Math Camp 508B Summer 2020

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Final Exam
Due on Sep. 11, 2020, 12:00 PM, US CST

September 3, 2020

1. (**OLS**) State OLS assumptions and Gauss-Markov theorem mathematically.

2. (**OLS and Consistency**) Prove that $\hat{\beta}_{OLS} \xrightarrow{p} \beta$, please state which law or theorem you apply in its proof?

3. (**Endogeneity**) State what endogeneity is and how do you solve this issue?

4. (**Estimation**) Please implement and show at least two different methods to obtain fixed effects estimators, $\hat{\beta}_{FE}$ in static panel data model.

5. (**Econometrics**) Standard textbooks treat Normal distributions with singular covariances matrices as objects of only theoretical interest. I beg to dier. So: have you worked with these probabilities for real-if yes, where?(Note the ne print: the explicit example I gave you does not count. You should not construct an example- the question asks whether somebody else(me)? gave you random variables with singular Gaussian distribution to work with. If the purpose was simply to give an example of such a distribution, it does not count).

6. (**Applied Econometrics**) A researcher models quarterly(=4 per year) data by a model
$$y_t = \alpha + x_t'\beta + e_t \tag{O}$$
and notices some seasonal patterns. She modifies the model to
$$y_t = a_1 S_{1,t} + a_s S_{2,t} + a_3 S_{3,t} + a_4 S_{4,t} + x_t'\beta + e_t; \tag{M}$$
where the $S_{i,t}$ are seasonal dummies (i.e. $S_{1,t}$ is 1 for t=1,5,9,... and 0 for the other t, $S_{2,t}$ is 1 of t = 2,6,10,.... and 0 for the other t, etc.),

   (a) Was it a good idea to drop the parameter α in the original model (O) when going to (M)?

   (b) Let us assume that the x_t are constants. Our research is interested in the "seasonally adjusted data". So she tries to do some "seasonal adjustment", by regressing

y_t and all the $x_{i,t}$ on $(S_{1,t}, S_{2,t}, S_{3,t}, S_{4,t})'$. Let us denote the residuals $\tilde{y_t}, \tilde{x_{i,t}}$. Then one wants to look at relations between them. One can try a linear model

$$\tilde{y_t} = \tilde{x_t}'\beta + \tilde{e_t}, \tag{SA}$$

where the $\tilde{x_t} = (\tilde{x_{1,t}}, \tilde{x_{2,t}}, ...)$. Show that the covariance of $\tilde{e_t}$ is not a multiple of the identity matrix. (Hint: Use geometry).

(c) Nevertheless, our researcher now estimates β by applying the an OLS estimator to the model(SA). Show that this procedure results in an efficient estimator for β. (Hint: Compare it to an OLS estimator for the model (M)).

7. (**Repeated Integration**) *Show that*

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

8. (**Statistical theory**) *Suppose that X is a random variable with pdf*

$$p(x;\theta) = (1-\theta)\theta^x$$

for $x \in \{0, 1, 2, ...\}$ and unknown $\theta \in [0, 1)$. Consider the loss function

$$L(\theta, d) = (\theta - d)^2/(1-\theta)$$

and the decision space $\mathcal{D} = [0, 1]$.
Show the risk function $R(\theta, \delta)$ as a power series in θ.

9. (**Optimal Data Reduction**) *Suppose that $X_1, ..., X_n$ are i.i.d. Poisson random variables with mean λ and we aim to estimate $g(\lambda) = \exp(-\lambda) = P_\lambda[X = 0]$.*

- *Show that $T_1 = \mathbb{1}(X_1 = 0)$ and $T_2 = \frac{1}{n}\sum_{i=1}^n \mathbb{1}(X_i = 0)$ are unbiased estimates of $g(\lambda)$.*
- *Show that $T = \sum_{i=1}^n X_i$ is sufficient.*

10. (**Stochastic Orders**) *Show that $X_n \sim N(0, 1/n)$ is $O_p(n^{-1/2})$ and $X_n \sim \chi^2_n$ is $O_p(n)$.*

11. (**LLN & Consistency**) *Consider $\{X_i\}$ are i.i.d. $Unif[0,1]$, denote $\tilde{X}_n = (\Pi_{i=1}^n X_i)^{\frac{1}{n}}$, show that $\tilde{X}_n \xrightarrow{p} e^{-1}$.*

12. (**Asymptotics**) *Suppose X_n converges to X in probability. Show that X_n converges to X in distribution,too.*

13. (**Markov inequality**) *Suppose for each $n \in \mathbb{N}$ you have given a linear model*

$$y^{(n)} = X^{(n)}\beta + e^{(n)}$$

where $e^{(n)} = (e_1, \dots, e_n)'$. Assume $X^{(n)}$ to be a constant $n \times k$ matrices, where k remains constant. Let the e_i to be independent, and $Ee_i^2 = \sigma^2, \sup Ee_i^4 < \infty$. Let

$$\widehat{e^{(n)}} = (I - X^{(n)}(X^{(n)'}X^{(n)})^{-1}X^{(n)'})y^{(n)}$$

be the the n-th OLS-residual. Show that

$$\frac{1}{n}\widehat{e^{(n)}}'\widehat{e^{(n)}}$$

is a consistent estimator for σ^2. **Note: You are also able to use the other way to solve this question rather than apply *Markov inequality.* Remark: We do not need to make any assumption about the $X^{(n)}$ in order to achieve consistency- in contrast to the OLS estimator for β.**

14. (**Delta Method**) *Let X_n be a sequence of random variables such that for some $\theta_0 \in \mathbb{R}$, we have*

$$\sqrt{n}(X_n - \theta_0) \xrightarrow{d} N(0,1).$$

Also assume that $X_n \xrightarrow{p} \theta_0$ and g is twice differentiable, $g'(\theta_0) = 0$, and $g''(\theta_0) \neq 0$. Show that

$$n(g(X_n) - g(\theta_0)) \xrightarrow{d} ?$$

15. (**Brain Teaser**) What is the 100th digit to the right of the deicmal point in the decimal representation of $(1+\sqrt{2})^{3000}$. (Hint:$(1+\sqrt{2})^2 + (1-\sqrt{2})^2 = 6.$)