Econ 508B: Lecture 3

Riemann, Lebesgue Integral and Repeated Integration

Hongyi Liu

Washington University in St. Louis

July 24, 2017

Outline

1. Review of Riemann Integral
2. Lebesgue Integral
3. Comparison between Lebesgue and Riemann Integral
4. Repeated integration

Outline

1. Review of Riemann Integral
2. Lebesgue Integral
3. Comparison between Lebesgue and Riemann Integral
4. Repeated integration

Let f be defined and finite on a finite interval $[a, b]$. If $\Gamma = \{a = x_0 < x_1 < ... < x_n = b\}]$ is a partition of $[a, b]$, let $|\Gamma|$, called the *norm* of Γ, be defined as the length or diameter of a longest subinterval of Γ:

$$|\Gamma| = \max_i (x_i - x_{i-1})$$

Then we arbitrarily choose intermediate points $\{\xi_i\}_{i=1}^n$ satisfying $x_{i-1} \leq \xi_i \leq x_i$. Let

$$m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x), \ M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$$

Definition 1.1 (Riemann sums)

The Riemann sum and the upper-and lower-Riemann sums of f w.r.t the partition Γ are, respectively, defined as

$$R_\Gamma = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

$$U_\Gamma = \sum_{i=1}^{n} M_i(x_i - x_{i-1}), L_\Gamma = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

Definition 1.2 (Riemann integral)

We then further define the **Riemann integral** by if $\lim_{|\Gamma|\to 0} R_\Gamma$ exists and $\lim_{|\Gamma|\to 0} R_\Gamma = \int f(x)dx$. And such the definition is equivalent to the statement that

$$\overline{\int} f = \inf_\Gamma U_\Gamma, \underline{\int} f = \sup_\Gamma L_\Gamma, \overline{\int} f = \underline{\int} f = \int f$$

Outline

1. Review of Riemann Integral
2. Lebesgue Integral
3. Comparison between Lebesgue and Riemann Integral
4. Repeated integration

Lebesgue integral of simple function

Recall from the last slide that a simple function $\varphi(x)$ is a finite sum

$$\varphi(x) = \sum_{k=1}^{N} a_k \chi_{C_k}(x)$$

The **canonical form** of ϕ is the unique decomposition of the above form, where the numbers a_k are distinct and non-zero values, and the sets C_k are disjoint, formulated by taking the set $C_k = \{x : \phi(x) = a_k\}$ where a_k are distinct values, it turns out the sets C_k are disjoint.

Definition 2.1

If ϕ is a simple function with canonical form $\phi(x) = \sum_{k=1}^{N} a_k \chi_{C_k}(x)$, then we define the **Lebesgue integral** of ϕ on a measurable set C by

$$\int_C \varphi(x)dx = \int_C \varphi(x)\chi_C(x)dx = \int_C \varphi(x)d\mu = \sum_{k=1}^{N} a_k \mu(C_k)$$

where $C = \cup_k C_k$

Monotone Convergence Theorem for nonnegative functions

Theorem 2.1 (Nonnegative functions)

If $\{f_k\}$ is a sequence of nonnegative measurable functions such that $f_k \nearrow f$ on C, then

$$\int_C f_k \to \int_C f.$$

This allows us to **interchange** the integral with limits.

Theorem 2.2

Let f be nonnegative and measurable on C, then

$$\int_C f = \sup \sum_j [\inf_{x \in C_j} f(x)] \mu(C_j)$$

We say, f is **Lebesgue integrable** or **integrable** if $\int f(x)dx < +\infty$.

Properties

The integral of nonnegative measurable functions satisfies the following properties:

(i) *Linearity:* If $f, g \geq 0$, and a, b are positive real numbers, then

$$\int (af + bg) = a \int f + b \int g$$

(ii) *Additivity:* If $f \geq 0$ and measurable on C and C is the countable union of disjoint measurable sets $C_j, C = \cup_j C_j$. Then

$$\int_C f = \sum \int_{C_j} f.$$

(iii) *Monotonicity:* If $0 \leq f \leq g$ a.e. in C and measurable on C, then $\int_C f \leq \int_C g$.

continued

(iv) Let C_1 and C_2 be measurable and $C_1 \subset C_2$. If f is nonnegative and measurable on C_2, then $\int_{C_1} f \leq \int_{C_2} f$.

(v) Let f be nonnegative on C. If $\mu(C) = 0$, then $\int_C f = 0$.

(vi) Let $f \geq 0$ and measurable on C. Then $\int_C f = 0$ if and only if $f = 0$ a.e. in C.

Is it always true that $\int f_n \to \int f$?

However, the following example provides a negative answer to this.

Example 2.1

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n \\ 0 & \text{o.w.} \end{cases}$$

Here $f_n(x) \to 0$ for all x while $\int f_n(x)dx = 1$ for all n. In particular, $\int f_n > \int f$.

Lemma 2.1 (Fatou's Lemma)

Suppose $\{f_k\}$ is a sequence of measurable functions with $f_n \geq 0$ defined on C, then

$$\int_C (\lim_{k\to\infty} \inf f_k) \leq \lim_{k\to\infty} \inf \int_C f_k$$

Lebesgue's Dominatd Convergence Theorem

Theorem 2.3

Let $\{f_k\}$ be a sequence of nonnegative measurable functions on E such that $f_k \to f$ a.e. in C. If $f_k \leq g$ a.e. for all k,where $\int g < +\infty$ or it is integrable. Then

$$\int_C f_k \to \int_C f.$$

Proof:

By *Fatou's lemma*, $\int_C f = \int_C (\lim_{k\to\infty} \inf f_k) \leq \lim_{k\to\infty} \inf \int_C f_k$. Then apply *Fatou's lemma* to the nonnegative functions $g - f_k$, obtaining

$$\int_C (g-f) = \int_C \liminf (g-f_k) \leq \liminf \int_C (g-f_k) = \int_C g - \limsup \int_C f_k.$$

Combining the above two inequalities completes the proof.

The integral of a measurable f

Theorem 2.4

Let $f \geq 0$ defined on a measurable set C. Then $\int_C f$ exists if and only if f is measurable.

Moreover, $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Since f^+, f^- are nonnegative and measurable, therefore, $\int_C f^+$ and $\int_C f^-$ exist and are nonnegative, possibly having value $+\infty$.Then define

$$\int_C f(x)dx = \int_C f(x)^+ dx - \int_C f(x)^- dx$$

As mentioned before, f is integrable if $\int_C f$ is finite. Usually we write $f \in L(C)$, $L(C) = \{f : \int_C f\}$ is finite.

Theorem 2.5

Let f be measurable on C. f is integrable over C if and only if $|f|$ is.

M.C.T & Uniform Convergence Theorem

Let $\{f_k\}$ be a sequence of measurable functions on C.

(i) If $f_k \nearrow f$ a.e. on C and $\exists g \in L(C)$ such that $f_k \geq g$ a.e. on C for all k, then $\int_C f_k \to \int_C f$.

(ii) If $f_k \searrow f$ a.e. on C and $\exists g \in L(C)$ such that $f_k \leq g$ a.e. on C for all k, then $\int_C f_k \to \int_C f$.

Their proofs are alike to the proof of the nonnegative case.

Theorem 2.6 (Uniform Convergence Theorem)

Let $f_k \in L(C)$ and suppose $\{f_k\}$ uniformly converge to f on $C, \mu(C) < \infty$. Then $f \in L(C)$ and $\int_C f_k \to \int_C f$.

Fatou's Lemma

Theorem 2.7 (Fatou's lemma)

Let $\{f_k\}$ be a sequence of measurable functions on C. If $\exists g \in L(C)$ such that $f_k \geq g$ a.e. on C for all k, then

$$\int_C \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int_C f_k.$$

Corollary 2.1

Let $\{f_k\}$ be a sequence of measurable functions on C. If $\exists g \in L(C)$ such that $f_k \leq g$ a.e. on C for all k, then

$$\int_C \limsup_{k\to\infty} f_k \geq \limsup_{k\to\infty} \int_C f_k.$$

Dominated Convergence Theorem & Bounded Convergence Theorem

Theorem 2.8 (Dominated Convergence Theorem)

Let $\{f_k\}$ be a sequence of measurable functions on C such that $f_k \to f$ a.e. in C. If $\exists g \in L(C)$ such that $|f_k| \leq g$ a.e. in C, then $\int_C f_k \to \int_C f$.

Corollary 2.2 (Bounded Convergence Theorem)

Let $\{f_k\}$ be a sequence of measurable functions on C such that $f_k \to f$ a.e. in C. If $\mu(C) < +\infty$ and $\exists$ a finite constant M such that $|f_k| \leq M$ a.e. in C, then $\int_C f_k \to \int_C f$.

Outline

1. Review of Riemann Integral
2. Lebesgue Integral
3. Comparison between Lebesgue and Riemann Integral
4. Repeated integration

Dirichlet function

- The Riemann and Lebesgue integral are defined in different ways, with the latter generally perceived as the more general.
- The example that follows shows that functions exist that are Lebesgue integrable but not Riemann integrable.

Example 3.1

Consider the characteristic function of the rational numbers in $[0, 1]$,

$$\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \text{ rational,} \\ 0, & \text{o.w.} \end{cases}$$

This function, known as the Dirichlet function, is not Riemann integrable.

To shed light on that, choose an arbitrary partition Γ of the interval $[0,1]$. It turns out that $m_i = \inf \mathbb{1}_{\mathbb{Q}} = 0, M_i = \sup \mathbb{1}_{\mathbb{Q}} = 1$.

$$\overline{\int} f = 1 \neq \underline{\int} f = 0.$$

Therefore, the Riemann integral cannot exist. On the other hand, $\mathbb{1}_{\mathbb{Q}} \in L([0,1])$ turns out to be Lebesgue integrable. Then enumerate the rationals in $[0,1]$ as $\{r_0, r_1, ...\}$. Each rational number is covered by an open set G_i of size $\epsilon/2^i$. It means that $\{r_0, r_1, ...\}$ contained in the set $G_\epsilon = \cup_{i=0}^{\infty} G_i$. It is well-known that the countable union of open sets is a Lebesgue measurable set. It turns to consider the following indicator function:

$$\mathbb{1}_{G_\epsilon}(x) = \begin{cases} 1, & \text{if } x \in O, \\ 0, & \text{o.w.} \end{cases}$$

O is any open set covering $[0,1]$. Therefore,

$$0 \leq \int \mathbb{1}_Q dx \leq \int \mathbb{1}_{G_\epsilon} \leq \epsilon \to 0.$$

.

Outline

1. Review of Riemann Integral
2. Lebesgue Integral
3. Comparison between Lebesgue and Riemann Integral
4. Repeated integration

Fubini's Theorem

In general, write $\mathbb{R}^d$ as a product $\mathbb{R}^d = \mathbb{R}^{d1} \times \mathbb{R}^{d2}$, where $d = d_1 + d_2$.

Theorem 4.1

Let $f(x, y) \in L(\mathbb{R}^d), \mathbb{R}^d = \mathbb{R}^{d1} \times \mathbb{R}^{d2}$. Then for almost every $y \in \mathbb{R}^{d2}$:

(i) $f(x, y)$ is measurable and integrable on $\mathbb{R}^{d1}$.

(ii) The function defined by $\int_{\mathbb{R}^{d1}} f(x, y)dx$ is integrable on $\mathbb{R}^{d2}$.

(iii) $\int_{\mathbb{R}^{d2}}(\int_{\mathbb{R}^{d1}} f(x, y)dx)dy = \int_{\mathbb{R}^d} f$.

By symmetry, we also have that $\int_{\mathbb{R}^{d1}}(\int_{\mathbb{R}^{d2}} f(x, y)dy)dx = \int_{\mathbb{R}^d} f$.

In particular,

$$\int_{\mathbb{R}^{d2}}(\int_{\mathbb{R}^{d1}} f(x, y)dx)dy = \int_{\mathbb{R}^{d1}}(\int_{\mathbb{R}^{d2}} f(x, y)dy)dx = \int_{\mathbb{R}^d} f.$$

It turns out that *Fubini's theorem* states that the integral of f on $\mathbb{R}^d$ can be computed by iterating lower-dimensional integrals, and the iterations can be taken in any order.

Tonelli's Theorem

Theorem 4.2

Let $f(x,y)$ be nonnegative and measurable on $\mathbb{R}^d = \mathbb{R}^{d1} \times \mathbb{R}^{d2}$. Then for almost every $y \in \mathbb{R}^{d2}$, we will have the same result to *Fubini's theorem.*

(i) $f(x,y)$ is measurable and integrable on $\mathbb{R}^{d1}$.

(ii) The function defined by $\int_{\mathbb{R}^{d1}} f(x,y)dx$ is integrable on $\mathbb{R}^{d2}$.

(iii) $\int_{\mathbb{R}^{d2}}(\int_{\mathbb{R}^{d1}} f(x,y)dx)dy = \int_{\mathbb{R}^d} f$.

By symmetry, we also have that $\int_{\mathbb{R}^{d1}}(\int_{\mathbb{R}^{d2}} f(x,y)dy)dx = \int_{\mathbb{R}^d} f$.

In particular,

$$\int_{\mathbb{R}^{d2}}(\int_{\mathbb{R}^{d1}} f(x,y)dx)dy = \int_{\mathbb{R}^{d1}}(\int_{\mathbb{R}^{d2}} f(x,y)dy)dx = \int_{\mathbb{R}^d} f.$$

An application

We will show if X is a non-negative random variable, then $E[X] = \int_{\mathbb{R}^+} P(X \geq x)dx$. *Proof:*

An application

We will show if X is a non-negative random variable, then $E[X] = \int_{\mathbb{R}^+} P(X \geq x)dx$. *Proof:*

Apply *Tonelli's theorem* to the function $(\omega, x) \mapsto \mathbb{1}_{X(\omega) > x}$. One obtains

$$\int_\Omega \int_{\mathbb{R}^+} \mathbb{1}_{X(\omega) \geq x} dx dP(\omega) = \int_\Omega \int_0^{X(\omega)} dx dP(\omega) = \int_\Omega X(\omega) dP(\omega) = E(X)$$

while integrating Ω firstly and defining $A_x = \{\omega \in \Omega | X(\omega) \geq x\}$,

$$\begin{aligned}\int_{\mathbb{R}^+} \int_\Omega \mathbb{1}_{X(\omega) \geq x} dP(\omega) dx &= \int_{\mathbb{R}^+} \int_\Omega \mathbb{1}_{\omega \in A_x} dP(\omega) dx \\ &= \int_{\mathbb{R}^+} P(A_x) dx = \int_{\mathbb{R}^+} P(X \geq x) dx\end{aligned}$$

continued

For the discrete case, $E(X) = \sum_{i=1}^{\infty} P(X \geq i)$. *Proof:*

For the discrete case, $E(X) = \sum_{i=1}^{\infty} P(X \geq i)$. *Proof:*

$$E[X] = \sum_{i=1}^{\infty} ip(i) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} p(i)$$
$$= \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} p(i) = \sum_{k=1}^{\infty} P(X \geq k)$$

Intuitively, we could add up probabilities column-by-column, though, and get the same result by adding row-by-row.

Convolution

If f and g are measurable in $\mathbb{R}^n$, their convolution $(f*g)(x)$ is defined by

$$(f*g)(x) = \int_{\mathbb{R}^n} f(x-t)g(t)dt,$$

provided the integral exists. And $f*g = g*f$ simply by changes of variable.

Theorem 4.3

If $f \in L(\mathbb{R}^n)$ and $g \in L(\mathbb{R}^n)$, then $(f*g)(x)$ exists for almost every $x \in \mathbb{R}^n$ and is measurable. Moreover, $f*g \in L(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n}(f*g)dx = (\int_{\mathbb{R}^n} fdx)(\int_{\mathbb{R}^n} gdx)$$
$$\int_{\mathbb{R}^n}|f*g|dx \leq (\int_{\mathbb{R}^n} |f|dx)(\int_{\mathbb{R}^n} |g|dx)$$

proved by Tonelli's theorem and Fubini's theorem