Econ 508B: Lecture 2

Lebesgue Measure and Lebesgue Measurable Functions

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Outline

Outline

Topological Space

Definition 1.1

A *topological space* is a pair $(\mathbb{S}, \mathcal{C})$, where $\mathcal{S}$ is a non-empty set and $\mathcal{C}$ is a collection of subsets of $\mathbb{S}$ such that

- $\emptyset, \mathbb{S} \in \mathcal{C}$,
- **(finite intersection:)**$C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cap C_2 \in \mathcal{C}$, and
- **(finite or infinite union:)**$\{\mathcal{C}_k : k \in N^+\} \subset \Rightarrow \cup_{k \in N^+} \in \mathcal{C}$

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Remark 1.1

- The elements of $\mathcal{C}$ are called open sets and the collection $\mathcal{C}$ is called a topology on $\mathbb{S}$.
- Using *de Morgan's laws*, the above axioms defining open sets become axioms defining **closed sets**.

Union/Intersection

Definition 1.2

A sequence $\{C_k\}$ of sets is said to *increase* to $\cup_k C_k$ if $C_k \subset C_{k+1}$ for all k and to *decrease* to $\cap_k C_k$ if $C_k \supset C_{k+1}$ for all k; we use the notations $C_k \nearrow \cup_k C_k$ and $C_k \searrow \cap_k C_k$ to denote these two possibilities.
If $\{C_k\}_{k=1}^{\infty}$ is a sequence of sets, we define

$$\lim sup C_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} C_k\right), \lim inf C_k = \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} C_k\right)$$

noting that

- $U_j = \bigcup_{k=j}^{\infty} C_k$ and $V_j = \bigcap_{k=j}^{\infty} C_k$ satisfy $U_j \searrow \lim sup C_k$ and $V_j \nearrow \lim inf C_k$,
- $\lim inf C_k \subset \lim sup C_k$.

Norm on $\mathbb{R}^N$

(1) $|x| \geq 0$ and $|x| = 0 \Leftrightarrow x = 0$,

(2) $|\alpha x| = |\alpha||x|, x \in \mathbb{R}^N, \alpha \in \mathbb{R}$,

(3) $|x + y| \leq |x| + |y|, \forall x, y \in \mathbb{R}$,

(4) (**Cauchy-Schwarz inequality:**)$|x \cdot y| \leq |x||y|$.

Proof for (4):

$$\forall x, y \in \mathbb{R}, xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$$

$$x \cdot y = \sum x_k y_k \leq \frac{1}{2}\sum x_k^2 + \frac{1}{2}\sum y_k^2 = \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2$$

$$x' = \lambda x, y' = \frac{1}{\lambda}y, \lambda \neq 0(to\ be\ chosen), x' \cdot y' = x \cdot y$$

$$\leq \frac{1}{2}|\lambda|^2|x|^2 + \frac{1}{2|\lambda|^2}|y|^2 = |x||y|\ (choose\ \lambda = \sqrt{\frac{|y|}{|x|}})$$

Metric Space

Definition 1.3

A *metric space* is a pair $(\mathbb{S}, d)$ where $\mathbb{S}$ is a nonempty set and d is a function from $\mathbb{S} \times \mathbb{S}$ to $\mathbb{R}^+$ (d is called a metric on $\mathbb{S}$) satisfying

(i) $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{S}$,

(ii) $d(x, y) = 0$ iff $x = y$,

(iii) (**triangle inequality:**)$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{S}$.

A *metric space* $(\mathbb{S}, d)$ is a topological space where a set C is open if for all $x \in C$, $\exists$ an $\epsilon > 0$ such that $B(x, \epsilon) \equiv \{y : d(y, x) < \epsilon\} \subset C$.

continued

Any one of the following metrics defined on any Euclidean space $\mathbb{R}^n, 1 \leq n < \infty$ is a metric space:

(1) For $1 < p < \infty$,

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

(2)

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

(3) For $0 < p < 1$,

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)$$

Question: why there does not exist power $\frac{1}{p}$ to (3)?

Outline

Lebesgue Outer (Exterior) Measure

Define closed n-dimensional intervals $I = \{\mathbf{x} : a_j \leq x_j \leq b_j, j = 1, ..., n\}$ and their volumes $v(I) = \prod_{j=1}^{n}(b_j - a_j)$. To define the outer measure of an arbitrary subset C of $\mathbb{R}^n$, cover C by a countable collection S of intervals I_k, and let

$$\sigma(S) = \sum_{I_k \in S} v(I_k)$$

The **Lebesgue outer measure** of C, denoted as $\mu^*(C)$, is defined by

$$\mu^*(C) = inf \sigma(S)$$

where the infimum is taken over all covers S of C. Thus, $0 \leq \mu^*(C) \leq +\infty$

Properties of Outer Measure

- For an interval I, $\mu^*(I) = v(I)$.
- **Monotonicity:** If $C_1 \subset C_2$, then $\mu^*(C_1) \leq \mu^*(C_2)$.
- **Countable sub-additivity:** If $C = \cup C_k$ is a countable union of sets, then $\mu^*(C) \leq \sum \mu^*(C_k)$.
- **Empty set:** The empty set has outer measure zero, e.g., $\mathbb{Q}$.

Remark 2.1

In particular, any subset of a set with outer measure zero has outer measure zero and that the countable union of sets with outer measure zero has outer measure zero as shown by the example of $\mathbb{Q}$.

Moreover, there are sets with outer measure zero that are not countable, e.g., Cantor Set.

continued

Theorem 2.1 (Outer Approximation)

Let $C \subset \mathbb{R}^n$. Then given $\epsilon > 0$, $\exists$ an open set G s.t. $C \subset G$ and $\mu^*(G) \leq \mu^*(C) + \epsilon$. Hence,

$$\mu^*(C) = inf \mu^*(G),$$

where the infimum is taken over all open sets G containing C.

Theorem 2.2

If $C \subset \mathbb{R}^n$, $\exists$ a set H of type G_δ s.t. $C \subset H$ and $\mu^*(C) = \mu^*(H)$

Outline

'Measurable' & 'Measure'

Definition 3.1 (Lebesgue measurable)

A subset C of $\mathbb{R}^n$ is defined to be ***Lebesgue measurable***, or ***measurable***, if given $\epsilon > 0$, $\exists$ an open set G s.t.

$$C \subset G, and\ \mu^*(G - C) < \epsilon$$

‘Measurable’ & ‘Measure’

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Definition 3.2 (Measure)

If C is measurable, its outer measure is referred to as its ***Lebesgue measure*** or simply its ***measure***, and denoted by $\mu(C)$ as previously illustrated:

$$\mu(C) = \mu^*(C), for\ measurable\ C.$$

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Example 3.1

- Every open set is measurable.
- Every set of outer measure zero is measurable.

Remark 3.1

Noting that the condition for measurability should not be confused with theorem 2.1, which only states that $\exists$ an open set G containing C such that $\mu^*(G) \leq \mu^*(C) + \epsilon$. Since $G = C \cup (G - C)$ when $C \subset G$, which only implies that $\mu^*(G) \leq \mu^*(C) + \mu^*(G - C)$. However, we cannot obtain from $\mu^*(G) \leq \mu^*(C) + \epsilon$ that $\mu^*(G - C) < \epsilon$.

Properties of Measurable Set

- (**Countable subaddtivity:**) The union $C = \cup C_k$ of a countable measurable sets is measurable and $\mu(C) \leq \sum \mu(C_k)$.
- Every closed set F is measurable.
- The complement of a measurable set is measurable.
- The intersection $C = \cap_k C_k$ of a countable measurable sets is measurable.
- If C_1 and C_2 are measurable, then $C_1 - C_2$ is measurable.
- The collection of measurable subsets of $\mathbb{R}^n$ is σ-algebra.
- Every Borel set is measurable.

Properties of Lebesgue Measure

Lemma 3.1

Set C in $\mathbb{R}^n$ is measurable if.f. given $\epsilon > 0$, $\exists$ a closed set $F \subset C$ such that $\mu^*(C - F) < \epsilon$.

Theorem 3.1

If $\{C_k\}$ is a countable collection of disjoint measurable sets, then $\mu(\bigcup_k C_k) = \sum_k \mu(C_k)$.

Corollary 3.1

Suppose C_1 and C_2 are measurable, $C_2 \subset C1$, and $\mu(C_2) < +\infty$. Then $\mu(C_1 - C_2) = \mu(C_1) - \mu(C_2)$.

Theorem 3.2

Let $\{C_k\}_{k=1}^{\infty}$ be a sequence of measurable sets.

(1) If $C_k \nearrow C$, then $\lim_{k\to\infty} \mu(C_k) = \mu(C)$.

(2) If $C_k \searrow C$ and $\mu(C_k) < +\infty$ for some k, $\lim_{k\to\infty} \mu(C_k) = \mu(C)$.

Proof: (1) Assume that $\mu(C_k) < +\infty$ for all k, otherwise both $\lim_{k\to\infty} \mu(C_k)$ and $\mu(C)$ are infinite and the statement holds. Break C via

$$C = C_1 \cup (C_2 - C_1) \cup ... \cup (C_k - C_{k-1}) \cup ...$$

By theorem 3.1,

$$\mu(C) = \mu(C_1) + \mu(C_2 - C_1) + ... + \mu(C_k - C_{k-1}) + \cdots.$$

By corollary 3.1,

$$\mu(C) = \mu(C_1)+(\mu(C_2)-\mu(C_1))+\cdots+(\mu(C_k)-\mu(C_{k-1}))+\cdots = \lim_{k\to\infty} \mu(C_k).$$

Proof: (2) clearly assume that $\mu(C_1) < +\infty$. Write

$$C_1 = C \cup (C_1 - C_2) \cup \cdots \cup (C_k - C_{k+1}) \cup \cdots.$$

Likewise,

$$\begin{aligned}\mu(C_1) &= \mu(C) + (\mu(C_1) - \mu(C_2)) + \cdots + (\mu(C_k) - \mu(C_{k+1})) + \cdots \\ &= \mu(C) + \mu(C_1) - \lim_{k\to\infty} \mu(C_k).\end{aligned}$$

Hence, $\mu(C) = \lim_{k\to\infty} \mu(C_k)$.

- Noting that the condition $\mu(C_k) < +\infty$ for some k is necessary.

Characterizations of Measurability

Theorem 3.3

- C is measurable if and only if $C = H - Z$, where H is of type G_δ and $\mu(Z) = 0$.
- C is measurable if and only if $C = H \cup Z$, where H is of type F_δ and $\mu(Z) = 0$.

Proof: (**sufficiency**) It is trivial that C is measurable because H and Z are both measurable sets.
(**necessity**): For the **first** one, suppose that C is measurable and choose an open set G_k such that $C \subset G_k$ and $\mu(G_k - C) < 1/k$ for $k = 1, 2, \ldots$. Let $H = \bigcap_k G_k$, which is a set of type $G_\delta, C \subset H$ and $H - C \subset G_k - C$ for every k. As $k \to \infty$, $\mu(H - C) = 0$.
Secondly, C is measurable, so is the complement set of C, denoted as C^c. Then applying the result of first one, we obtain that $C^c = \bigcap_k G_k - Z$, and using *de Morgan's laws* we have $C = (\bigcup_k G_k^c) \cup Z$.

Nonmeasurable set

Construct a nonmeasurable subset of $\mathbb{R}^1$, and the construction in $\mathbb{R}^d, d > 1$ is similar. The construction of a nonmeasurable set uses the following axiom of choice and rests on equivalence relation among real numbers in $[0, 1]$.

Axiom 3.1 (Zermelo's Axiom:)

A family of arbitrary nonempty disjoint sets indexed by a set $A, \{C_\alpha : \alpha \in A\}$, $\exists$ a set consisting of exactly one element from each $C_\alpha, \alpha \in A$.

Lemma 3.2

Let C be a measurable subset of $\mathbb{R}^1$ with $\mu(C) > 0$. Then the set of differences $\{d : d = x - y, x \in C, y \in C\}$ contains an interval centered at the origin.

Vitali Theorem: There exist nonmeasurable sets.

Proof:
An **equivalence relation**, defined as $x \sim y$ on the real line if $x - y$ is rational, which can be formulated by $C_\alpha = \{\alpha + r : r \ is \ rational\}$. It means that any two classes C_α and C_β are either identical or disjoint.

Hence, one equivalence class consists of all the rational numbers, and the other distinct classes are disjoint sets of irrational numbers.

Using *Zemelo's axiom*, construct the set C consisting of exactly one element from each distinct equivalence class, therefore, any two points of C must differ by an irrational number, which implies that the set $\{d : d = x - y, x \in C, y \in C\}$ cannot contain an interval. According to ***Lemma 3.2***, it suffices that either C is not measurable or $\mu(C) = 0$. Let $C_r = C + r, r \in \mathbb{Q}$, then $\cup_{r \in \mathbb{Q}} C$ representing the union of the translation of C by every rational is $\mathbb{R}^1$, $\mathbb{R}^1$ would have measure zero if C. Then it completes the proof that C is nonmeasurable.

Outline

Measurable Functions

Definition 4.1 (Measurable function)

In general, let Ω_1 be a set with a σ-algebra $\mathcal{C}_1$, and Ω_2 be a set with a sigma-algebra $\mathcal{C}_2$, and T be a function from Ω_1 to Ω_2. Say T is $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle -$**measurable** if the inverse image $\{x \in \Omega_1 : Tx \in C_2\} \in \mathcal{C}_1$ for each $C_2 \in \mathcal{C}_2$.

In particular, if $\Omega_2 = \mathbb{R}$, $\mathcal{C}_2$ becomes the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

Example 4.1

Recall the definition for *random variable* in the previous slide.

$\langle \mathcal{C}, \mathcal{B}(\mathbb{R}) \rangle$-measurable

In particular, to establish $\langle \mathcal{C}, \mathcal{B}(\mathbb{R}) \rangle$-measurability of a map into the real line, it is simplified to check the inverse images of intervals of the form (a, ∞) as follows.

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Definition 4.2

Let C be a measurable set in $\mathbb{R}^n$ and f be a real-extended function (i.e., $f(\mathbf{x}) \in [-\infty, +\infty], \mathbf{x} \in C$) defined on C. f is referred to as a ***Lebesgue measurable*** function on C or ***measurable*** function if for every finite a, the set

$$\{\mathbf{x} \in C : f(\mathbf{x}) > a\}$$

is a ***measurable*** subset of $\mathbb{R}^n$, which is often simply denoted as $\{f > a\}$. ***why?***

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$$C = \{f = -\infty\} \cup (\bigcup_{k=1}^{\infty} \{f > -k\})$$

Elementary properties of measurable functions

The definition for measurable function is equivalent to any of the following statements hold for finite a:

(i) $\{f \geq a\}$ is measurable.

(ii) $\{f < a\}$ is measurable.

(ii) $\{f \leqslant a\}$ is measurable.

Corollary 4.1

$\{f > -\infty\}, \{f < +\infty\}, \{f = +\infty\}, \{a \leqslant f \leqslant b\}, \{f = a\}$, etc, are all measurable if f is measurable.

Properties

Theorem 4.1

The **finite-valued** function f is measurable if and only if $f^{-1}(G)$ is measurable for every open set G of $\mathbb{R}^1$, and if and only if $f^{-1}(F)$ is measurable for every closed set F of $\mathbb{R}^1$.

Theorem 4.2

Let A be a dense subset of $\mathbb{R}^1$. Then f is measurable if $\{f > a\}$ is measurable for all $a \in A$.

Remark 4.1 (dense)

A set $C \subset C_1$ is said to be *dense* in C_1 if $\forall x_1 \in C_1$ and $\epsilon > 0, \exists$ a point $x \in C$ such that $0 < |x - x_1| < \epsilon$. Example: $\mathbb{Q}$ is dense in $\mathbb{R}$.

almost everywhere or a.e.: A property or a statement holds in C except in some subset of C with measure zero. For instance, the statement "$f = 0$ a.e. in C" is abbreviated of $f(x) = 0$ in C with the exception of those x in some subset Z of C with $\mu(Z) = 0$.

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Theorem 4.3

If f is measurable and if $g = f$ a.e., then $g = f$ a.e., then g is measurable and $\mu(\{g > a\}) = \mu(\{f > a\})$.

Operations on measurable functions

Theorem 4.4

If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable a.e. in C, and Φ is continuous, then $\Phi \circ f$ or $\Phi(f)$ is measurable.

Remark 4.2

The cases that arise frequently are

$$\Phi(f) = |f|, |f|^p (p > 0), e^{cf}$$

are measurable if f is measurable.
Noting another special case is that of

$$f^+ = max\{f, 0\},\ f^- = -min\{f, 0\}$$

Operations on measurable functions

Suppose f and g are measurable, and $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions, and λ is any real number, then so are

- $\{f > g\}$.
- $f + \lambda$ and λf.
- $f + g$.
- fg, and f/g if $g \neq 0$ a.e.
- $\sup_n f_n(x)$, $\inf_n f_n(x)$, $\limsup_{n\to\infty} f_n(x)$, and $\liminf_{n\to\infty} f_n(x)$.

Plus, if $\lim_{n\to\infty} f_n(x) = f(x)$ and $\{f_n\}_n^{\infty} = 1$ is a collection of measurable functions, then f is measurable.

Characteristic function or indicator function

Our attention to the objects that lie at the heart of integration theory: measurable functions. The starting point is the notion of a ***characteristic function*** or ***indicator function*** of a set C, which is defined by

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$

Clearly, χ_C is measurable if and only if C is measurable. χ_C is an example of what is referred to as a **simple function** on $\mathbb{R}^n$: a simple function on a set $C \subset \mathbb{R}^n$ is one that is defined on C and supposes **only** a finite number of finite values on C. If f is a simple function on C taking distinct values $a_1, ..., a_N$ on disjoint subsets $C_1, ..., C_N$ of C, and $C = \bigcup_{k=1}^{N} C_k$, then

$$f(x) = \sum_{k=1}^{N} a_k \chi_{C_k}(x), \; x \in C.$$

Approximation by simple function

Theorem 4.5

- Every function can be represented as the convergence of a sequence $\{f_k\}$ of simple functions.
- If $f \geq 0$, the sequence can be chosen to increasingly converge to f, i.e. $f_k \leq f_{k+1}, \forall k$.
- If f is measurable, then $\{f_k\}$ can be chosen to be measurable.

Proof

We begin by approximating pointwise, non-negative measurable functions by simple functions.

Firstly, start with a truncation within $[0,k]$. Suppose $f \geq 0$, $k = 1, 2, ...,$ subdivide the values of f by partitioning $[0,k]$ into subintervals $[(j-1)2^{-k}, j2^{-k}], j = 1, ..., k2^k$. Then

$$f_k(x) = \begin{cases} \frac{j-1}{2^k} & \text{if } \frac{j-1}{2^k} \leq f(x) \leq \frac{j}{2^k}, j = 1, ..., k2^k \\ k & \text{if } f(x) \geq k. \end{cases}$$

Then, $f_k(x) \to f(x)$ as k goes to infinity for all x. Clearly, $f_k \leq f_{k+1}$ and $f_k \to f$ because of $0 \leq f - f_k \leq 2^{-k}$ as k tends to infinity wherever f is finite and $f_k = k \to +\infty$ wherever $f = +\infty$. It completes the proof for the second theorem of nonnegative case. In fact,

$$f_k(x) = \sum_{j=1}^{k2^k} \frac{j-1}{2^k} \chi_{\{\frac{j-1}{2^k} \leq f(x) \leq \frac{j}{2^k}\}} + k\chi_{\{f \geq k\}}$$

Approximation by simple function

For the first theorem, using the **decomposition** of the function f: $f = f^+ - f^-$. Since both f^+ and f^- are nonnegative, then it trivially yields to apply the above theorem twice.

Theorem 4.6

Suppose f is measurable on $\mathbb{R}^d$. Then $\exists$ a sequence of simple function $\{f_k\}_{k=1}^{\infty}$ that satisfies

$$|f_k(x)| \leq |f_{k+1}(x)|, \ \lim_{k \to \infty} f_k(x) = f(x), \forall x$$

In particular, $|f_k(x)| \leq |f(x)|, \forall x, k$.

Theorems of Egorov and Lusin

Theorem 4.7 (Egorov's Theorem)

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set C with $\mu(C) < \infty$ and assume that $f_k \to f$ a.e. on C. Given $\epsilon > 0$, $\exists$ a closed set $F_\epsilon \subset C$ such that $\mu(C - F_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on F_ϵ.

Theorem 4.8 (Lusin)

Suppose f is measurable and finite valued on C with $\mu(C) < \infty$. $\forall \epsilon > 0, \exists$ a closed set F_ϵ with $F_\epsilon \subset C$ and $\mu(C - F_\epsilon) \leq \epsilon$ and such that $f|_{F_\epsilon}$ is continuous.