

# Math Camp 508B Summer 2020

Instructor: Hongyi Liu  
Homework 2: Due on Sep 4, 2020

August 23, 2020

1. **(Fundamentals of Measure)** Let  $\mathcal{C} = \{[0, 1]\} \cup \{[\frac{1}{2^{n+1}}, \frac{1}{2^n}) : n = 0, 1, 2, \dots\}$ . Find out which of the following sets included in  $\sigma\langle\mathcal{C}\rangle$ :

- $\{0\}, \{1\}, \{0, 1\}, \{\frac{1}{2}\}, \{\frac{1}{3}\}, [0, \frac{1}{2}], [\frac{1}{4}, 1), (\frac{1}{4}, 1], (0, \frac{1}{2})$ .

2. **(Set theory)** Prove De Morgan's Law:

3. **(Real analysis)** Find  $\limsup_{n \rightarrow \infty} C_n$  and  $\liminf_{n \rightarrow \infty} C_n$  where

$$C_n = \begin{cases} (\frac{1}{3} - \frac{1}{n+1}, 1 + \frac{1}{n+1}) & \text{if } n = 1, 3, 5, \dots \\ (\frac{1}{n+1}, \frac{2}{3} - \frac{1}{n+1}) & \text{if } n = 2, 4, 6, \dots \end{cases}$$

4. **(Inequalities)** What is Tchebysheff's inequality? And its proof?

5. **(Convergence & Inequalities)** For i.i.d. Bernoulli trials with success probability  $p = 1/2$ , let  $T_n$  denote the number of times in the first  $n$  trials that a success is followed by a failure. Denote  $I_i = I\{\text{ith trial is a success and } (i+1)\text{st trial is a failure}\}$ ,  $T_n = \sum_{i=1}^{n-1} I_i$ , then prove that  $\frac{T_n}{n} \xrightarrow{P} \frac{1}{4}$ .

6. **(MGF)** A standard example of two distinct distributions with the same moment is based on the density of lognormal distribution (Billingsley, Probability and Measure, chapter 30.)

$$f(x) = \frac{1}{\sqrt{2\pi}} 1/x \exp(-(\log x)^2/2)$$

And its perturbed density:

$$f_a(x) = f(x)(1 + a \sin(2\pi \log x))$$

Show that they have the same moments and the  $n^{\text{th}}$  moment of each of them is  $\exp(n^2/2)$ .

7. **(MGF)** Let  $M_X(t) = \frac{9}{(3-t)^2}$ ,  $E[X^n]$ ?

8. **(MLE)** Find the maximum likelihood estimator of

- $p_\theta(x; \theta) = \exp\{-(x - \theta)\}, 0 < \theta < x < \infty$ .