Econ 508A Homework

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August 5 2019

Exercise I

The set up of the problem is borrowed from Prof. Manuelli's Econ 501 slides.

Environment

The economy is populated by a large number of identical households with preferences given by:

$$u(c_1, c_2) = u_1(c_1) + \beta u_2(c_2),$$

where c_1 represents the consumption of the household in the first period of their life and c_2 is the corresponding consumption in their second life period. Assume $u_i(\cdot)$ are twice continuously differentiable and increasing.

Each household is endowed with e units of the single good available in the economy in the first period of their life. This good can be both consumed and invested in the form of capital, denoted by k. If invested, the household has access to a reproductive technology that yields $f(k)$ units of output in the second period. Assume $f(\cdot) : \mathbb{R}^2_+ \to \mathbb{R}$ is twice continuously differentiable and $f(0) = 0$.

Thus, the resource constraints and the non-negativity constraints are given by, respectively:

$$e \geqslant c_1 + k \tag{1}$$
$$f(k) \geqslant c_2 \tag{2}$$
$$c_1 \geqslant 0 \tag{3}$$
$$c_2 \geqslant 0 \tag{4}$$
$$k \geqslant 0 \tag{5}$$

Questions

Consider the following cases:

1. $u_1 = ln(c_1)$, $u_2 = ln(c_2)$, $f(k) = \alpha k \quad (\alpha > 0)$
2. $u_1 = ln(c_1)$, $u_2 = c_2$, $f(k) = \alpha k \quad (\alpha > 0)$

3. $u_1 = c_1$, $u_2 = c_2$, $f(k) = \alpha k$ $(\alpha > 0)$

Assume there is a Central Planner that has perfect knowledge of the environment. For each case, answer the following questions:

a. Express the maximization problem of the Central Planner in normal form.

b. Does the constraint qualification hold?

c. Do the sufficient conditions for an optimum hold (i.e. these are the conditions required so that the KKT condition is both necessary and sufficient)?

d. Set up the Lagrangian. Denote $\lambda_1, \lambda_2, \ldots, \lambda_5$ the Lagrange multipliers associated with constraints (1) to (5), respectively.

e. Compute the candidates for a maximum, considering the possibility of corner solutions. In particular, compute the allocations and Lagrange multipliers for the following cases:

   e.1. $c_1 > 0$, $c_2 > 0$, $k > 0$.

   e.2. $c_1 = 0$, $c_2 > 0$, $k > 0$.

   e.3. $c_1 > 0$, $c_2 = 0$, $k = 0$.

   Which of these cases is a maximum? Does it depend on the value of the endowment e? Does it depend on the value of α?

d. Denote the optimum by $x^\star = (c_1^\star, c_2^\star, k^\star)$ and show that the KKT condition at this point.