Central Banking Without Fiscal Support: Honest (and Dishonest) Optimal Monetary Policy*

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Abstract

In the absence of fiscal support central banks with large balance sheets can experience negative cash flows. In this paper I ask how should a central bank choose monetary policy when it faces a *temporary negative cash flow of uncertain duration*. I consider different levels of commitment to smooth price processes or, equivalently, commitment to not resort to bursts of inflation. Under full commitment to a smooth price level process (honest policy) interest rates and inflation are increasing while net income is negative, and they stabilize when net income returns to its normal value (zero). The real interest rate equals the discount factor, and the economy is Fisherian. In the absence of constraints on price level jumps (dishonest policy) the nominal interest rate is low (and constant) but there is deflation while net income is negative and real interest rates are high and increasing over time. When cash flow returns to normal the central bank defaults on its nominal debt by engineering a large surprise (but not unexpected) inflation. This policy delivers higher welfare. A hybrid policy that seeks to minimize the size of unexpected jumps in the price level (semi-dishonest policy) resembles a combination of the previous two cases. Optimal policies can display nonmonotonicities with respect to cash flows and are characterized by a negative trade off between short and long run inflation.

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1 Introduction

How should a central bank conduct monetary policy when it faces a temporary negative cash flow? Even though up until a few years ago it would have seem impossible for an *independent* central bank to be in such situation, the changes in the conduct of monetary policy in the last decade have made the case for a temporary period of negative net income a distinct possibility.

In the aftermath of the 2008 financial crisis many central banks have significantly expanded their balance sheets that now include interest earning assets and interest paying liabilities. As suggested by Hall and Reis (2015) a central bank that holds different asset types and that has issued interest earning liabilities (e.g. interest earning reserves or, as has been proposed for the ECB, interest earning bonds) faces a variety of risks, including maturity transformation risk, interest rate risk and exchange rate risk. In all cases, the key issue is to determine the implications of a negative cash flow —for example, a situation in which interest payments plus expenses exceed income from assets— for the conduct of monetary policy and the economic viability of a central bank.

Del Negro and Sims (2015) describe a model that distinguishes between the central bank balance sheet and the rest of the government budget constraint and study the conditions under which it is feasible to support a given policy that they model as a version of the interest smoothing version of the Taylor rule. They define fiscal support as a transfer from the treasury to the central bank and show conditions under which a temporary negative cash flow *does not* require fiscal support to implement the given, arbitrary, policy. They parameterize a version of their model and argue that for the U.S. the Federal Reserve Bank is not likely to need fiscal support in the near future since they estimate that the present value of seigniorage is large. Alternative assumptions about the demand for money yield different estimates of the present value of resources available to the Fed (assets plus seigniorage) and, hence, potentially different views on the sustainability of a given policy in the absence of fiscal support. Cavallo et. al. (2018) discuss the fiscal implications of alternative strategies concerning the Fed's balance sheet normalization. Reis (2015) provides a summary.

Hall and Reis (2015) also explore the sustainability of what they call a "dividend policy" which amounts to a rule that determines a transfer from the central bank to the treasury (which can be negative) as a function of the state of the economy. They emphasize that the "stability of a central bank depends crucially on what happens to its dividends when net income is negative." They study several policies that deal with strategies to manage the Fed's and the European Central Bank's balance sheet, and they emphasize different approaches to designing a financial policy with an emphasis toward what they view as a *financially stable policy* defined as a strategy that rules out explosive paths of interest paying liabilities issued by the central bank. They point out that they do not view their model as a useful way to study the inflationary consequences of negative net income. They state that "a small loss for the Fed would require very large increases in inflation." They claim that higher seigniorage is associated with a loss of central bank independence.

Benigno and Nisticò (2015) describe a variety of neutrality and non-neutrality results associated with open market operations in a context where the central bank has a large balance sheet. They show that neutrality obtains only under relatively stringent conditions and that, if the central banks face the risk of a negative net income it is likely that a traditional monetary policy will require fiscal support in the form of transfers from the treasury to the central bank.

A common starting point of this literature is the existence (or the possibility) of a negative cash flow. Given this primitive assumption, existing research explores the implications of different transfer (or dividend) schemes, and of given monetary policy rules for inflation and interest rates. Even though the motivation for the negative net income in the literature is related to the potential losses that a central bank can incur given that, effectively, it behaves as a hedge fund, knowledge of the actual source of the negative net income does not appear to be essential to study the optimal policy. The basic problem is *how* to finance a negative cash flow and this can be separated from the reason *why* the central bank finds itself in this predicament when it comes to analyzing policy options.

Since the defining feature of the problem is the existence of a negative cash flow, the design of the optimal monetary policy applies to *non-independent* central banks as well to the extent that they are required to "monetize" some or all of the deficit incurred by the rest of the government. For example, in early 2016 the Argentinean central bank agreed to transfer to the treasury a certain amount of resources over the following twelve months to finance the central government deficit. This, from the point of view of the bank's balance sheet is a negative net income shock and the problem faced the Central Bank of Argentina was how to finance such a transfer.[1] Ignoring differences in institutions, the situation faced by the Central Bank of Argentina (and many other central banks is emerging markets) is similar to the potential problems that could be faced by the Fed or the ECB as a consequence of their unconventional monetary policies.

[1] For additional details see Manuelli and Vizcaino (2017).

In this paper I complement the existing literature on the impact of negative net income on prices and economic activity. Instead of asking whether a specific monetary policy rule (or a dividend policy) is sustainable in the face of negative net income, I study what is the *optimal response* of a central bank to a *temporary* negative net income that will last for a *random* length of time.

Since I rule out any form of fiscal support my setting violates the neutrality conditions in Wallace (1981) and Sargent and Smith (1987). By assumption, in my framework any monetary policy will have an impact on inflation and interest rates. When the problem is narrowly defined as how to finance a temporary negative net income shock by a central bank whose only source of revenue is seigniorage then the policy problem is clearly defined, and it boils down to how to choose the stochastic process for seigniorage in an optimal way.

I study the case in which the monetary authority can issue two types of *nominal liabilities*: a non-interest earning asset (money) and an interest earning bond (either interest-earning reserves or a T-Bill like asset). I limit myself to the case in which these choices do not have an impact on private consumption. There are many ways to describe the policy choices faced by the monetary authority but, as mentioned above, they all boil down to selecting the policy that minimizes the present value of distortions associated with inflation.

To understand the essence of the problem is easiest to consider first the case of a *permanent* transfer of *known* value. In this case —and if the real side of the economy is constant— the optimal policy is one of printing money to finance the payments. The reason is simple: There is no gain associated with smoothing over time the distortions associated with inflation since the environment is completely stationary.

When the transfer is *temporary* and the length of the transfer period is deterministic —which implies that the present value of the transfer is *known*— the answer depends on the assumptions that are made about the central bank's ability to commit. Uribe (2016) shows that the optimal policy is to keep the inflation rate constant forever. During the negative net income period the monetary authority issues bonds and, when the net income returns to zero, it uses the revenue from printing money to finance the interest on the existing debt. The optimal path of inflation is slightly different if the monetary authority has limited commitment. In the case that the monetary authority can default at the time that net income returns to its normal level, Manuelli and Vizcaino (2017) show that inflation is high for as long as the central bank's net income is negative and it drops to zero when net income is normalized.

I this paper I consider the case in which the negative net income —which I sometimes label a transfer— can be either high or low. The first phase is characterized by high negative net income and, although the level of net income is certain, the duration of this phase is *uncertain*. I assume that when the high transfer (negative net income) phase ends the economy returns to its normal level (zero net income).

In order to determine the central bank's optimal policy it is necessary to be explicit about the constraints that it faces. As usual in monetary models with nominally denominated assets the monetary authority has a huge incentive to engineer a once and for all large inflation that wipes out its liabilities whenever that is possible. Thus, to make progress on the nature of "good" policies it is necessary to lay out the constraints —imposed outside the model— about the (implicit) default opportunities. In addition to the constraints on default I restrict the central bank to policies in which the nominal interest rate is no lower than the discount factor. This implies that in a steady state with zero net income inflation is zero.[2]

I find that under what Auernheimer (1974) labeled an "honest" policy —which corresponds to a policy that displays no jumps in the price level— the optimal policy is characterized by increasing inflation and an increasing ratio of bonds to money during the period in which the central bank's net income is negative. The amount of seigniorage falls short of the required transfer and the difference is financed by selling interest earning liabilities. The reason for this less than full monetization lies in the temporary and random nature of the negative net income: As soon as the period of negative net income ends the central bank can distort individual choices (relative to the first best) and raise enough revenue to service the debt issued in the first phase. As time goes by and the high transfer phase continues, total liabilities of the central bank keep increasing and this requires higher interest rates and higher inflation. When the phase of negative net income ends, the central bank engages is a reverse open market operation: it lowers interest rates and inflation, and the real demand for money increases. The economy displays standard Fisherian features and the real interest rates is always equal to the discount factor.

At the other end on the commitment continuum is the case in which there are *no restrictions on implicit default*. Put differently, the central bank is free to follow a policy that results in an arbitrary jump in the price level at the time that the central bank's net income returns to its normal —normalized to zero in the model— level. I label such a policy as a "dishonest" government policy. It displays a form of *strong commitment* in terms of timing —since the central bank is restricted from creating a large inflation during the negative net transfer period— and *weak commitment* in terms of implicit default rates —since the central bank can inflate away some or all

[2]Of course this rules out the Friedman rule but allowing permanent deflation would not change the qualitative results but would make the presentation somewhat more cumbersome.

of its liabilities when net income returns to its normal level.

In this case, the optimal monetary policy is such that the nominal interest rate is constant and equal to the discount factor permanently. During the high transfer (negative income) period the inflation rate is negative and decreasing and the ex-post real interest exceeds the discount factor and is increasing. The reason for this is that during this phase the real value of the liabilities of the central bank is growing and the optimal policy calls for defaulting on this debt —at the random time at which net income returns to its normal level— by creating a large inflation. In order for the private sector to hold the debt issued by the central bank it has to be compensated in the form of a high (and increasing over time as the size of the default increases) real return. In the long run, inflation is zero.

I label a third case as a policy of *limited commitment,* a "semi-dishonest" policy[3]. As in the previous case, I assume that the central bank is restricted to a form of strong commitment in terms of timing (no default until net income returns to normal) but if the monetary authority finds it necessary to default, then it chooses the *smallest* default rate, that is, the smallest decrease in the value of its liabilities consistent with feasibility. I view this as a form of limited commitment.

The optimal monetary policy in this case blends elements of the two extreme cases: During the negative net income phase the nominal interest rate and the inflation rate are weakly increasing. Moreover, even though the central bank liabilities are increasing without bound there is a range of possible values that have the property that are unreachable in the context of a policy that guarantees zero default but, if reached under the policy that allows defaults, do not trigger an actual default. For sufficiently high levels of central bank debt —which will happen with positive probability— inflation is at the level that maximizes seigniorage and constant and real interest rates increase over time. If the duration of the negative net income period is long enough that the economy has issued a large amount of debt, then there is a burst of inflation at the time net income normalizes. In the long run, inflation is constant but much higher than in the case of dishonest policy.

In a final section I extend the basic setting to allow for multiple stochastic increases in (negative) net income. In this case, and depending on the details about the timing of commitment there is a (trivial) degree of indeterminacy of the dishonest policy. The honest policy is such that interest rates increase for a while only to decrease each

[3] I chose "semi-dishonest" rather than "semi-honest" since following the much used (and abused) Tolstoian analogy there is only one way for the government to be honest but there are infinite ways —indexed by the size of the default– in which it can behave in a dishonest manner in the context of this model.

time that a new (higher) level of negative net income is realized. Thus, even though there is a clear sense that the level of transfers that the central bank has to make is weakly decreasing, the optimal policy displays non-monotonicities.

The paper most closely related to this is the work by Diaz-Gimenez et. al. (2008) where it is also shown that time consistency limits the ability of the monetary authority to implement optimal policies. The key difference with this paper is that the path of deficits is not decreasing (and random), and that there is coordination between the fiscal and monetary authorities which implies that taxes other than the inflation tax are used to finance the service of the debt. Martin (2013) has a thorough discussion of optimal policies in economies with different frictions. He restricts himself to the case in which bonds have payoffs in real terms. In the context a the New Keynesian models, Leeper et. al. (2016) discuss optimal monetary and debt policy imposing time consistency. For a recent survey of monetary models and optimal policies see Canzoneri (2011)

2 The Economy

2.1 The Environment

I study the simplest model that allows me to capture the best strategy to finance a temporary transfer. Net income is given by $-x_t$ where

$$x_t = \begin{cases} x & t \in [0, T] \\ 0 & t \geq T \end{cases} , \tag{1}$$

where $x > 0$, and T is random and has an *exponential distribution* with parameter η. This implies that the expected duration of the negative net income phase is $1/\eta$. In section 5 I extend the model to allow for multiple phases of increasing net income. I label the random period of time in which $x_t > 0$ phase I, while the period in which $x_t = 0$ phase II.

I assume a pure exchange economy. Income per period is given by y, and consumption is $c = y - g$, where g is government spending. I simplify the real side of the economy so that I can concentrate on the optimal monetary policy. Extending the model to accommodate correlation between the negative net income period and real variables is part of ongoing research.

2.2 Households

I assume that there is a representative dynasty that has preferences over consumption and real money balances given by

$$U = E\left[\int_0^\infty e^{-\rho t}\left[u(c_t) + v(m_t)\right] dt\right],$$

where c_t is consumption at time t and $m_t = M_t/P_t$ is real money balances. The functions u and v are assumed strictly concave and twice continuously differentiable. The private sector budget constraint is given by

$$dM_t + dB_t^T + dB_t^M = \left(i_t(B_t^T + B_t^M) + P_t y_t - P_t c_t - P_t \tau_t\right) dt + d\tilde{N}_t, \tag{2}$$

where $\tilde{N}_t$ is the Poisson process that captures the (potential) jump in the price level associated with the end of the transfer period, that is, a potential jump that occurs at time T .

On the income side the representative dynasty earns interest at the nominal rate i_t on its holdings of treasury-issued debt, B_t^T, as well as the stock of monetary authority-issued debt, B_t^M.[4] In addition, the representative household earns income, spends resources purchasing consumption and pays taxes (if $\tau_t < 0$, the household receives a transfer). The notation emphasizes that it is possible to separate the policy decisions made by the treasury —choices of B_t^T and τ_t— from those made by the central bank.

It useful to consider what happens if the economy is in the first phase (that is $x_t = x$), and hence there are no jumps in the price level. In this case, the appropriate version of equation (2) is

$$\dot{M}_t + \dot{B}_t^T + \dot{B}_t^M = i_t(B_t^T + B_t^M) + P_t y_t - P_t c_t - P_t \tau_t,$$

where a dot denotes time derivative. Let W_t be total financial wealth. Thus,

$$W_t = B_t^T + B_t^M + M_t,$$

and with this notation the budget constraint in phase I is

$$\dot{W}_t = i_t W_t - i_t M_t + P_t y_t - P_t c_t - P_t \tau_t.$$

[4] If one views B_t^M as interest earning reserves, then money, M_t, is the set of liabilities that are used in transactions. For a transactions based model that derives a stable demand see Lucas and Nicolini (2013).

Let lower case letters denote real values of every variable. The real version of the budget constraint is then

$$\dot{w}_t = (i_t - \pi_t)\, w_t - i_t m_t + y_t - c_t - \tau_t.$$

What is the potential impact of the arrival of phase II? Since the only state variable is wealth, a jump in the price level can affect the real value of wealth. Let the post jump price level be denoted P'_T and the pre-jump simply $P_T = P_{T^-}$ which corresponds to the left limit of the price process. Then real wealth before and after the jump are

$$w'_T = \frac{W_T}{P'_T}, \text{ and } w_T = \frac{W_T}{P_T},$$

This implies that

$$w'_T = w_T \frac{P_T}{P'_T}.$$

In order to avoid specifically using the (somewhat cumbersome) left limit notation I will denote

$$w_T = w',\, w_{T^-} = w,\;\; P_T = P' \text{ and } P_{T^-} = P.$$

The stochastic process for wealth is then

$$dw_t = ((i_t - \pi_t)\, w_t - i_t m_t + y_t - c_t - \tau_t)\, dt + ((i_t - \pi_t)\, w'_t - (i_t - \pi_t)\, w_t) dN_t. \quad (3)$$

The problem faced by the representative consumer is simply

$$U = E\left[\int_0^\infty e^{-\rho t}\left[u(c_t) + v(m_t)\right] dt\right],$$

subject to equation (3), where the expectation is taken over the realization of the time of the first jump of the Poisson process.

Before analyzing the optimal monetary policy it is necessary to derive the optimal behavior on the part of the private sector since the household's decision rules are part of the constraints faced by the central bank.

The HJB equation corresponding to the household's optimization problem is

$$\begin{aligned} \rho H(w) &= \max_{c,m} \{u(c) + v(m) + H'(w)\,((i - \pi)\, w - im + y - c - \tau) \\ &\quad + \eta\left[H(w\frac{P}{P'}) - H(w)\right]\right\}. \end{aligned}$$

The first order conditions are

$$u'(c) = H'(w),$$

and

$$v'(m) = H'(w)i.$$

Thus, the demand for money is simply given by

$$\frac{v'(m)}{u'(c)} = i. \tag{4}$$

Equation (4) completely summarizes optimal behavior on the part of the household. As such, it must be imposed as a constraint on the problem solved by the monetary authority.

Since consumption is constant so is $H'(w)$. This implies that, in equilibrium, $H''(w) = 0$. Differentiating the HJB equation with respect to w and imposing the envelope condition we get that

$$(\rho + \eta) H'(w) = H'(w) (i - \pi) + \eta H'(w \frac{P}{P'}) \frac{P}{P'}.$$

However, since equilibrium consumption is constant, it follows that

$$H'(w) = H'(w') = H'(w \frac{P}{P'})$$

and the previous equation implies that

$$i = \rho + \eta (1 - P/P') + \pi,$$

and, hence, that the demand for money is

$$\frac{v'(m)}{u'(c)} = \rho + \eta (1 - P/P') + \pi. \tag{5}$$

In this economy the realized real interest rate *contingent on no arrival* of the phase II (normal net income) is $r = \rho + \eta (1 - P/P')$. It follows that the *higher the expected jump in the price level the higher the realized real interest rate.* If, on the other hand, the optimal policy is restricted to continuous paths of the price level (that is, no default), then the real interest rate is equal to the discount factor. Moreover, it also follows that once the economy enters phase II the real interest rate satisfies $r = \rho$.

2.3 Central Bank

To determine the optimal monetary policy I need to be explicit about the objective of the central bank as well as the constraints that it faces. What is the monetary authority's budget constraint? Here, as the literature correctly emphasizes, it is important to separate the central bank's balance sheet from the consolidated government's budget constraint. The key difference is that, in this model, the central bank's only source of income that can be used to service any interest paying liability is seigniorage. It is possible, with some complication of notation, to add earnings from the assets in the central bank's portfolio to its budget constraint. In this case, the policy that I describe implies that whenever income is positive it is transferred to the treasury, while negative net income has to financed using seigniorage. The only qualitative difference with a model that allows for an explicit return on the portfolio is that the limit on central bank's liabilities must take into account the future earnings on its assets.[5] Thus, in what follows I use a simplified version of the central bank's budget constraint that embeds this transfer policy.

The central bank's budget constraint after time T, that is, in phase II ($x_t = 0$) is

$$\dot{M}_t + \dot{B}_t^M = i_t B_t^M .$$

With a slight abuse of notation I use W_t to denote total liabilities of the central bank. $W_t = M_t + B_t^M$. The budget constraint is then given by

$$\dot{W}_t = i_t W_t - i_t M_t,$$

or, in real terms,

$$\dot{w}_t = (i_t - \pi_t)\, w_t - i_t m_t.$$

Since $i_t - \pi_t = \rho$ and $i_t = v'(m_t)/u'(c)$, then the phase II budget constraint that incorporates optimal behavior by the private sector is

$$\dot{w}_t = \rho w_t - \frac{v'(m_t)m_t}{u'(c)}. \tag{6}$$

A similar derivation shows that the budget constraint faced by the monetary authority in phase I (that is when net income is negative) is simply

$$\dot{w}_t = \rho w_t + x - \frac{v'(m_t)m_t}{u'(c)}. \tag{7}$$

[5] A more involved exercise would allow the central bank to accumulate precautionary reserves or to liquidate some good assets to make up the negative net income. However, I ignore this possibility since my objective is to determine how to deal with negative net income rather than how to prevent the possibility of negative cash flow.

3 Optimal Policy: Phase II

It is convenient to work backwards to characterize the policy. Let time T be the realization of the random duration time corresponding to phase I. The central bank is faced with a stock of liabilities, w_T and wants to maximize the utility of the representative agent. Since consumption is constant and there is no residual uncertainty after time T, the problem faced by the monetary authority is simply

$$\max_{m_t} \int_T^\infty e^{-\rho(t-T)} v(m_t) dt,$$

subject to

$$\dot{w}_t = \rho w_t - \frac{v'(m_t)m_t}{u'(c)}, \text{ with } w_T \text{ given.}$$

At this point I need to make some assumption about the demand for money function to render the problem tractable. To simplify notation let

$$z(m) \equiv \frac{v'(m)m}{u'(c)}.$$

Condition 1 *Assume that the function $z(m)$ is such that $z(0) = 0$, there exists a unique m^* such that $z(m^*) = \rho m^*$, and there is a value $m^+ \in (0, m^*)$ such that $z(m^+) > z(m)$ for all $m \neq m^+$ (and $z'(m^+) = 0$). Moreover, $z'(m) > 0$ for $m < m^+$, and $z'(m) < 0$ for $m > m^+$.*

Condition 2 *Let*

$$\phi(m) \equiv \frac{v'(m)}{z'(m)} = \frac{z(m)}{mz'(m)} u'(c),$$

and assume that $\phi'(m) > 0$.

The economic interpretation of these conditions is straightforward. The assumptions about the demand for money summarized in $z(m)$ amount to assuming that the seigniorage function is single peaked. In this notation m^* is the first best real money balances. It is the policy that guarantees zero inflation and a nominal interest rate equal to the discount factor.[6]

[6] In this economy, the Friedman rule holds. The optimal policy calls for deflation. However, since the objective is to understand the timing of inflation it is simpler to pick zero inflation as the best option. This is also the long run inflation rate that is most commonly prescribed in monetary papers (see Diercks (2017)).

The assumption that $\phi'(m) > 0$ implies that the relative prudence coefficient of the function $v(m)$ is greater than the elasticity of marginal utility of money balances minus one. In particular any prudence coefficient greater than or equal to minus one satisfies this condition.

Let's now determine the optimal policy after time T. It is immediate that, if feasible, it is optimal to keep real money balances constant since the function $v(m)$ is strictly concave. This requires that $\dot{w}_t = 0$ in equation (6) Hence, for an arbitrary $w_T = w$, define $\hat{m}(w)$ as the largest value of m —if it exists— such that

$$\rho w = z(\hat{m}(w)). \tag{8}$$

Note that because $v'(m) > 0$, $\hat{m}(w)$ is greater than $\hat{m}(w + x/\rho)$ which is the value of real money balances that guarantee that $\dot{w} = 0$ in phase I as determined by equation (7).

The value of the problem in phase II is given by

$$\hat{V}(w) = \frac{v(\hat{m}(w))}{\rho}, \tag{9}$$

and

$$\hat{V}'(w) = \frac{v'(\hat{m}(w))}{\rho}\hat{m}'(w) = \frac{v'(\hat{m}(w))}{z'(\hat{m}(w))} = \phi(\hat{m}(w)), \tag{10}$$

where the second equality uses the value of the derivative $\hat{m}'(w)$ from equation (8).

Under what conditions is this a feasible policy? Let

$$\rho w^+ = z(m^+) \tag{11}$$

denote the maximum level of total liabilities that can be financed. Thus, for any $w_T \leq w^+$, the optimal policy is as described above. What happens if $w_T > w^+$? In this case the central bank must default on its debt to reduce its value to (at least) w^+. Can the monetary authority do this? Yes. Since its liabilities are nominally denominated the central bank can increase the money supply to pay off part of the debt. This will result in a high level of inflation and in the reduction of the real value of the debt. This is the sense in which a central bank cannot be insolvent: it has a simple way of defaulting by reducing the real value of its liabilities via inflation.

4 Optimal Policy: Phase I

In order to study the optimal monetary policy it is important to be explicit about the constraints faced by the monetary authority. If the central bank is required to be "honest" in the sense of Auernheimer (1974) —which he defined as choosing a policy that keeps the price level from jumping at any time— then this forces the monetary authority to avoid high inflations at random times. The alternative is to let the central bank implement a policy that, depending on the state of the economy, decreases the real value of its liabilities.

4.1 "Honest" Monetary Policy

As indicated above, if the price level is a continuos function of time (no jumps at time T), then so is the level of total liabilities. In terms of the notation in the previous section we impose that

$$w' = w.$$

This, in turn, implies that $P = P'$ and that the real interest rate is $r = \rho$. The law of motion for the liabilities of the central bank implies that the maximum sustainable level of total debt is given by w^1 which is defined by

$$\rho w^1 + x = z(m^+).$$

If $w > w^1$ then $\dot{w} > 0$ and it is unbounded. This implies that, for sufficiently long T, w_T will exceed the maximum feasible level of liabilities consistent with no default in the low transfer phase. This maximum level is given by w^+ that is defined in equation (11). Thus, it is *not feasible* for an honest government to let the level of total liabilities exceed w^1 during the period in which net income is negative. This must be the case even though $w^1 < w^+$, that is, the maximum value of liabilities is strictly below the maximum sustainable without default. As indicated before, the reason for this tighter cap is that for higher levels of debt it is not possible to guarantee that total debt will be bounded. If the central bank followed a policy that results in $w_t \in (w^1, w^+)$ at some time t, then such a policy has a non-zero probability of requiring a default when the economy switches to the normal, zero net income, phase.

The HJB equation that characterizes the value of the program solved by the central bank is[7]

$$\rho V(w) = \max_m \left\{ v(m) + V'(w) \left(\rho w + x - z(m) \right) + \eta \left[\hat{V}(w) - V(w) \right] \right\} \tag{12}$$

The optimal choice of real money balances satisfies

$$\frac{v'(m)}{z'(m)} = \phi(m) = V'(w). \tag{13}$$

[7] Existence of a solution follows from standard arguments. I assume that the value function is C^2 but the conditions for htis need to be verified.

This equation defines $m(w)$. It follows that

$$\phi'(m)m'(w) = V''(w)$$

Since in this phase all quantities are deterministic it is useful to view them as functions of time. Totally differentiating both sides when both w and m are viewed as functions of time we get —after imposing the first order conditions— that

$$(\rho + \eta) V'(w)\dot{w} = V''(w)\dot{w} (\rho w + x - z(m)) + V'(w)\rho\dot{w} + \eta\hat{V}'(w)\dot{w}.$$

Since

$$\rho w + x - z(m) = \dot{w},$$

then either $\dot{w} = 0$ or $\dot{m} = m'(w)\dot{w}$

$$\dot{m} = \eta \frac{\left(V'(w) - \hat{V}'(w)\right)\dot{w}}{\phi'(m)} = \eta \frac{\phi(m) - \phi(\hat{m}(w))}{\phi'(m)}. \tag{14}$$

Thus, the optimal policy is given by the solution of a pair of ordinary differential equations

$$\dot{w}_t = \rho w_t + x - z(m_t) \tag{15}$$

and

$$\dot{m}_t = \eta \frac{\phi(m_t) - \phi(\hat{m}(w_t))}{\phi'(m_t)}. \tag{16}$$

subject to two conditions: The initial (given) level of liabilities, w_0,[8] and the condition that $\lim_{t\to\infty} w_t = w^1$. Note that this requires that

$$\lim_{t\to\infty} \eta \frac{\phi(m_t) - \phi(\hat{m}(w_t))}{\phi'(m_t)} = 0$$

so that real money balances have to be constant if liabilities are constant as well. Given the assumptions about $\phi(m)$ the previous condition is satisfied (and $w_t = w^1$) if and only if $m_t \to m^+$ as

$$\lim_{m_t\to m^+} \eta \frac{\phi(m) - \phi(\hat{m}(w))}{\phi'(m)} = 0,$$

for any feasible w

[8] As in any monetary model the Central Bank has an incentive to engineer a large inflation at $t = 0$ to erase as much debt as possible. We view w_0 as the level of debt after whatever degree of default —in the form of higher inflation— has been achieved.

Proposition 3 (Honest Optimal Policy) *Any optimal policy during the high transfer phase has the property that, if $w < w^1$, then $\dot{m}_t < 0$ and $\dot{w}_t > 0$. Moreover $w_t \to w^1$ and $m_t \to m^+$.*

Proof. In order to study the dynamics implied by the system of equations (15)-(16) it is convenient to consider two loci. The first corresponds to $\dot{w}_t = 0$. This defines a locus $\alpha(w; x)$ given by

$$\alpha(w; x) = \{(w, m) : z(m) = \rho w + x\}.$$

The second locus of interest is the set of pairs (w, m) such that $\dot{m}_t = 0$. The construction in this case is a little more delicate. Consider first the situation when $m_t < m^+$. From equation (16) and the monotonicity of $\phi(m_t)$, $\dot{m}_t = 0$ if and only if $m_t = \hat{m}(w_t)$. This defines the locus $\alpha(w; 0))$ given by

$$\alpha(w; 0) = \{(w, m) : z(m) = \rho w\}$$

which is, effectively, a translation of $\alpha(w; x)$. As I argued before, the condition that $w_t \to w^1$ requires that $m_t \to m^+$. However, $m_t \to m^+$ implies that $\dot{m}_t \to 0$. Thus, the locus of points such that $\dot{m}_t = 0$ has a discontinuity at $m = m^+$. It is given by $\alpha(w; 0)$, for $m < m^+$, and it collapses to m^+ when $w = w^1$.

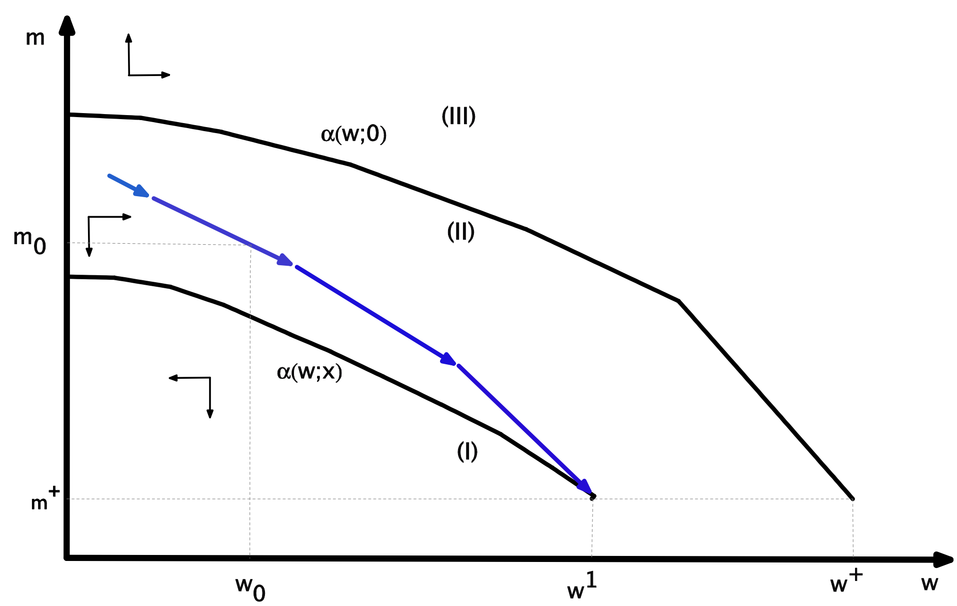

Figure 1: "Honest" Monetary Policy Dynamics

Figure 1 describes the dynamics associated with the system of ordinary differential equations. There are three regions in the (m, w) plane. I claim that a solution to the problem cannot enter either regions I or III. To see this, note that if the solution entered region I, the level of real money balances would, eventually, fall below m^+ and this implies that the candidate solution would be on the wrong side (the upward sloping side) of the Laffer curve. This is inconsistent with maximizing behavior. On the other hand, if the candidate solution enters region III then the stock of central bank liabilities increases without bound in the high transfer phase (phase I) and this violates feasibility since, for t long enough, w_t will exceed w^1.

It follows that any solution must remain in region II. Qualitatively the dynamic properties of the optimal policy are given by a manifold in (m, w) space similar to the blue path in Figure 1. ■

The proposition characterizes the optimal monetary policy when net income is negative. For a given initial level of liabilities, w_0, the optimal policy picks an initial level of real money balances (m_0 in Figure 1) and from then on, the level of liabilities increase and real money balances are decreasing. This implies that the nominal interest rate given by

$$\frac{v'(m_t)}{u'(c)} = i_t$$

is increasing. The inflation rate is also increasing and given by

$$\pi_t = i_t - \rho.$$

The economy displays the standard Fisherian result: Inflation and nominal interest rates move one for one.

Along the optimal path total liabilities of the central bank are increasing. Since real money balances are decreasing, the ratio of bonds to money has an upward trend. Under the optimal policy the central bank relies increasingly on debt issues to finance the negative cash flow. Thus, along the optimal path the central bank is actively engaged in open market operations that change the composition of its liabilities between non-interest bearing money and interest bearing reserves (or bonds).

Why is inflation increasing? Both Uribe (2016) and Manuelli and Vizcaino (2017) find that optimal inflation (at least within a phase) is constant. The reason for that result is that, as shown by Barro (1979), optimal smoothing of distortions over time requires that, whenever feasible, the deadweight loss associated with inflation is minimized when the wedges are constant.

However, when there is uncertainty about the size (or, equivalently, duration of the transfer) there is an option value associated with picking initially "too low" an inflation rate, which implies that the level of seigniorage falls short of financing need (the value of the negative net income). The reason is that the central bank chooses higher real money balances (and lower interest rates and relatively smaller debt-money ratio) taking into account that there is a positive probability —constant in my Poisson setting— that the transfer will be small (short duration). As times goes by, the optimal policy relies on increasing debt issues as the best option to smooth out the distortion associated with non-constant real money balances. under the honest policy there is an upper bound on the total liabilities of the central bank.

As indicated above, at (random) time T the central bank engages in an open market operation that reverses the previous trend: Real money balances increase and nominal interest rates decrease as the ratio of interest earning debt to money decreases. The long term inflation depends on the level of central bank liabilities at time T. The longer the duration of the high transfer phase the higher the accumulated central bank debt and the higher the level of seigniorage (and hence inflation) that must be raised to service that debt.

4.2 "Dishonest" Monetary Policy

In this section I allow for the possibility that the monetary authority will choose to engineer a short-lived burst of inflation at the time that net income returns to zero. In this case it is clear that the monetary authority can attain the (restricted) first best: Real money balances can be kept at $m = m^*$ and hence the nominal interest rate equals $i = \rho$ at all times.

Since it is possible for the price level to jump at time T, then in phase II the government can choose $w' = m^*$ which corresponds to the highest allowable (best) level of real money balances. If total government liabilities at time T (before the switch) are equal to w and since $w' = w \times (P/P')$, it follows that the jump in the price level at time T is given by

$$\frac{P'}{P} = \frac{w}{m^*}.$$

To complete the description of the equilibrium it is necessary to determine the evolution of the central bank's liabilities. During the negative net income phase, total liabilities evolve according to

$$\dot{w}_t = \rho w_t + x - z(m^*).$$

If at time $t = 0$ the central bank was following a policy consistent with zero inflation this implies that $w_0 = m^*$. In this case the solution for the path of total liabilities during the high transfer phase is

$$w_t = \left(m^* - \frac{x}{\rho}\right) + \left(\frac{x}{\rho}\right) e^{\rho t}.$$

Thus, total liabilities increase at a geometric rate. The longer the duration of the high transfer phase the higher the level of total liabilities at the time of the switch and, consequently, the higher the burst of instantaneous inflation.

It is of interest to trace the behavior of inflation in this equilibrium. Recall that optimality on the part of the household implies that the nominal interest rate is given by

$$i = \rho + \eta\left(1 - P/P'\right) + \pi,$$

which in this case corresponds to

$$i = \rho + \eta\left(1 - m^*/w\right) + \pi,$$

Since in this equilibrium $i = \rho$, it follows that inflation is given by

$$\pi_t = \eta\left(m^*/w_t - 1\right) < 0.$$

Thus, the optimal policy induces deflation and, in the limit as $t \to \infty$, the inflation rate equals $-\eta$.

During the negative net income phase the realized real interest rate —that is, the ex-post real interest rate conditional on no switching to phase II— is

$$r_t \equiv i - \pi_t = \rho + \eta\left(1 - m^*/w_t\right).$$

Thus, the real interest rate during the transition is higher than the discount factor and it is *increasing* over time. The higher return compensates the household for the capital loss associated with a switch to the low transfer phase that is associated with the central bank defaulting on its debt in real terms. To an outsider who does not have the correct model, the behavior of the nominal interest rate and inflation would look "non-Fisherian" since those two variables do not move one for one. Rather, the lack of perfect synchronization is due to the decrease in the real interest rate along an equilibrium path. Again, to an outsider the real interest rate would be identified from the change in consumption and the discount factor. Since all those objects are constant along an equilibrium path, the uninformed conclusion would be that the real interest rate is constant.

In this case the monetary authority decreases the nominal quantity of money over time so that the real value is constant in the face of continuous decreases in the price level. These decreases in the stock of money are instrumented by sales of interest earning bonds. Thus, along the equilibrium path the ratio of bonds to money is increasing.

At the time that the economy switches to the low transfer phase, the policy requires a jump in the price level —a burst of inflation— that decreases the real value of liabilities to the level consistent with zero inflation in the long run.

I summarize this discussion in the following proposition

Proposition 4 (Dishonest Optimal Policy) *The optimal monetary policy is such that real money balances are constant in both phases and equal to m^*. The nominal interest rate is also constant and equal to the discount factor. The stock of central bank liabilities increases geometrically during phase I and drops to m^* at the beginning of phase II and remains constant thereof. In phase I the real interest rate exceeds the discount factor and is increasing and the inflation rate is negative and decreasing. In phase II inflation is zero and the real interest rate equals the discount factor.*

Proof. The discussion above describes the construction. Note that if $m_t = m^*$ is feasible then this is the optimal policy since —given the constraint to keeping the nominal interest rate no lower than the discount factor— it corresponds to the first best. Thus, to prove the result all I need to do is to check that the household is willing to hold the central bank debt which, given the choice of zero inflation increases without bound in phase I. However, the household is willing to hold any asset with nominal risk free return given by $i = \rho + \eta (1 - P/P') + \pi$. Setting $i = \rho$ the implications for inflation and real interest rates follow. ■

The result might appear somewhat surprising as complete default supports the first best. The extreme welfare property depends heavily on the assumption of independence of the real sector from the policy choice. To see this, note that in phase I the real interest rate on safe investments is different form the discount factor (and increasing). This implies that, in an economy with investment opportunities, the option of default implies that investment is lower relative to the honest policy. Thus, the welfare comparison in this case will depend on the specification of the technology. The two policies —honest and dishonest— are characterized by positive and negative properties. More work is needed to handle the general case.

4.3 "Semi-Dishonest" Monetary Policy

In this section I study an intermediate case: I assume that the monetary authority can default —in the sense of engineering a jump in the price level— at the time that the economy switches to the low transfer phase but that the size of the default must be the minimum necessary to ensure feasibility. In other words, if at time T total liabilities are w_T, then the post-switch liabilities are w' with

$$w' = \min\{w_T, w^+\},$$

where, as defined before, w^+ is the highest level of total liabilities that can be financed raising seigniorage. We find that giving the monetary authority the extra flexibility associated with default enlarges the set of values of the liabilities of the central bank that *do not trigger* a default.

To be precise, in the case that the monetary authority is restricted to never default, the level of liabilities during the negative net income phase can never exceed w^1 which is strictly less than w^+. However, when default is an option, the monetary authority will implement a lower interest rate (and lower inflation) policy relative to the no default case and when total liabilities lie in the set (w^1, w^+), a switch to the low transfer phase will not trigger a default. Rather, the central bank that is constrained to minimize the size of the implicit default will increase long run inflation to raise enough seigniorage to service the existing debt.

The optimal policy implies that if the duration of phase I is long enough —an event with positive probability— the level of liabilities of the central bank will exceed w^+. In that case, and for any $w_t \geq w^+$, the monetary authority chooses a constant interest rate independent of w_t and, when the switch occurs it defaults by increasing the price level so that the real value of its liabilities is w^+, the maximum sustainable in the low transfer phase.

The formal result is in the following proposition

Proposition 5 (Semi-Dishonest Optimal Policy) *The optimal policy in phase I is given by*

1. *If $w < w^+$, then $\dot{m}_t < 0$ and $\dot{w}_t > 0$. In this region, the inflation rate is given by*

$$\pi_t = \frac{v'(m_t)}{u'(c)} - \rho > 0,$$

the nominal interest rate is

$$i_t = \rho + \pi_t,$$

and total liabilities of the central bank evolve according to

$$\dot{w}_t = \rho w_t + x - \frac{v'(m_t)m_t}{u'(c)}$$

2. *If $w \geq w^+$, then $\dot{m}_t = 0$ and $\dot{w}_t > 0$. In this region the inflation rate, the nominal interest rate and the evolution of central bank debt are*

$$\pi_t = \frac{v'(m^+)}{u'(c)} - \rho - \eta\left(1 - w^+/w_t\right),$$

$$i_t = \rho + \eta\left(1 - w^+/w_t\right) + \pi_t,$$

$$\dot{w}_t = (\rho + \eta)\, w_t - \eta w^+ + x - \frac{v'(m^+)m^+}{u'(c)}.$$

Proof. In the case $w < w^+$ the same arguments used in Proposition 3 apply to this case. A possible path is displayed in Figure 2 in red. It differs from the honest optimal policy that is shown in blue since it only requires that

$$m_t \rightarrow m^+ \text{ as } w_t \rightarrow w^+,$$

since this is the bound that minimizes the difference between w_T —the level of liabilities when the economy switches to phase II— and w^+ the highest level that can be financed in phase II. If real money balances converged to a value greater than m^+ this gap would be larger. If real money balances were below m^+, the economy would be on the upward sloping part of the Laffer curve and this cannot be optimal.

I conjecture that the value function corresponding to the region $w \geq w^+$ is an *affine* function of w. To see this note that in this region of the state space

$$\hat{V}(w) = \frac{v(\hat{m}(w^+))}{\rho}$$

is independent of w. Thus, imposing the conjecture on the functional form of the value function in the HJB equation we obtain that

$$(\rho + \eta)(V_0 + V_1 w) = \max_m \left\{ v(m) + V_1\left((\rho + \eta)\, w - \eta w^+ + x - z(m)\right) + \eta \frac{v(\hat{m}(w^+))}{\rho} \right\}.$$

Under this conjecture real money balances are constant and given by

$$\frac{v'(m^{++})}{z'(m^{++})} = \phi(m^{++}) = V_1.$$

Since both m^{++} and V_1 are endogenous I now show to to determine them.

Imposing the optimal choice of m be m^{++} on the HJB equation above we get that V_0 must satisfy

$$(\rho+\eta)\, V_0 = v(m^{++}) + V_1\left(-\eta w^{+} + x - z(m^{++})\right) + \eta\frac{v(\hat{m}(w^{+}))}{\rho}.$$

Let $V(w)$ be the solution to the HJB equation for $w \leq w^{+}$. For the continuation affine value function —the conjectured function $V_0 + V_1 w$— to be the actual solution I need to impose both the value matching and the smooth pasting conditions at $w = w^{+}$. These conditions amount to imposing that

$$\phi(m^{++}) = V_1 = V'(w^{+}) = \phi(m(w^{+})) = \phi(m^{+}),$$

where the first equality is the first order condition derived above, the second is the smooth pasting condition and the other two follow from the optimal path in Figure 2. Thus, the optimal choice of money balances is $m^{++} = m^{+}$ and $V_1 = \phi(m^{+})$.

The HJB equation satisfied by the function V evaluated at $w = w^{+}$ is

$$(\rho+\eta)\, V(w^{+}) = v(m^{+}) + V_1\left(\rho w + x - z(m^{+})\right) + \eta\frac{v(\hat{m}(w^{+}))}{\rho},$$

and it follows that, given our choice of V_1,

$$V(w^{+}) = V_0 + V_1 w^{+},$$

which shows that the value matching condition is automatically satisfied. ■

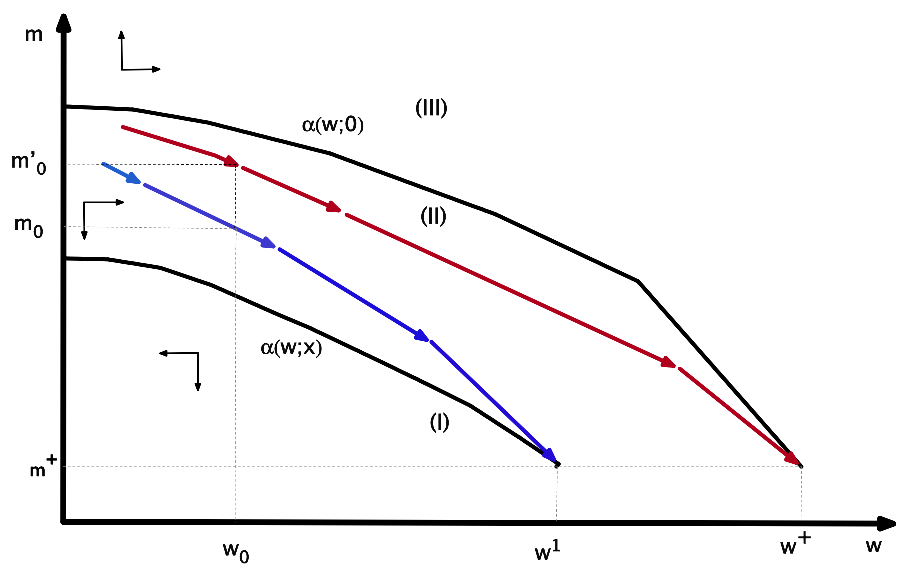

Figure 2: Optimal Policies. Blue ("Honest") and Red ("Semi-Dishonest")

Even though when the total value of central bank liabilities is low the qualitative properties of the honest and semi-dishonest solutions are similar, the government that has the option of minimal default will choose higher real money balances, lower nominal interest rates and, hence, lower inflation conditional on the level of total liabilities. In Figure 2 the red path —which is also the equilibrium relationship between total central bank liabilities, w, and real money balances, m— shows that for any level of w, the optimal choice of real money balances under the semi-dishonest policy (on the red stable manifold) is strictly higher than under the honest policy (the blue manifold). For a given initial level of liabilities, w_0, the honest policy requires that the central bank set real money balances at m_0 while the semi-dishonest policy permits the central bank to select a higher level of real money balances, m_0', and, consequently lower nominal interest rates. The reason for this is that the semi-dishonest government has the option —even though it is not always exercised— of partially defaulting on its nominal liabilities by engineering a bout of inflation at the time that phase II arrives (see Figure 2). Formally, note that when $w = w^1$ the honest government must choose $m = m^+$. However, the semi-dishonest government can afford —it is optimal— to let total liabilities increase and, hence, will select $m > m^+$. This argument extends to all $w < w^1$.

5 Multiple Phases

In this section I extend the model to allow for multiple phases. I restrict the presentation to just two but it is very clear how to extend the analysis to any finite number of rounds. As before, net income is $-x_t$ where x is given by

$$x_t = \begin{cases} x_0 & t \in [0, T_0] \\ x_1 & t \in [T_0, T_1], \\ 0 & t \geq T_1, \end{cases}$$

where $x_1 < x_0$ and, as before, I maintain the Poisson assumption with parameters η_0 and η_1. Thus, the environment is such that an initially high level of transfers (of uncertain duration) will be followed by a lower level (also of uncertain duration) and then a return to the normal (zero) net cash flow.

In the case of a dishonest government the optimal policy is exactly the same as in the case of one phase in terms of the choice of real money balances: the optimal choice is $m_t = m^*$. In order to fully describe the policy it is necessary to specify any constraints on the timing of default. The most natural extension of the one phase case is to permit a default only when net income has returned to normal (i.e. $x = 0$). In this case the solution is as described in the one phase case.

If the ability to default is extended to *every time* that there is a change in net income then the model has a continuum of equilibria. The reason is simple: Any combination of default at $t = T_1$ and $t = T_2$ is an equilibrium. The particular selection matters for the time path of inflation and real interest rates and without further restrictions, the model has few implications for observables. Put differently, there isn't a strong case for reading properties of a monetary policy from looking at nominal interest rates and inflation in this setting. The economy is very non-Fisherian.

The implications are tighter in the case of an honest government. Let $V_j(w)$ be the central bank's value function when $x_t = x_j$. Then $V_1(w)$ coincides with the value function described in the case of the honest policy with just one level of x (if $x = x_1$). The optimal path described in Figure 1 in blue is also the optimal choice of real money balances given total liabilities which I denoted $m_1(w)$. The key property is that $m_1(w) < \hat{m}(w)$. Thus, if T_0 is the realization of the time at which net income changes from $-x_0$ to $-x_1$ and w_{T_0} is the level of central bank total liabilities at that time, the continuation optimal honest policy is as described in Proposition 3: Inflation and interest rates are positive and increasing for as long as $w_t \leq w^1$. At time T_1 the central bank engages in the same reverse open market operation I described in section 4.1. What happens during the first phase, that is, for $t \in [0, T_0)$? A derivation that follows the same logic of the proof of Proposition 3 shows that the relevant system of differential equations is

$$\dot{w}_t = \rho w_t + x_0 - z(m_t) \tag{17}$$

and

$$\dot{m}_t = \eta \frac{\phi(m_t) - \phi(m_1(w_t))}{\phi'(m_t)}. \tag{18}$$

subject to two conditions: The initial (given) level of liabilities, w_0,[9] and the condition that

$$m_t \to m^+ \text{ as } w_t \to w^0,$$

where

$$\rho w^0 = z(m^+) - x_0 < \rho w^1 = z(m^+) - x_1.$$

[9] As in any monetary model the Central Bank has an incentive to engineer a large inflation at $t = 0$ to erase as much debt as possible. We view $\tilde{w}$ as the level of debt after whatever degree of default —in the form of higher inflation— has been achieved.

Figure 3 displays the functions $m_0(w)$ (in dark green) and $m_1(w)$ (in blue). Moreover, it also shows one possible realization (in green) that assumes that the first phase ends when $w_0 = w_{T_0}$ and the second phase ends when $w_1 = w_{T_1}$.

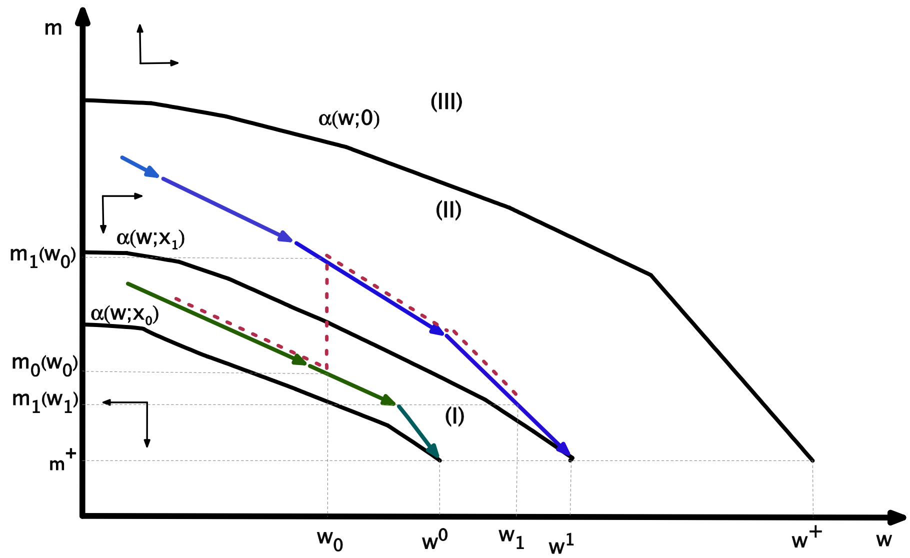

Figure 3: "Honest" Policy with Mutliple Phases

During the first phase real money balances decrease (and total liabilities increase) until time T_0. Right before the switch from the $-x_0$ to the $-x_1$ regime real money balances are $m_0(w_0)$. At $t = T_0$, the central bank —responding to the good news that the level of transfers has decreased— performs an open market operation that increases real money balances (decreases nominal interest rates) from $m_0(w_0)$ to $m_1(w_0)$. From then on, the path follows the blue line. At $t = T_1$, cash flow returns to normal (zero). In Figure 3 w_{T_1} is denoted w_1. At that point, The new level of real money balances is given by $\hat{m}(w_1)$ (not shown) that is greater than $m_1(w_1)$.

money balances (a decrease in the interest rate) from $m(w_1)$ to $m^+(w_1)$. From then on, inflation and nominal interest rates are constant.

It is interesting to note that even though net income is in a very clear sense weakly increasing (transfers are weakly decreasing) inflation and interest rates do not display a monotone path. Figure 4 shows that time path of inflation (nominal interest rates are simply $\rho + \pi_t$).

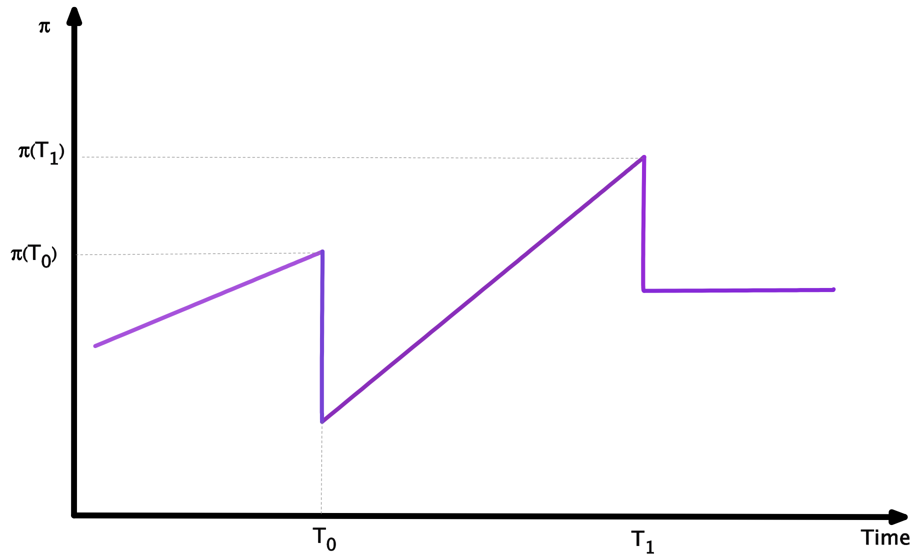

Figure 4: Time Path of Inflation

As in the case of a single stage, the optimal policy initially displays low nominal interest rates and low inflation. As time goes by and the central bank finds itself in the need to keep making transfers, it chooses a higher nominal interest policy as well as a higher ratio of interest bearing liabilities to non-interest bearing money.

At the time that "good news" arrive in the form of higher (less negative) net income, the central bank adjusts its portfolio discretely: it repurchases some bonds by issuing money. Alternatively, this action can be described as lowering the nominal interest rate that decreases the demand for interest bearing debt and increases the demand for non-interest bearing money. After the decrease in the nominal interest rate the central bank keeps increasing the nominal yield on its interest bearing liabilities and accommodates higher inflation. This pattern continues until time T_1. At the point the net income is zero and the central bank picks a nominal interest rate that is consistent with raising sufficient seigniorage to finance the interest payments on the interest earning liabilities.

6 Optimal Unpleasant Monetarist Arithmetic

In this section I study the trade off between inflation in the long run and inflation in the short run. Since this trade off does not arise in the case of possible default I concentrate on the "honest" government case. Moreover, to keep the analysis simple

I consider the two phase case.

As in the previous analysis, real money balances in Phase II satisfy

$$\rho w = z(\hat{m}(w)),$$

while inflation during this phase (which I view as long run inflation) is a function of the stock of central bank liabilities at the (random) time that the economy switched from Phase I to Phase II. Formally, $\pi^L(w)$ is given by

$$\rho + \pi^L(w) = \frac{v'(\hat{m}(w))}{u'(c)}. \tag{19}$$

So if the central bank commits to a policy that limits that size of its liabilities, say a policy that sets that in Phase I $w_t \leq \bar{w} \leq w^1$, this corresponds to committing to an upper bound on the level of long run inflation.

Let $\bar{w}' < \bar{w}$, then this corresponds to a commitment to lower inflation in the long run since $\pi^L(\bar{w}') < \pi^L(\bar{w})$. What happens in Phase I? Let inflation in the short run as a function of the central bank liabilities under a particular cap $\bar{w}$ be denoted $\pi^S(w; \bar{w})$, then the model implies that $\pi^S(w; \bar{w}') > \pi^S(w; \bar{w})$, that is, there is an intertemporal trade off between short run and long run inflation. The intuition is that the expected present discounted value of the transfer has to be financed by seigniorage. A lower level of inflation in the long run is associated with smaller seigniorage. This decrease must be compensated with a higher level of seigniorage in the short run and this requires that the inflation tax be higher. Note that

$$\rho + \pi^S(w; \bar{w}) = \frac{v'(m(w; \bar{w}))}{u'(c)},$$

where $m(w; \bar{w})$ is the optimal level of real money balances in Phase I when the current level of central bank liabilities is w and the upper bound to which the central bank is commits is $\bar{w}$. Then, $\pi^S(w; \bar{w}') > \pi^S(w; \bar{w})$ if and only if $m(w; \bar{w}') < m(w; \bar{w})$. The following proposition summarizes the result.

Proposition 6 *Let the maximum level of liabilities under an honest policy be $\bar{w} \leq w^1$. The function $m(w; \bar{w})$ is increasing in $\bar{w}$.*

Proof. Define the function $H(m, w)$ by

$$H(m, w) \equiv v(m) + \phi(m)(\rho w + x - z(m)),$$

where $\phi(m) = v'(m)/z'(m)$ under an optimal policy. It follows that if m is chosen optimally

$$\frac{\partial H}{\partial m}(m, w) = \phi'(m)(\rho w + x - z(m)).$$

Thus, since I assumed that $\phi'(m) > 0$ then H is increasing in m whenever the liabilities of the central bank are increasing as well, that is, $\rho w + x - z(m)$. It is straightforward to argue that in an optimal solution this must be the case. The argument is similar to that of Proposition 3. The dynamics of the solution can be described using Figure 1 since the two loci that describe the inaction regions for the state and costate variables —$\alpha_1(m)$ and $\alpha_2(m)$— do not depend on the upper bound on w. The only difference is that w_t cannot exceed $\bar{w}$ which can be strictly less than w^1. Thus, the same argument used in Proposition 3 shows that the solution must stay in region II, and this implies that $\rho w + x - z(m(w; \bar{w})) > 0$.

Let $\bar{w}' < \bar{w}$. Since the $\bar{w}'$ economy is more constrained than the $\bar{w}$ economy the value of the central bank problem in the $\bar{w}'$ economy ($V(w; \bar{w}')$) is lower than the value in the $\bar{w}$ economy ($V(w; \bar{w})$). Since the HJB equation (12) can be written as

$$V(w; \bar{w}) = H(m(w; \bar{w}), w) + \eta \hat{V}(w),$$

it follows that

$$V(w; \bar{w}') < V(w; \bar{w}) \Leftrightarrow H(m(w; \bar{w}'), w) < H(m(w; \bar{w}), w).$$

Since the function $H(m, w)$ is monotone increasing in m, this implies that $m(w; \bar{w}') < m(w; \bar{w})$ which completes the proof. ■

Under the optimal policy there is no free lunch: the price of lower inflation today is higher inflation tomorrow. In particular governments that "revise" their long run inflation targets in the direction of accepting higher inflation (and higher nominal interest rates) can implement a policy of lower interest rates (and lower inflation) in Phase I.

7 Concluding Comments

What are the lessons for the central banks that could find themselves in a situation in which net income is negative and that there is no forthcoming fiscal support? The first lesson is that the ability to issue interest bearing liabilities —reserves in the case of the Fed and also some proposals sympathetic to the possibility that the ECB will issue Eurobonds (see, for example, Corsetti et. al. (2016))— implies that there is a disconnect between inflation and the need to transfer resources to the rest of the economy if the central bank chooses its policy optimally. The central bank's ability to issue both interest bearing and non-interest bearing liabilities is an essential feature of the optimal policy.

The structure of the "honest" optimal monetary policy shows that uncertainty about the duration of the negative cash flow phase creates an option value for keeping inflation low that, as a first approximation, seems contrary to the principles of intertemporal smoothing of distortions. A central bank that needs to finance a given level of transfers to some other sector, in this case it does not matter whether the other sector is the private banking industry —the potential situation that the ECB and the Fed can find themselves in— or some other government agency —this is the case corresponding to a central bank that chooses to finance part of the government deficit, as in the case of Argentina—, should start with a policy of relatively low nominal interest rates that will increase as the period with negative cash flow continues. Upon reversion to normal times —that is to zero net income— the central bank picks the lowest interest rate consistent with financing its liabilities.

The "dishonest" optimal policy —which means that the central bank is allowed to engineer large surprise inflations if necessary— implies a much lower (and constant) nominal interest rate, continued deflation and high (and increasing) real interest rates. When the net income flow returns to its normal level the optimal policy creates a large inflation that dilutes the real value of the debt to the level that it consistent with the low nominal interest rate.

In the intermediate case —the central bank can create a sudden burst of inflation but it is restricted to the smallest possible change in the price level— there are a continuum of equilibria depending on the details of when the defaults associated with jumps in the price level can occur.

The optimal policy viewed as an interest rate policy does not resemble a Taylor rule. In the case of the honest policy (no surprise inflation) the nominal interest rate is an increasing function of the real value of the total liabilities of the central bank. An outside observer just looking at the data will find a positive relationship between inflation and nominal interest rates. In the case of the honest policy the model implies that $i = \rho + \pi$ so that the coefficient on the observed policy rule is one. In the case of a dishonest policy the coefficient is zero since inflation keeps decreasing and the nominal interest rate is unchanged.

Finally, under the "honest" optimal policy if the central bank can commit to a maximum level of interest rates in the long run (Phase II), then increases in this level are associated with decreases in the interest rate in the short run (Phase I). Under the optimal policy the monetary arithmetic is unambiguously unpleasant.

References

[1] Auernheimer, L., 1974, "The Honest Government's Guide to the Revenue from the Creation of Money," *Journal of Political Economy*, Vol. 82, No. 3, pp. 598-606 "

[2] Barro, Robert, 1979, "On the Determination of the Public Debt." *Journal of Political Economy,* 87(6): 940-971.

[3] Benigno, P. and S. Nisticò, 2015, "Non-Neutrality of Open Market Operations," CEPR Discussion Paper N0. 10594.

[4] Canzoneri, M., R. and B. Diba, 2011, "The Interaction Between Monetary and Fiscal Policy," In Benjamin M. Friedman, and Michael Woodford, editors: **Handbook of Monetary Economics**, Vol. 3B, The Netherlands: North-Holland, 2011, pp. 935-999.

[5] Cao, Q, 2015, "Optimal Fiscal and Monetary Policy with Collateral Constraints," manuscript.

[6] Cavallo, M., M. Del Negro, W. S. Frame, J. Grassing, B. A. Malin and C. Rosa, 2018, "Fiscal Implications of the Federal Reserve's Balance Sheet Normalization," Federal Reserve Bank of New York, Staff Report No. 833 (January)

[7] Corsetti, G., L. Dedola, M. Jarocziński, B. Maćkowiak and S. Schmidt, 2016, "Macroeconomic Stabilization, Monetary-Fiscal Interactions, and Europe's Monetary Union," ECB working paper No. 1988.

[8] Del Negro, and C. Sims, 2015, "When Does a Central Bank's Balance Sheet Require Fiscal Support?" *Journal of Monetary Economics*, 73, pp: 1-19.

[9] Díaz-Giménez, J., Giovannetti, G., Marimon, R. and P. Teles, 2008, "Nominal Debt as a Burden on Monetary Policy," *Review of Economic Dynamics*, Volume 11, Issue 3, pp: 493–514.

[10] Diercks, A. M., 2017, "The Reader's Guide to Optimal Monetary Policy," working paper.

[11] Leeper, E.,L. Campbell and L. Ding, 2016, "Optimal Time-Consistent Monetary, Fiscal and Debt Maturity Policy," working paper.

[12] Lucas,R. Jr. and Nicolini, Juan, 2013. "On the stability of money demand," 2013 Meeting Papers 353, Society for Economic Dynamics.

[13] Manuelli,R. and J.I. Vizcaino, 2017,"Monetary Policy with Declining Deficits: Theory and an Application to Recent Argentine Monetary Policy," *Federal Reserve Bank of Saint Louis Review*, 99(4), pp: 351-375.

[14] Martin, F., 2013, "Government Policy in Monetary Economies," *International Economic Review*, Volume 54, Issue 1, pp: 185–217.

[15] Reis, R., 2015, "Comment on: 'When Does a Central Bank's Balance Sheet Require Fiscal Support?'," *Journal of Monetary Economics*, 73, pp: 20-25.

[16] Sargent, T. and B. Smith, 1987, "Irrelevance of Open Market Operations in Some Economies with Government Currency Being Dominated in Rate of Return," *American Economic Review*, 77, pp: 78-92.

[17] Sargent, T. and N. Wallace, 1981, "Some Unpleasant Monetarist Arithmetic," Federal Reserve Bank of Minneapolis *Quarterly Review*, Fall, pp:1-17.

[18] Uribe, Martin, 2016, "Is the Monetarist Arithmetic Unpleasant?," NBER working paper 22866.

[19] Wallace, N., 1981, "A Modigliani-Miller Theorem for Open Market Operations," *American Economic Review*, 71 (3), pp: 267-74.