ECON 508-A

Matrix Algebra and Linear Transformations

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The material here is based on the slides on Matrix Algebra and Linear Transformations prepared by Carmen Aston-Figari.

All errors are mine.

Section 1: Notation

Notation

Let A be a matrix of M rows and N columns.

$$A_{M\times N} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix}$$

The element in row *i* and column *j* is

$$= [a_{ij}]$$

It can also be written as a collection of N M-sized column vectors

$$= \left[a^1, a^2, \dots, a^N\right], \text{ where } a^i \in R^M$$

Section 2: Review of Matrix Algebra

Scalar multiplication

$$\alpha A = \left[\alpha a_{ij}\right]$$

Addition

$$A + B = \left[a_{ij} + b_{ij}\right]$$

Multiplication

$$A_{M\times N} \cdot B_{N\times Q} = D_{M\times Q} = \left[d_{ij}\right]$$

$$\text{where } d_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$$

Laws of Matrix Algebra

Theorem 1

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$

Proof: Omitted. □

Definition

Transpose of

$$A_{M\times N} = \left[a_{ij}\right]$$

$$A'_{N\times M} = \left[a'_{ij}\right] = \left[a_{ji}\right]$$

Theorem 2

1. $(A+B)' = A' + B'$
2. $(AB)' = B'A'$
3. $(\alpha A)' = \alpha A'\ ,\ \ \alpha \in \mathbb{R}$

Proof: Omitted. □

Square matrix

$$A_{N \times N} = [a_{ij}]$$

Diagonal matrix

$$A_{N \times N} = [a_{ij}]$$

$$a_{ij} = 0 \text{ for } i \neq j$$

Identity matrix

$$I_N = [a_{ij}]$$

$$a_{ii} = 1, a_{ij} = 0 \text{ for } i \neq j$$

$$I_M A_{M \times N} = A_{M \times N} = A_{M \times N} I_N$$

Symmetric matrix

$$A_{N \times N} = [a_{ij}]$$

$$a_{ij} = a_{ji} \text{ for every } i, j$$

$$A' = A$$

Trace

Definition

Given: $A_{N \times N'}$

$$\operatorname{tr}(A) = \sum_{i=1}^{N} a_{ii}$$

Theorem 3

Given: $A_{N \times N}$, $B_{N \times N}$

1. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$
2. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

Proof: Omitted. □

Given $A_{N\times N'}$, $|A|$ denotes the **determinant** of A.

Definition: (i,j)-th minor of A

$$A_{ij} : (N-1)\times(N-1)$$

matrix obtained by deleting row i and column j.

Definition: (i,j)-th cofactor of A

$$c_{ij}(A) = (-1)^{(i+j)}\left|A_{ij}\right|$$

Definition: (i,j)-th cofactor of A

$$|A| = \sum_{k=1}^{N} a_{ij}c_{ij}(A)$$

expansion along i-th row (k=j) or j-th col (k=i)

Let A be:

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 7 & 0 \\ 1 & 9 & 2 \end{bmatrix}$$

Exercise: Compute the determinant by performing the expansion of A both along rows and columns.

Properties of Determinants

Theorem 4

1. $|A| = \left|A^T\right|$
2. $|AB| = |A||B|$
3. $|I_N| = 1$
4. $|kA| = k^N|A|$
5. $|A| = 0$ if A has a row or col of $0'\mathrm{s}$
6. If we multiply a row (col) of A by k to get $\hat{A}$ $|\hat{A}| = k|A|$
7. If we add a multiple of a row (col) to another row (col) to get $\hat{A}$, $|\hat{A}| = |A|$

Proof: Omitted. □

Definition

Given $A_{N\times N}$, its inverse, denoted A^{-1}, is the $N \times N$ matrix such that

$$A^{-1}A = AA^{-1} = I_N$$

Theorem 5

1. A^{-1} exists $\Leftrightarrow |A| \neq 0$.
2. If A^{-1} exists, it is unique.
3. We say that A is **non-singular** if A^{-1} exists.

Proof: Omitted. □

Properties of the Inverse

Theorem 6

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $|A^{-1}| = \frac{1}{|A|}$
4. $(A')^{-1} = (A^{-1})'$

Proof: Omitted. □

Definition

The **Adjoint** of an $N \times N$ matrix A is given by:

$$Adj\,(A) = [C(A)]^{'}$$

where

$$C_{ij}(A) = \left[c_{ij}(A)\right].$$

In other words $C(A)$ is a matrix whose element i, j is the (i, j)-th cofactor of A

$$c_{ij}(A) = (-1)^{(i+j)}\left|A_{ij}\right|$$

The inverse of A, A^{-1} is given by:

$$A^{-1} = \frac{1}{|A|}Adj\,(A)$$

Inverse: Example

Let A be:

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 4 \end{bmatrix}$$

Exercise: Compute the inverse of A.

Section 3: Linear independence and simultaneous linear equations

Linear Dependence and Independence

Definition
Non-zero vectors $\{x^1, \ldots, x^K\} \in \mathbb{R}^N$ are **linearly dependent** if $\exists \;\; \alpha_1, \ldots, \alpha_K$ not all zero such that:

$$0_N = \sum_{i=1}^{K} \alpha_i x^i$$

where 0_N is the N-dimensional zero vector.

Definition
$\{x^1, \ldots, x^K\} \in \mathbb{R}^N$ are **linearly independent** if

$$0_N = \sum_{i=1}^{K} \alpha_i x^i \implies \alpha_k = 0 \;\; \forall \;\; k = 1, \ldots, K$$

Rank

Definition
The column (row) rank of $A_{M \times N}$ is the max number of linearly independent cols (rows).

$$\text{col rank}\,(A_{M \times N}) \leq N$$
$$\text{row rank}\,(A_{M \times N}) \leq M$$

Theorem 7

$\text{col rank}(A) = \text{row rank}(A) = \text{rank}(A)$

Lemma $\text{rank}(A) \leq \min\{N, M\}$
Corollary $\text{rank}\,(A') = \text{rank}(A)$

Definition

$$A_{M \times N},\; M \leq N \text{ is of full rank iff } \text{rank}(A) = M$$

Theorem 8

$|A| \neq 0 \Leftrightarrow \text{rank}\,(A_{N\times N}) = N \Leftrightarrow A^{-1}$ exists.

Proof: Omitted. □

Simultaneous Linear Equations

A linear equation:

$$a_1 x_1 + a_2 x_2 + \ldots + a_N x_N = b$$

A system of M simultaneous linear equations:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1N} x_N &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2N} x_N &= b_2 \\ &\vdots \\ a_{M1} x_1 + a_{M2} x_2 + \ldots + a_{MN} x_N &= b_M \end{aligned}$$

Can be represented in matrix terms:

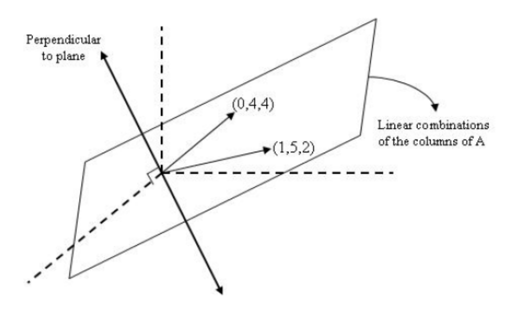
$$A_{M \times N} x_{N \times 1} = b_{M \times 1}$$

Interpretation of Ax

$$
\begin{aligned}
Ax = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N \\
\vdots \\
a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_N
\end{aligned}
$$

$$
= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{M1} \end{bmatrix} x_1 + \ldots + \begin{bmatrix} a_{1N} \\ a_{2N} \\ \vdots \\ a_{MN} \end{bmatrix} x_N
$$

$$
= a^1 x_1 + \ldots + a^N x_N
$$



$Ax = b$ can be solved iff b lies in the plane that is **spanned** by the column vectors of A .
The plane is called the **column space** of A.
The line perpendicular to the plane is called the **null space** of A.

Column space and null space

- The **column space** of a matrix $A_{M \times N}$ is the set of vectors that can be written as a linear combination of the columns of A.
- The dimension of the column space is the rank of A.
- The **null space or kernel** of A is the set of vectors x such that $Ax = 0$.
- The dimension of the null space of A is equal to $N - rank(A)$.