Moderate Expected Utility

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Main Results: two representation theorems

\( \varphi : \mathbb{Z}^2 \to [0,1] \) binary choice rule on a finite \( \mathbb{Z} \).

Theorem 1. \( \varphi \) is a MUM if and only if it satisfies MST+.

\( \varphi : \Delta^2 \to [0,1] \) binary choice rule on lotteries over \( \mathbb{Z} \).

Theorem 2. \( \varphi \) is a MEM if and only if it is non-constant, continuous, linear, convex, and satisfies MST+. 
Halff’s (1976) formula: Moderate Utility Model (MUM) probability of choosing $x$ over $y$

$$
\rho(x, y) = F\left[ \frac{\mu(x) - \mu(y)}{d(x, y)} \right]
$$

- utility difference
- strictly increasing
- dissimilarity (distance metric)

Testable implication: choice is moderately transitive
Transitivity is observable, directly testable

- Single notion of transitivity in deterministic models

- Different strengths of transitivity in random choice

If \( p(a, b) \geq \frac{1}{2} \) and \( p(b, c) \geq \frac{1}{2} \), then

\[
\begin{align*}
p(a, c) & \geq \frac{1}{2} \\
p(a, c) & \geq \min \{ p(a, b), p(b, c) \} \quad (WST) \\
p(a, c) & \geq \max \{ p(a, b), p(b, c) \} \quad (MST)
\end{align*}
\]
Useful compromise: accommodate common failures of SST; retain more empirical bite than WST.

Ex: \( Z = \{a, b, c\} \)

\[
\begin{align*}
\rho(a, b) &> \rho(b, c) > \rho(a, c) > \frac{1}{2} \\
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\rho(a, c) &> \rho(b, c) > \rho(a, b) > \frac{1}{2}
\end{align*}
\]
Failures of SST and degree of comparability

L.J. Savage’s example:

\[ \text{Trip to Rome} \quad \text{vs. Trip to Paris:} \quad \rho(R, P) = \frac{1}{2} \]

\[ \text{Trip to Rome + €5} \quad \text{vs. Trip to Rome:} \quad \rho(R^+, R) = 1 \]

\[ \text{Trip to Rome + €5} \quad \text{vs. Trip to Paris:} \quad \rho(R^+, P) = ? \]

SST requires \( \rho(R^+, P) \geq 1 \) **unreasonable**

MST only requires \( \rho(R^+, P) \geq \rho(R, P) \)
Evidence

Mellers, Chang, Birnbaum & Ordoñez (1992):

“WST and MST are often satisfied, although a few exceptions have been noted ...
... while SST is frequently violated”

Early examples: Chipman (1960)

Tversky & Russo (1969)

Recent examples: Lea & Ryan (2015)

Soltani, De Martino, Camerer (2012)
Data example I: Lea and Ryan (2015)
Female Tüngara Frogs
Mating Decisions

\begin{align*}
\text{speed} & \\
\text{attractiveness} & \\
\end{align*}

\begin{align*}
\text{WST} & \checkmark \\
\text{MST} & \checkmark \\
\text{SST} & \times \\
\end{align*}
Data example II: Soltani, De Martino, Camerer (2012)
Male Caltech students
Choice over money lotteries
Data example III: Tversky and Russo, 1969

Prisoners in Detroit

Perceptual task: choose larger rectangle

90% 50%

77%
Useful compromise: accommodate common failures of SST; retain more empirical bite than WST.

Ex: \( Z = \{a, b, c\} \)

\[
\begin{align*}
\rho(a, b) &> \rho(b, c) > \rho(a, c) > \frac{1}{2} \\
\rho(b, c) &> \rho(a, b) > \rho(a, c) > \frac{1}{2} \\
\rho(a, b) &> \rho(a, c) > \rho(b, c) > \frac{1}{2} \\
\rho(b, c) &> \rho(a, c) > \rho(a, b) > \frac{1}{2} \\
\rho(a, c) &> \rho(a, b) > b, c) > \frac{1}{2} \\
\rho(a, c) &> \rho(b, c) > \rho(a, b) > \frac{1}{2}
\end{align*}
\]
Proposition 1. \( \lim_{n \to \infty} \#MST(n)/\#WST(n) = 0. \)
$Z \neq \emptyset$ finite set of choice options

$\rho: Z^2 \rightarrow [0,1]$, $\rho(x, y) + \rho(y, x) = 1$ choice rule

Def: $\rho$ is a moderate utility model (MUM) if

- $\exists \mu: Z \rightarrow \mathbb{R}$ utility
- $\exists d: Z^2 \rightarrow \mathbb{R}^+$ distance metric
- $\exists F: [-T, T] \rightarrow [0,1]$ strictly increasing, $F(t) = 1 - F(-t)$

such that for all $x \neq y$,

$$\rho(x, y) = F \left( \frac{\mu(x) - \mu(y)}{d(x, y)} \right).$$

Easy implication: $\rho(x, y) \geq \frac{1}{2}$ iff $\mu(x) \geq \mu(y)$. 

$\rho(x, x) = \frac{1}{2}$
Ex: Thurstone's binary Probit is a MUM

\[(U_x)_{x \in \mathbb{Z}} \sim \mathcal{N}(\mu, \Sigma), \quad \Sigma \text{ full rank}\]

\[p(x | y) = \Pr\{ U_x > U_y \} \]

\[= \Pr \left\{ \frac{U_x - U_y - (\mu_x - \mu_y)}{\text{Std}(U_x - U_y)} > -\frac{(\mu_x - \mu_y)}{\text{Std}(U_x - U_y)} \right\} \]

\[= \Phi \left( \frac{\mu_x - \mu_y}{\text{Std}(U_x - U_y)} \right) \]

\[= F \left[ \frac{\mu(x) - \mu(y)}{d(x, y)} \right] \]
Ex: Tversky's Elimination-by-Aspects is a MUM

Every a mapped to a set of “aspects” \( A \); \( m \) is a measure

\[ m(a) = m(A) \]

\[ d(a,b) = m(A \setminus B) + m(B \setminus A) \]

\[ F(t) = \frac{1}{2} + \frac{1}{2} t \]

\[ \Rightarrow \rho(a,b) = \frac{1}{2} + \frac{1}{2} \left( \frac{m(A) - m(B)}{m(A \setminus B) + m(B \setminus A)} \right) \]

\[ = \frac{m(A \setminus B)}{m(A \setminus B) + m(B \setminus A)} \]
Ex: a function of angle in attribute space

\[ \rho(x, y) = F \left( \frac{\mu(x) - \mu(y)}{d(x, y)} \right) \]
Strengthen MST to MST+

\[ [\text{MST}] \quad \rho(a, b) \land \rho(b, c) \geq 1/2 \]
\[ \Rightarrow \rho(a, c) > \rho(a, b) \land \rho(b, c) \]

\[ [\text{MST+}] \quad \rho(a, b) \land \rho(b, c) \geq 1/2 \]
\[ \Rightarrow \begin{cases} 
\rho(a, c) > \rho(a, b) \land \rho(b, c) \\
\text{OR}\\
\rho(a, c) = \rho(a, b) = \rho(b, c)
\end{cases} \]
Theorem 1. A choice rule $\rho$ on a finite $Z$ is a MUM if and only if it satisfies MST$^+$. 

Steps of the proof:

$(\Rightarrow)$ Easy.

$(\Leftarrow)$ Construct $u, d$

Show $d$ is a metric

Show $u, d$ represent $\rho$
(⇒) Step 1 (Halff, 1976): Every MUM satisfies MST

Proof: Suppose MST fails \[ \left\{ \begin{array}{l} \rho(x, y) \land \rho(y, z) \geq \frac{1}{2} \\ \rho(x, z) < \rho(x, y) \land \rho(y, z) \end{array} \right. \]

\[ m(x) - m(z) < d(x, z) \cdot \left[ \frac{m(x) - m(y)}{d(x, y)} \land \frac{m(y) - m(z)}{d(y, z)} \right] \]

\[ \leq \left[ d(x, y) + d(y, z) \right] \cdot \left[ \frac{m(x) - m(y)}{d(x, y)} \land \frac{m(y) - m(z)}{d(y, z)} \right] \]

\[ \leq d(x, y) \cdot \left[ \frac{m(x) - m(y)}{d(x, y)} \right] + d(y, z) \cdot \left[ \frac{m(y) - m(z)}{d(y, z)} \right] \]

\[ = m(x) - m(z) \quad (⇒⇔) \]
(⇒) Step 2: Every MUM satisfies MST+.

Let \( p(x,y) \land p(y,z) \geq 1/2 \).

By Step 1, \( p(x,z) \geq p(x,y) \land p(y,z) \).

Suppose \( p(x,z) = p(x,y) \land p(y,z) = p(x,y) \). Then,
\[
\mu(x) - \mu(y) + \mu(y) - \mu(z) = d(x,z) \left[ \frac{\mu(x) - \mu(y)}{d(x,y)} \right]
\]
\[
\leq [d(x,y) + d(y,z)] \left[ \frac{\mu(x) - \mu(y)}{d(x,y)} \right]
\]
\[
= \mu(x) - \mu(y) + d(y,z) \left[ \frac{\mu(x) - \mu(y)}{d(y,z)} \right].
\]
Construct \( m \) and \( d \):

\[
\text{MUM} \iff \text{MST} \Rightarrow \text{WST} \Rightarrow \exists m: \mathbb{Z} \to \{1, \ldots, k\} \text{ onto such that}
\]

\[
\rho(x, y) \geq \frac{1}{2} \iff m(x) \geq m(y)
\]

Order pairs \( x \neq y \) by \( |\rho(x, y) - \frac{1}{2}| \):

\[
\rho(x^1, y^1) > \rho(x^2, y^2) > \rho(x^3, y^3) > \ldots > \rho(x^m, y^m) > \frac{1}{2}
\]

\[
i \quad 1 \quad 2 \quad 3 \quad \ldots \quad m
\]

\[
D_i \quad 0 \quad 1 \quad (n-1) \quad (n-1)^{m-2}
\]

Let \( d(x, y) := \begin{cases} 0, & x = y \\ C_i, & x \neq y, \rho(x, y) = \frac{1}{2} \\ \left(\frac{C}{2} + D_i\right)|m(x) - m(y)|, & \rho(x, y) = \rho(x^i, y^i) > \frac{1}{2} \end{cases} \]

\[
C_i = (n-1)^{[n(n-1)/2 + 1]}
\]
(MUM \iff \textit{MST+}) Show \ d(x, z) \leq d(x, y) + d(y, z)

\[
\begin{align*}
d(x, y) &= \begin{cases} 
0, & x = y \\
C, & x \neq y, \ v(x, y) = \frac{1}{2} \\
(C/2 + D_i) |m(x) - m(y)|, & v(x, y) = v(x', y') > \frac{1}{2}
\end{cases}
\end{align*}
\]

\[C = (n-1)^{\lceil n(n-1)/2 \rceil}
\]

Case 5 of 10: \( m(x) > m(y) > m(z) \)

\[
d(x, y) + d(y, z) - d(x, z) = (C/2 + D_i) |m(x) - m(y)| \\
+ (C/2 + D_j) |m(y) - m(z)| \\
- (C/2 + D_e) |m(x) + m(y) - m(y) - m(z)|
\]

\[
= (D_i - D_e) |m(x) - m(y)| + (D_j - D_e) |m(y) - m(z)|
\]

\[
\geq (D_i \lor D_j - D_e) \cdot 1 + (D_i \land D_j - D_e) (n-2)
\]

\[
\geq (n-1)^{\ell - 1} - (n-1)^{\ell - 2} + [0 - (n-1)^{\ell - 2}] (n-2) = 0
\]
(MUM ⇐ MST+) Show $m, d$ represent $\rho$

\[ d(x, y) = \begin{cases} 0, & x = y \\ C, & x \neq y, \rho(x, y) = \frac{1}{2} \\ (C/2 + D_i) |m(x) - m(y)|, & \rho(x, y) = \rho(x', y') > \frac{1}{2} \end{cases} \]

Case 1: $\rho(w, x) \geq \rho(y, z) > \frac{1}{2}$

\[ \Leftrightarrow m(w) > m(x), m(y) > m(z), \]

\[ d(w, x) = (C/2 + D_i) (m(w) - m(x)) \]

\[ d(y, z) = (C/2 + D_j) (m(y) - m(z)) \]

\[ \Leftrightarrow \frac{m(w) - m(x)}{d(w, x)} = \frac{1}{C/2 + D_i} \geq \frac{1}{C/2 + D_j} = \frac{m(y) - m(z)}{d(y, z)} \]
Recall $\mathcal{Z} = \{1, 2, \ldots, n\}$ is finite.

**Proposition 2.** If $\rho(x, y) = F\left(\frac{m(x) - m(y)}{d(x, y)}\right)$ is a MUM, then:

(i) \( \rho(x, y) \geq \frac{1}{2} \iff m(x) \geq m(y) \)

(ii) \( \rho(x, y) > \rho(x, z) > \rho(y, z) \geq \frac{1}{2} \implies d(x, y) < d(x, z) \).

- Analyst recovers ordinal information about $m, d$.

**Identification for $m, d, F$?**

- We need a "sufficiently rich" set of options
Rich setting: lotteries (e.g. Gul & Pesendorfer 2006)

\[ Z = \{1, 2, \ldots, n\} \text{ finite set of prizes} \]

\[ \Delta = \Delta(Z) \text{ lotteries over } Z \]

\[ \rho: \Delta^2 \to [0,1], \quad \rho(x,y) + \rho(y,x) = 1 \text{ choice rule} \]

Def: \( \rho \) is a moderate expected utility model (MEM) if

\[ \exists \quad U: \Delta \to [0,1] \text{ linear, onto} \]

\[ \exists \text{ norm } \|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle} \]

\[ \exists \quad F: [-T,T] \to [0,1] \text{ strictly increasing and continuous} \]

such that \( \rho(x,y) = F \left[ \frac{\|x-y\|}{\|x-y\|} \right] \)
Necessary conditions for MEM:
MST+, continuous on $\Delta^2 \setminus \text{Diagonal}$

Def: $\rho$ is

linear $\rho(x, y) = \rho(\alpha x + (1-\alpha) z, \alpha y + (1-\alpha) z)$

convex $x \neq y$, $\rho(x, y) = \frac{1}{2}$, $\rho(x, z) = \rho(y, z) > \frac{1}{2}$
implies $\rho\left(\frac{1}{2} x + \frac{1}{2} y, z\right) > \rho(\alpha y + (1-\alpha) x, z) \quad \forall \alpha \neq \frac{1}{2}$
Convex $p$:
Theorem 2. $\rho$ is a MEM iff it is non-constant, continuous, linear, convex, and MST+.

Steps of the proof: $(\Rightarrow)$ Easy.

$(\Leftarrow)$ $\rho$ has unique linear extension to hyperplane $H \supseteq \Delta$.

$vNM \Rightarrow U$, parallel stochastic indifference hyperplanes $I$.

$B(x,y,p) := \{ z \sim x : \rho(z,y) \geq p \}$ is symmetric, convex, compact, non-empty interior $\Rightarrow$ unit ball for $\| \cdot \|_B$.

Minkowski functional is a norm on $I$:

$\| z - \bar{x} \|_B := \inf \{ \lambda > 0 : z \in \bar{x} + \frac{\lambda}{B-\bar{x}} \}$

(see e.g. Thompson (1996))
Define a norm \( \equiv \) choose a unit ball

\[
\{ x \sim \hat{x} : \rho(y, x) \geq \rho \}
\]

symmetric, convex, compact, non-empty interior
Uniqueness

**Proposition 3.** Let $\rho$ be a MEM represented by $(U_1, \| \cdot \|_1, F_1)$ and let $\bar{\rho} := \max_{x,y} \rho(x,y)$.

Then $(U_2, \| \cdot \|_2, F_2)$ represent the same MEM if and only if there exist $A, B > 0$ such that:

(i) $U_2 = U_1$

(ii) If $\rho(x,y) = 1/2$ and $\rho(w,z) = \bar{\rho}$, then

\[
\begin{align*}
\|x - y\|_2 &= A \|x - y\|_1, \\
\|w - z\|_2 &= B \|w - z\|_1, \\
\langle x - y, w - z \rangle_2 &= \langle x - y, w - z \rangle_1 = 0;
\end{align*}
\]

(iii) $F_2(t) = F_1 \left( \frac{T A}{\sqrt{T^2 + (A^2 - B^2)t^2}} \right)$ for all $t \in [-T/B, T/B]$, where $T := F_1^{-1}(\bar{\rho})$.

Normalize $\|\hat{x} - \hat{y}\| = 1$ for some $\hat{x} \neq \hat{y}$ with $\rho(\hat{x}, \hat{y}) = 1/2$

$\|\hat{w} - \hat{z}\| = 1$ for some $\hat{w}, \hat{z}$ with $\rho(\hat{w}, \hat{z}) = \max \rho$

$\Rightarrow U, \| \cdot \|, F$ uniquely pinned down
Hence $U$ is unique (already normalized)

$\|\cdot\|, F$ are unique up to scaling by $A, B > 0$

Normalize $\|x - \hat{x}\| = 1$ and $\|\bar{y} - \hat{x}\| = 1 \Rightarrow U, \|\cdot\|, F$ unique!

$\vec{y} - \hat{x} \perp I$

$U(x) = U(\hat{x}) = U(x')$
Estimation: 

\[ p(x,y) = F \left[ \frac{\mu'(x-y)}{\sqrt{(x-y)\overline{\mathbf{M}}(x-y)}} \right] \]

By Proposition 3, 

\( \mu \) is unique 

\( F, \tilde{\mathbf{M}} \) unique (after normalization)

Another way: 

Fix an \( F \) \( \Rightarrow \mu, \tilde{M} \) point identified. (e.g. \( F = \Phi \)).
Extending a MUM on $\mathbb{Z}$ to a MEM on $\Delta(\mathbb{Z})$

Proposition 4.

Let $(\mu, d, F)$ be a MUM representation of $\rho > 0$ on $\mathbb{Z}$.

$\exists$ MEM rep. $(U, \| \cdot \|, F')$ extending $(\mu, d, F)$ to $\Delta$ iff:

(i) $\mu$ satisfies $\min_{x} \mu(x) = 0$, $\max_{x} \mu(x) = 1$

(ii) $d$ satisfies $\sum_{i=1}^{n} \sum_{j=1}^{n} d(x^i, x^j) x_i x_j < 0 \ \forall \ 0 \neq \alpha \in \ker 1$

(iii) $F$ is continuous
Observable Properties

- Weak Transitivity
- Moderate Transitivity+
- Strong Transitivity+ (and Positivity)
- Acyclicity (and Positivity)
- Product Rule (and Positivity)

Models

- Proposition 5
- Theorem 1
- Tversky and Russo (1969)
- Fudenberg et al. (2015)
- Luce (1959)

- Weak Utility
- Moderate Utility
- Simple Scalability
- Fechnerian Utility
- Logit

\[
F \left( \frac{u(x) - u(y)}{s(x,y)} \right) \\
\text{relax } d(x,z) \leq d(x,y) + d(y,z)
\]

\[
F \left( \frac{u(x) - u(y)}{d(x,y)} \right)
\]

\[
F(\delta(x,y)) + \delta(x,y)
\]

\[
F(u(x) - u(y))
\]

\[
\frac{1}{1 + e^{u(y) - u(x)}}
\]
Luce and Suppes (1965):

Fig. 5. A summary of the implications (→) and failures of implication (→) among models and observable properties on the assumption that the probabilities are different from 0 and 1 and that the set of all alternatives is finite. The number beside a line indicates the relevant theorem. The relation between any two concepts can be deduced using only the transitivity of implication from these relations plus the fact that a binary model or property never implies a nonbinary one (see text).
Relation to Random Utility Model (RUM)

Relation to Random Expected Utility Model (GP 2006)
Relation to Random Utility Model (RUM)

Def: $\rho: \mathbb{Z}^2 \to [0,1]$ is a binary RUM if

$$\exists \mu \text{ probability on the set of strict orderings on } \mathbb{Z} \text{ such that } \rho(x,y) = \mu(\{ >: x > y \})$$

Results:

1. MUM $\neq$ RUM
2. RUM $\neq$ MUM
1. MUM & RUM. Example: \( Z = \{1, 2, 3, 4, 5, 6\} \), \( 0 < \epsilon < \frac{3}{10} \)

\[
\begin{align*}
\rho(4, 5) &= \rho(4, 6) = \rho(2, 5) = \rho(2, 3) = \rho(1, 6) = \rho(1, 3) = 1 \\
\rho(2, 6) &= \rho(1, 5) = \frac{1}{2} + \epsilon \\
\rho(2, 4) &= \rho(1, 4) = \rho(3, 5) = \rho(3, 6) = \frac{1}{2} + \frac{\epsilon}{2} \\
\rho(3, 4) &= \rho(1, 2) = \rho(5, 6) = \frac{1}{2} + \frac{\epsilon}{3}
\end{align*}
\]

\[
2 > 3 > 4 > 6 \implies \mu\left(\{x: 3 > 4 \text{ and } 6 > 2\}\right) = 0 \\
\mu\left(\{x: 6 > 2 \text{ and } 5 > 1\}\right) = 0 \\
\mu\left(\{x: 3 > 4 \text{ and } 5 > 1\}\right) = 0 \\
\]

Should be \( \rho(3, 4) + \rho(5, 1) + \rho(6, 2) \leq 1 \)

\[
\text{but } \quad \frac{1}{2} + \frac{\epsilon}{3} + \frac{1}{2} - \epsilon + \frac{1}{2} - \epsilon = \frac{3}{2} - \frac{5}{3} \epsilon > 1 \quad (\implies \iff)
\]
2. RUM & MUM. Ex: Condorcet cycle

Let \( P(a \succ b \succ c) = P(b \succ c \succ a) = P(c \succ a \succ b) = \frac{1}{3} \).
Then, \( p(a,b) = p(b,c) = p(c,a) = \frac{2}{3} \)
I conclude with some speculations concerning measurement. With regard to representation, it is not unlikely that MST is sufficient as well as necessary for moderate utility models. With regard to uniqueness, clearly, similarity and linear transformations on $d$ and $u$, respectively, are permissible, and these may be the only allowable transformations in sufficiently rich systems. In support of this conjecture, note that the number of restraints imposed by the triangle inequality grows much more rapidly with the number of stimuli than does the number of free parameters.
Distance metric

\[ d(x,y) \geq 0 \]
\[ d(x,y) = 0 \text{ if and only if } x = y \]
\[ d(x,y) = d(y,x) \]
\[ d(x,z) \leq d(x,y) + d(y,z) \]

Norm

\[ \|x\| \geq 0 \]
\[ \|x\| = 0 \text{ if and only if } x = 0 \]
\[ \|ax\| = |a| \cdot \|x\| \quad \forall a \in \mathbb{R} \]
\[ \|x + y\| \leq \|x\| + \|y\| \]

Norm induced by inner product

\[ \|x\| = \sqrt{\langle x, x \rangle} \]