Problem Set 1

Econ 508B Summer 2020

Due on Aug 28, 2020

Problem 1 *Let X be income and Y be the savings rate. Let Z be savings (in thousands of dollars), so $Z = XY$. The savings of a family with income x_i and savings rate y_j is $z_{ij} = x_i y_j$, so that mean savings for the families in our sample is given by*

$$m_Z = \sum_i \sum_j x_i y_j p(x_i,\ y_j)$$

Will this equal $m_X m_Y$? That is, can mean savings be obtained by multiplying mean savings rate by mean income? Explain. In general, suppose $Z = f(X, Y)$. What characteristic(s) must satisfy f so that $m_Z = f(m_X, m_Y)$? Show your work.

Problem 2 *Consider two events A and B with $\Pr(A) = 0.5$ and $\Pr(B) = 0.7$. Determine the minimum and maximum values of $\Pr(A \cap B)$ and the conditions under which each is attained.*

Problem 3 *For any events A, B, and C defined on a sample space, show that*

1. *Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.*
2. *DeMorgan's Laws: $(A \cap B)^c = A^c \cup B^c$.*

Problem 4 *Prove that if A and B are mutually exclusive, then*

$$P\left(A|\left(AUB\right)\right) = \frac{P\left(A\right)}{P\left(A\right) + P\left(B\right)}$$

Problem 5 *Let $\mathcal{X} = \{1, 2, 3, 4\}$. Find the smallest σ-algebra that contains the sets $\{1\}$ and $\{1, 2, 3\}$*

Problem 6 *Show that the intersection of two σ-algebra's is a σ-algebra.*

Problem 7 *Suppose* ν *is a measure on Borel sets of* $(0, \infty)$ *and that* $\nu((x, 2x]) = \sqrt{x}$ *for all* $x > 0$. *Find* $\nu((0, 1])$

Problem 8 *Let* ν *be a measure on* $(\mathcal{X}, \Sigma)$. *Show that* f *is integrable if and only if* $\int |f| d\nu < \infty$

Problem 9 *Suppose that a sample space* $\mathcal{X}$ *has* n *elements. Show that the number of subsets that can be formed from the elements of* $\mathcal{X}$ *is* 2^n

Problem 10 *Let* X *be a continuous r.v. with pdf* $f(x)$ *and cdf* $F(x)$. *For a fixed number* x_0 *such that* $F(x_0) < 1$ *define*

$$g(x) = \begin{cases} f(x) / [1 - F(x_0)] & x \geq x_0 \\ 0 & x < x_0 \end{cases}$$

Show that $g(x)$ *is a pdf.*

Problem 11 *An electronic device has lifetime denoted by* T. *The device has a value* $V = 5$ *if it fails before* $t = 3$; *otherwise, it has a value* $V = 2T$. *Find the cdf of* V, *if* T *has pdf*

$$f_T(t) = \frac{1}{1.5} e^{-t/1.5} \ , t > 0$$

Problem 12 *Let* Z *be a discrete r.v. with pmf* $g(z)$. *We have defined the expectation of* Z *as* $\sum_i z_i g(z_i)$. *Suppose* $Z = h(X)$, *where* X *is another r.v. with pmf* $f(x)$. *Starting from the definition of* $E(Z)$, *show that in fact we can write* $E(Z) = \sum_i h(x_i) f(x_i)$.

Problem 13 *For the random variable* X *with pdf*

$$f(x) = \frac{1}{\sigma^2} x \exp\left\{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right\}$$

for $0 < x < \infty$ *and a positive number* σ^2, *find the pdf of* $Y = \exp(X)$

Problem 14 *Let X be a continuous, nonnegative r.v. ($f(x) = 0$ for $x < 0$). Show that*

$$E[X] = \int_0^\infty [1 - F_X(x)]dx$$

where $F_X(x)$ is the cdf of X.

Problem 15 *For the following random variables compute the variance: (a) X is distributed Poisson with parameter λ and (b) X is distributed Power on $[0,1]$ with parameter θ; this is, pdf is $f(x) = \theta x^{\theta-1}$ for $x \in [0,1]$, and $f(x) = 0$ elsewhere.*

Problem 16 *Suppose $X \sim U[0,3]$. Let $Z = F(X)$ where $F(\cdot)$ is the cdf of X. Find $E[Z]$. Can you generalize this result for any r.v. X with continuous pdf given by $f(x)$?*

Problem 17 *Show that if X is a continuos random variable, then*

$$m = \arg\min_a E|X - a|$$

where m is the median of X.

Problem 18 *Let $X \sim N(0,1)$. Find EX^4*