of a differentiable maximization problem

$$\max_{\{x\in\mathbb{R}^N\}} f(x,\theta)$$

$$\text{subject to}$$

$$g_1(x,\theta) \le 0$$

$$\vdots$$

$$g_K(x,\theta) \le 0$$

where $\theta \in \mathbb{R}^L$ is a vector of parameters.
Let $x^\star(\theta)$ be the optimal x for a given θ.
The value function is

$$f^v(\theta) = f(x^\star(\theta),\theta)$$

Theorem 2

Fix a parametrized differentiable optimization problem. For each $\theta \in \mathbb{R}^L$, let $x^\star(\theta)$ be a solution to the problem. Let $f^v : \mathbb{R}^L \to R$ be defined by $f^v(\theta) = f(x^\star(\theta), \theta)$. Let $J(\theta)$ be the indices of constraints that are binding at $x^\star(\theta)$. Suppose that the KKT conditions hold at $x^\star(\theta)$ for every $\theta \in U \subseteq \mathbb{R}^L$.
Fix any $\hat{\theta} \in \mathbb{R}^L$, let $J^\star = J(\hat{\theta})$, and let $\lambda_k^\star \geq 0$ be the associated KKT multipliers for $k \in J$. If $x^\star(\cdot)$ is differentiable at $\hat{\theta}$, then:

$$\frac{\partial f^v(\hat{\theta})}{\partial \theta} = \frac{\partial f\left(x^\star(\hat{\theta}), \hat{\theta}\right)}{\partial \theta} - \sum_{k \in J^\star} \lambda_k^\star \frac{\partial g_K(x^\star(\hat{\theta}), \hat{\theta})}{\partial \theta}$$

Proof: Omitted. □

Section 3: Application

Application: The Optimal -*Ramsey, Cass, Koopmans*- Growth Model.

Structure

Infinitely-lived household maximizes the present discounted value of future utility:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

Output is produced using capital as the only input:

$$y_t = f(k_t) \tag{2}$$

The law of motion for the capital stock is:

$$k_{t+1} = k_t(1-\delta) + i_t \tag{3}$$

There is a single good that can be either consumed or invested. The resource constraint is:

$$c_t + i_t = f(k_t) \tag{4}$$

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} = f(k_t)$$

- The solution to this problem is an infinite sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.
- **Shortcut given by RECURSIVE STRUCTURE**: the Planner faces the same problem every period!
- The solution is characterized by policy functions $c_t = g_c(k_t)$ and $k_{t+1} = g_k(k_t)$.
- Given the recursive structure, these policies apply each and every period.

- Let $v(k_0)$ be the **Value Function**.
- It is the maximized value of the objective function given an initial level of capital k_0.

$$v(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

Which can be re-written as:

$$v(k_0) = \max_{\{c_0, k_1\}} \left[U(c_0) + \max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t U(c_t) \right]$$

- The problem can be expressed as a recursive definition of the value function, or **Bellman Equation**:

$$v(k_0) = \max_{\{c_0,k_1\}} \Big[U(c_0) + \beta v(k_1)\Big]$$

s.t.

$$c_0 + k_1 = f\left(k_0\right)$$

- Replacing the constraint into the Bellman Equation

$$v\left(k_0\right) = \max_{\{k_1\}} \Big[U\left(f\left(k_0\right) - k_1\right) + \beta v\left(k_1\right)\Big]$$

- Instead of two infinite sequences now we need to find a policy function of the form: $k_1 = g(k_0)$.

- The F.O.C. is:

$$U'[f(k_0) - k_1] = \beta v'[k_1]$$

- Problem: we do not know $v(\cdot)$ or $v'(\cdot)$.
- Strategy: Assume the solution exists and is of the form $k_1 = g(k_0)$

$$v(k_0) = U(f(k_0) - g(k_0)) + \beta v(g(k_0))$$

- Totally differentiating:

$$\begin{aligned} v'(k_0) &= U'(f(k_0) - g(k_0)) f'(k_0) \\ &\quad - U'(f(k_0) - g(k_0)) g'(k_0) + \beta v'\big(g(k_0)\big) g'(k_0) \\ &= U'(f(k_0) - g(k_0)) f'(k_0) \\ &\quad - \left[U'\big(f(k_0) - g(k_0)\big) - \beta v'\big(g(k_0)\big)\right] g'(k_0) \end{aligned}$$

Solution Characterization Continued.

- Now we use the **ENVELOPE THEOREM**.
- Remember the FOC was $\rightarrow U'[f(k) - k_1] = \beta v'[k_1]$
- Thus:

$$v'(k_0) = U'(f(k_0) - k_1) f'(k_0)$$

- Updated one period (this is a.k.a. Benveniste-Scheinkman condition):

$$v'(k_1) = U'(f(k_1) - k_2) f'(k_1)$$

- Now we know what $v'(\cdot)$ in the FOC is:

$$U'[f(k_0) - k_1] = \beta U'(f(k_1) - k_2) f'(k_1)$$

- Intuition...
- Are we done?