Moderate Expected Utility

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Abstract

Accounting for product differentiation is a key concern in the analysis of consumer demand: any consumer who abandons their first choice after a price increase is more likely to substitute it with options that are "close" in characteristics to the original choice than with more "distant" options. We provide a behavioral foundation for using a distance metric to measure product differentiation in discrete choice. This approach is captured by a simple formula in the *moderate utility model*. We identify a single, directly testable property that characterizes the model: choices are moderately transitive. We show that the model allows the analyst to accommodate well-known failures of strong transitivity, while retaining significantly more empirical bite than weak transitivity, achieving a useful compromise. Extending the analysis to the domain of risky choice, we introduce and characterize the *moderate expected utility model*, and we show how the analyst can measure both differentiation and utility from observed choice behavior.

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1 Introduction

A consumer who abandons her first choice when facing a price increase is more likely to substitute it with options that are "close" in characteristics to the original choice than with more "distant" options. A change in the price of a coupe automobile, for example, should affect the market share of other coupes proportionally more than the market share of minivans. Accounting for these substitution patterns in consumer choice is a key concern for the analysis of markets with differentiated products, as can be seen in empirical analyses of demand for automobiles (Berry, Levinsohn and Pakes, 1995), ready-to-eat cereal (Nevo, 2001), online newspapers (Gentzkow, 2007), health plans (Einav et al., 2013), and so on.

In this paper, we provide a discrete choice behavioral foundation for the measurement of product differentiation as a *distance metric* over the options. As early as Domencich and McFadden (1975) and Hausman and Wise (1978), the discrete choice literature has allowed for options to vary in their degree of substitutability using a myriad of parametric models. We show that a wide variety of these models —including the familiar nested logit and covariance probit— are just restricted versions of a more general *moderate utility model* proposed in psychology by Halff (1976). In this model, the probability that an option x is preferred to option y is given by:

$$\rho(x, y) = F\left(\frac{u(x) - u(y)}{d(x, y)}\right) \tag{MUM}$$

The utility function u in the MUM represents the value of each option. The distance metric d captures the *differentiation* between the options and reflects their substitutability and their comparability. The ratio $[u(x) - u(y)]/d(x, y)$ can be interpreted as the strength of preference for option x over option y. The increasing transformation F maps strength of preference to choice probabilities and satisfies $F(t) = 1 - F(-t)$ for all t.

Our first contribution is to determine the empirical content of the MUM formula. We identify a single, directly testable, non-parametric condition that fully characterizes the model. Theorem 1 shows that a binary choice rule ρ over a finite set of alternatives can be represented by the MUM if and only if it satisfies *a moderate degree of transitivity.* This condition requires that if x is preferred to y with probability larger than one-half and, in turn, y is preferred to z with probability larger than one-half, then the probability that x is preferred to z must lie above the *minimum* of those two probabilities. For example, if $\rho(x, y) = 0.6$ and $\rho(y, z) = 0.8$, moderate transitivity requires that $\rho(x, z) > 0.6$.

Moderate transitivity is a less studied, intermediate condition between the two well-known postulates of *weak stochastic transitivity* (which for the same antecedent requires the less demanding conclusion $\rho(x, z) \geq 0.5$ in the example above) and *strong stochastic transitivity* (which requires the more demanding $\rho(x, z) \geq 0.8$).

The transitivity postulates above are directly testable in data by checking simple moment inequalities. In Section 2, we present the formal definitions and apply them to data examples drawn not only from economics, but also from psychology and biology (Examples 2–4). We use these examples throughout the paper for illustrative purposes. They belong to a robust class of empirical phenomena in which strong transitivity fails, but moderate transitivity still holds. By Theorem 1, all these examples can be accommodated by the MUM. In addition, we show that moderately transitive models retain significantly more empirical bite and predictive power out-of-sample than weakly transitive models. Thus, Theorem 1 implies that the MUM provides the analyst with a valuable modeling compromise between flexibility and predictive power.

Theorem 1 also allows us to determine how parametric restrictions commonly used by practitioners in applied work map on to restrictions on observable choice behavior (see Figure 1). For instance, specializing the distance metric in the MUM formula to the discrete metric is equivalent to going from moderately transitive behavior to acyclic* behavior. Converserly, we use Theorem 1 to show that the triangle inequality property of the distance metric in the MUM is exactly responsible for the 'gap' between weakly transitive and moderately transitive choice behavior.

Finally, Theorem 1 addresses a theoretical question that arises in previous work in economics and psychology. Chipman (1958) and Georgescu-Roegen (1958) formulated two slightly different versions of the moderate transitivity postulate, while Halff (1976) proposed the MUM formula. Halff (1976) showed that every MUM must satisfy Chipman's version of moderate transitivity and left open the question of sufficiency. Theorem 1 answers Halff's question by showing that Georgescu-Roegen's slightly stronger condition is both necessary and sufficient for the MUM representation.

Our second contribution is to address the important question of measurement. We ask: under what conditions can the analyst separately measure utility and differentiation from observed choices? We provide an answer by extending the MUM to the domain of risky choice, using a linear extension that is common in demand estimation with random coefficients (Hausman and Wise, 1978; Nevo, 2000). Theorem 2 shows that

*Acyclicity, which we define in Section 5, is an even more restrictive postulate than strong stochastic transitivity (Fudenberg, Iijima and Strzalecki, 2015).

a binary choice rule over lotteries is a *moderate expected utility model* (MEM) if and only if it satisfies continuity, linearity, and a weak independence postulate, in addition to moderate transitivity. Our MEM characterization identifies (i) a von Neumann-Morgenstern expected utility function which is unique up to an affine transformation; (ii) a norm, induced by an inner product, that is unique up to two scaling factors; and (iii) a continuous and monotonic transformation F that is unique up to the same factors used to rescale utility and norm. Our identification results matter for estimation: we show that all the parameters of the MEM formula are uniquely pinned down with straightforward normalizations of scale.

MUM and MEM nest several classic binary choice models in the literature as special cases. Some special cases of the MUM are also instances of the random utility model (RUM). We show that, despite having a non-empty intersection, neither the MUM nor the RUM nest each other. Likewise, we show that the MEM neither nests nor is nested in the random expected utility model studied by Gul and Pesendorfer (2006).

To obtain our results, we need only assume that the analyst is able to observe binary comparison data. This approach has two main advantages. First, richer kinds of data are not always readily available. For example, experimental designs are often deliberately restricted to binary comparisons (Hey and Orme, 1994; Halevy, Persitz and Zrill, 2018). Our characterization and uniqueness results can be readily applied to work with such data. Second, by not taking a specific stance on how behavior in pairwise comparisons relates to behavior in comparisons with more than two options, we are able to draw lessons for a much wider variety of empirical settings. Take, for example, the basic postulate of *regularity* which links behavior across choice problems of different sizes. It says that the market share of an existing option cannot increase when a new option is introduced. By focusing on binary comparisons, we don't take a stance on regularity. Thus, our results apply to model and measure product differentiation in the context of a population of rational consumers with heterogeneous tastes (which satisfies regularity) as well as in the context of individual stochastic choice (which can violate regularity).

Section 2, below, introduces the setup and the transitivity postulates. Sections 3 and 4 present the MUM and MEM and contain our main results. Section 5 places our results in the literature. Section 6 concludes with a discussion of the role of our differentiation parameter in non-binary choice contexts. We show that differentiation is the key to address several phenomena that have attracted much recent attention in the random choice literature.

2 Stochastic choice and transitivity

We begin with some definitions. Let Z be a finite set of choice options. A (binary, stochastic) *choice rule* on Z is a function $\rho : Z^2 \to [0,1]$ such that $\rho(i,j) + \rho(j,i) = 1$ for every $i, j \in Z$. When ρ represents demand in a population of standard rational consumers, the number $\rho(i,j)$ is the proportion of the population that prefers i to j. When ρ represents individual stochastic choice, $\rho(i,j)$ is the probability that the decision maker selects option i in a binary comparison against j.

The most familiar notions of transitivity for binary choice data are weak stochastic transitivity (WST) and strong stochastic transitivity (SST):

(WST) $\quad \rho(i,j) \geq 1/2$ and $\rho(j,k) \geq 1/2 \Rightarrow \rho(i,k) \geq 1/2$

(SST) $\quad \rho(i,j) \geq 1/2$ and $\rho(j,k) \geq 1/2 \Rightarrow \rho(i,k) \geq \max\{\rho(i,j), \rho(j,k)\}$

Both postulates are well-studied in the literature (see Rieskamp et al., 2006). In this paper we focus on a less studied, intermediate form of transitivity called moderate stochastic transitivity (MST):

(MST) $\quad \rho(i,j) \geq 1/2$ and $\rho(j,k) \geq 1/2 \Rightarrow \rho(i,k) \geq \min\{\rho(i,j), \rho(j,k)\}$

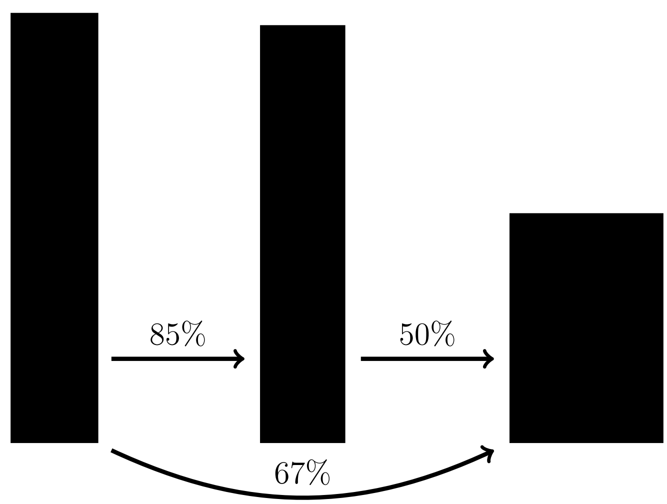
These definitions clearly imply that SST $\Rightarrow$ MST $\Rightarrow$ WST. The three postulates are directly testable in data by checking simple moment inequalities.

The classic example below illustrates how product differentiation drives systematic violations of SST by affecting the comparability and the substitutability of the options. In what follows, we show that MST often holds in empirical settings where SST is systematically violated. We then show that a model that imposes MST affords the analyst significantly more predictive power out-of-sample than just imposing WST.

Example 1 (attributed to L. J. Savage, adapted from Tversky (1972a))**.** *An individual has a difficult time comparing a trip to Paris, denoted P and a trip to Rome, denoted R, so that she is equally likely to pick either option $\rho(P,R) = 1/2$. The individual still has trouble deciding if the trip to Paris is enhanced by a €5 bonus, denoted by P^+. In other words, $\rho(P^+,R)$ is still approximately $1/2$. But when pressed to decide between the two Paris trip options, the individual clearly prefers the bonus, so that $\rho(P^+,P) = 1$. SST requires that $\rho(P^+,R) \geq \rho(P^+,P)$ which is intuitively violated in this case, while MST only requires the more plausible inequality $\rho(P^+,R) \geq \rho(P,R)$.*

Models that ignore the degree of differentiation of the options, such as the classic logit model (Luce, 1959; McFadden, 1974), fail to capture the behavior in Example 1. A small monetary bonus makes the choice comparison between the two Paris trips very easy. The same monetary bonus has negligible impact, however, on the difficulty of comparing a trip to Paris and a trip to Rome. Thus, the lesson we glean from Savage's example is that utility values alone cannot explain the difficulty of comparing two options.

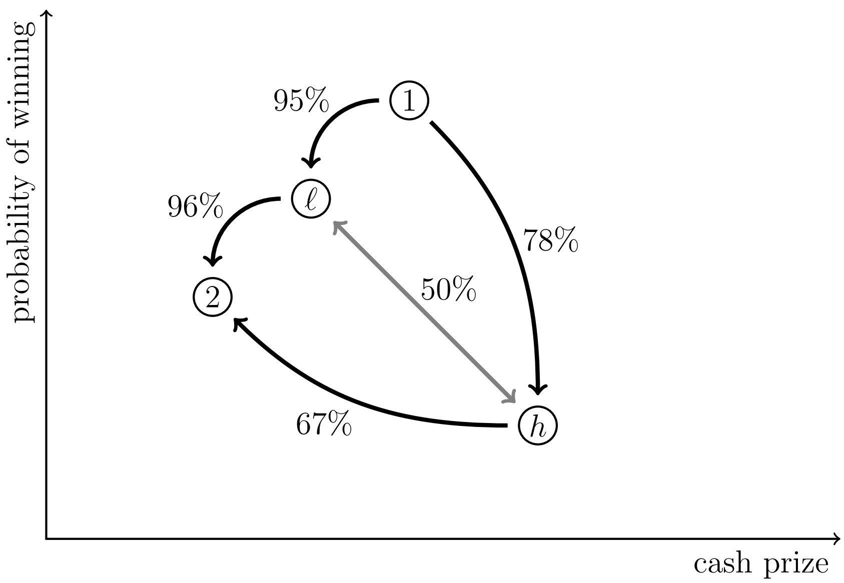
The accumulated evidence for systematic violations of SST in choice data is very robust. Reviewing some of this evidence, Mellers, Chang, Birnbaum and Ordonez (1992, p. 348) note that "weak and moderate stochastic transitivity are often satisfied, although a few exceptions have been noted," while "[s]trong stochastic transitivity is frequently violated." Chipman (1960) provides perhaps the earliest empirical demonstration of Savage's intuition in economics. Tversky and Russo (1969) provide the following visually compelling demonstration of this same intuition in psychology:



Example 2 (Perceptual task data)**.** *Tversky and Russo (1969) recorded hundreds of decisions by prison inmates in Michigan in perceptual choice tasks. Subjects were shown many different pairs of rectangles and were asked to pick the rectangle with the largest area in each pair. Three of these rectangles and their pairwise relative frequencies of choice are shown above. The middle and right rectangles have equal areas, so that each one is chosen 50% of the time in a binary comparison. The left rectangle is slightly larger than the others. The same difference in area was more easily detected in pairs with less differentiated shapes (85% correct answers) than in pairs with more differentiated shapes (67% correct answers). These frequencies violate SST but satisfy MST.*

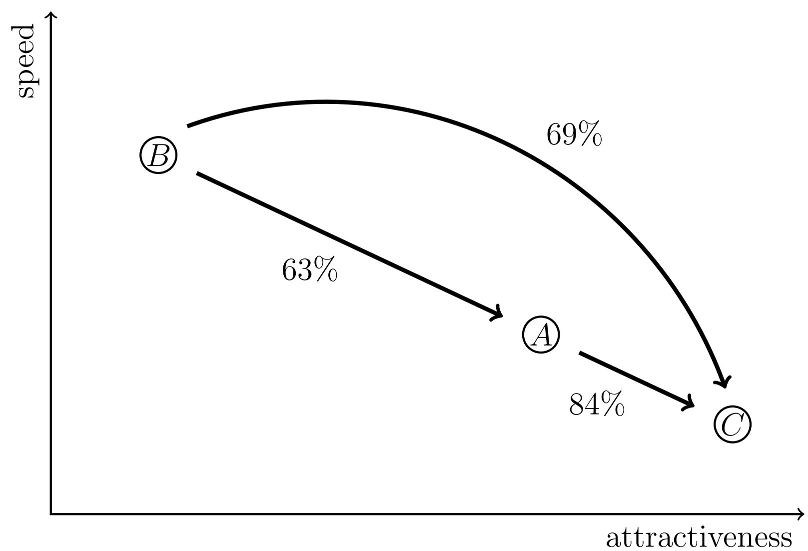
Note that the empirical violation of SST in Example 2 follows the recipe suggested by Savage's Example 1. Like the two trips to Paris, the first and second rectangles in Example 2 are less differentiated and therefore easier to compare. And like the trip to Rome, the third rectangle in Example 2 is more differentiated from the other options and therefore harder to compare.

The different levels of differentiation between the options also drive a violation of SST in Example 3 below.



Example 3 (Choice over lotteries)**.** *Soltani, De Martino and Camerer (2012) recorded thousands of choices by 21 male Caltech undergraduates using simple lotteries that pay a cash prize of* m *dollars with probability* p *in the lab. A high risk lottery* h *and a low risk lottery* ℓ*, depicted above, were fine-tuned to each individual to be approximately indifferent, (i.e., equally likely to be chosen in a binary comparison). Slightly perturbed versions of* h *and* ℓ *were then offered for comparison against several types of 'decoy' lotteries. Above we depict the relative location of two decoy lotteries* 1 *and* 2 *with respect to* h *and* ℓ*. Decoy lottery* 1 *dominates* ℓ *and was chosen 95% of the time against* ℓ *but only 78% of the time against* h*. Thus, choice frequencies violate SST in the direction* $1 \to \ell \to h$*. Decoy lottery* 2*, on the other hand, is dominated by* ℓ *and was chosen 4% of the time against* ℓ *and 33% of the time against* h*. Hence, choice frequencies also violate SST in the direction* $h \to \ell \to 2$*. It is easy to verify that MST holds in both cases.*

Our final example is taken from experiments with frogs, further broadening the empirical reach of Savage's intuition.



Example 4 (Animal studies)**.** *Lea and Ryan (2015) recorded hundreds of mating decisions by female túngara frogs. Female túngara frogs choose mates based on the sound of their call. Above we depict how the calls of the three male options A, B and C were differentiated along two desirable attributes. The horizontal axis represents a measure of static attractiveness, and the vertical axis represents speed measured in calls per second. In binary comparisons, option B was chosen in 63% of the trials against A; option A was chosen in 84% of the trials against C; and option B was chosen in 69% of the trials against C. Choices therefore satisfy MST but violate SST.*

Taken together, these examples illustrate the diversity of behavioral contexts in which the level of differentiation between the options affects their substitutability and their ease of comparison, causing SST to fail. In all these settings, however, MST still holds.

We have just seen that the strong transitivity postulate is too strong: relaxing SST helps address a robust range of empirical phenomena illustrated by the examples above. Conversely, we now argue that the weak transitivity postulate is too weak: it allows that $\rho(i,k) = .51$, for example, even if we observe that $\rho(i,j) = \rho(j,k) = .95$. Imposing moderate transitivity, in this case, leads to the sharper and arguably more sensible prediction $\rho(i,k) \geq 95$.

To quantify this additional predictive power, let ρ be a choice rule that satisfies WST. Enumerate the options in $Z = \{1,\ldots,n\}$ in such a way that $\rho(i,j) \geq 1/2$ whenever $i \leq j$. For the sake of simplicity, let us assume that choice probabilities differ whenever possible, so that the set $\{\rho(i,j) \in [0,1] : i \neq j\}$ has maximum cardinality with $n(n-1)$ elements.

When $Z = \{1, 2, 3\}$ has three alternatives, WST allows ρ to have six strict orderings:

$$\text{WST} \left\{ \begin{array}{l} \left. \begin{array}{l} \rho(1,3) > \rho(1,2) > \rho(2,3) \\ \rho(1,3) > \rho(2,3) > \rho(1,2) \\ \rho(1,2) > \rho(1,3) > \rho(2,3) \\ \rho(2,3) > \rho(1,3) > \rho(1,2) \end{array} \right\} \text{MST} \\ \rho(1,2) > \rho(2,3) > \rho(1,3) \\ \rho(2,3) > \rho(1,2) > \rho(1,3) \end{array} \right.$$

MST rules out the last two of the six strict orderings, where $\rho(1,3) < \min\{\rho(2,3), \rho(1,2)\}$. Let $\#WST(n) = [n(n-1)/2]!$ denote the number of strict orderings allowed by WST when Z has n options, and likewise, let $\#MST(n)$ denote the number of strict orderings allowed by MST. The ratio $\#MST(n)/\#WST(n)$ provides a measure of how restrictive is MST compared to WST. In the case $n = 3$ we just showed the ratio $\#MST(3)/\#WST(3)$ is equal to 2/3. This ratio decreases to less than 1/4 when $n = 4$ and less than 1/17 when $n = 5$. In fact, the ratio is arbitrarily small when n is large:

Proposition 1. $\lim_{n\to\infty} \#MST(n)/\#WST(n) = 0.$

We prove Proposition 1 in the Appendix. For completeness, we also show in the Appendix that the ratio between $\#SST(n)$ and $\#MST(n)$ goes to zero when n is large. In our main results, below, we characterize choice rules that satisfy a slightly stronger version of moderate transitivity introduced by Georgescu-Roegen (1958):

(MST+) $\rho(i,j) \geq 1/2 \text{ and } \rho(j,k) \geq 1/2 \Rightarrow \left\{ \begin{array}{l} \rho(i,k) > \min\{\rho(i,j), \rho(j,k)\} \\ \text{or} \\ \rho(i,k) = \rho(i,j) = \rho(j,k) \end{array} \right.$

The only difference between MST and MST+ is that the knife-edge case

$$\max\{\rho(i,j), \rho(j,k)\} > \rho(i,k) = \min\{\rho(i,j), \rho(j,k)\}$$

is allowed by MST and ruled out by MST+. For instance, MST+ requires $\rho(P^+, R) > \rho(P, R)$ in Example 1, where MST only requires $\rho(P^+, R) \geq \rho(P, R)$. This difference is critical to prove our characterization results, but it is empirically undetectable: no finite amount of data allows an analyst to tell MST and MST+ apart. Thus MST+ automatically holds together with MST in Examples 2–4. Moreover, Proposition 1 automatically holds with MST+ in place of MST.

To summarize, a parametric model that covers the range of choice behavior allowed by MST/MST+ is useful in two ways: first, it provides the flexibility that is needed to accommodate empirical violations of SST. Second, it imposes significant restrictions out of sample —allowing the analyst to make sharper predictions— than the more lenient WST. We describe such a model in the next section.

3 Moderate utility model

A choice rule ρ on a finite set Z is a *moderate utility model (MUM)* if there is a utility function $u : Z \to \mathbb{R}$, a distance metric $d : Z^2 \to \mathbb{R}_+$ and a strictly increasing function F, such that for all $i \neq j$,

$$\rho(i,j) = F\left(\frac{u(i) - u(j)}{d(i,j)}\right) \tag{1}$$

where F satisfies $F(t) = 1 - F(-t)$ for all t. This last requirement automatically follows from $F(t) = \rho(i,j) = 1 - \rho(j,i) = 1 - F(-t)$. The utility u represents the value of each option. It is easy to see that $\rho(i,j) \geq 1/2$ if and only if $u(i) \geq u(j)$. The distance d measures the *differentiation* between i and j. For any fixed utility difference, increasing the differentiation $d(i,j)$ drives choice probabilities closer to 1/2, capturing the fact that more differentiated objects are less substitutable and harder to compare. The ratio $[u(i) - u(j)]/d(i,j)$ can be interpreted as the strength of preference for option i over option j, while the function F maps strength of preference to choice probabilities.

The differentiation metric d gives the MUM the flexibility that is needed to deal with empirical violations of strong transitivity. Consider how the MUM accommodates the choice over trips in Example 1. Let the trip to Paris and the trip to Rome have the same utility $u(P) = U(R) = 0$, and let the trip to Paris enhanced with the €5 bonus have utility $u(P^+) = 1$. Let the distance metric be given by $d(P, P^+) = \varepsilon > 0$ and $d(P, R) = d(P^+, R) = 1/\varepsilon > 0$. Then, let $F = \Phi$ be the standard Gaussian cdf. Applying (1) we have $\rho(P, R) = 1/2$, $\rho(P^+, P) = \Phi(1/\varepsilon)$ and $\rho(P^+, R) = \Phi(\varepsilon)$. Taking $\varepsilon > 0$ small, we obtain $\rho(P^+, P)$ close to one and $\rho(P^+, R)$ close to 1/2 as desired. The small distance $d(P^+, P)$ captures the fact that there is almost no differentiation between the two Paris trips, which makes them easy to compare. The Rome trip is more differentiated from the other options and more difficult to compare.

The MUM formula can also fit the choice data in Examples 2–4. This follows directly from our first characterization result:

Theorem 1. *A choice rule ρ on a finite Z is a MUM if and only if it satisfies MST+.*

Halff (1976) proposed the MUM formula, and showed the first step we need to prove the necessity part of Theorem 1:

Lemma 1 (Halff, 1976)**.** *Every MUM satisfies MST.*

Proof. Suppose that $\rho(i,j) \geq 1/2$ and $\rho(j,k) \geq 1/2$ but, contrary to MST, $\rho(i,k) < \min\{\rho(i,j), \rho(j,k)\}$. A MUM representation of ρ and the triangle inequality property of the distance d would imply the contradiction

$$\begin{aligned} u(i) - u(k) &< d(i,k) \min\left\{\frac{u(i)-u(j)}{d(i,j)}, \frac{u(j)-u(k)}{d(j,k)}\right\} \\ &\leq [d(i,j) + d(j,k)] \min\left\{\frac{u(i)-u(j)}{d(i,j)}, \frac{u(j)-u(k)}{d(j,k)}\right\} \\ &\leq d(i,j)\frac{u(i)-u(j)}{d(i,j)} + d(j,k)\frac{u(j)-u(k)}{d(j,k)} \\ &= u(i) - u(k). \end{aligned}$$

□

To complete the necessity part of the proof, we now show that every MUM satisfies the stronger MST+ condition. Suppose $\rho(i,j) > \rho(j,k) = \rho(i,k) \geq 1/2$. The MUM representation would require

$$\frac{u(i)-u(j)}{d(i,j)} > \frac{u(j)-u(k)}{d(j,k)} = \frac{u(i)-u(k)}{d(i,k)},$$

which, in turn, would imply a contradiction to the triangle inequality:

$$d(i,k) = \frac{u(i)-u(j)+u(j)-u(k)}{u(j)-u(k)} d(j,k) > d(i,j) + d(j,k).$$

The proof of sufficiency in Theorem 1 is left for the Appendix. For any given ρ that satisfies MST+, we explicitly construct the utility u, the distance d and the transformation F; we show that d satisfies the properties of a metric (the key property being the triangle inequality); and we show that the representation holds.

To obtain full identification of the MUM parameters, in the next section we enrich the choice domain to include all lotteries over the finite set Z. With a finite set of options, however, choices reveal ordinal information about u and d:

Proposition 2. *Let ρ be a MUM with parameters (u, d, F). Then*

(i) $\rho(i, j) \geq 1/2$ *if and only if* $u(i) \geq u(j)$;

(ii) $\rho(i, j) > \rho(i, k) > \rho(j, k) \geq 1/2$ *implies* $d(i, j) < d(i, k)$.

Proof. From the MUM formula (1) it follows that $\rho(i, j) > 1/2$ if and only if $[u(i) - u(j)]/d(i, j) > 0$ if and only if $u(i) > u(j)$ proving (i). Suppose the assumption in item (ii) holds. Then, item (i) implies $u(j) \geq u(k)$ hence $u(i) - u(j) \leq u(i) - u(k)$. The MUM formula (1) implies $[u(i) - u(j)]/d(i, j) > [u(i) - u(k)]/d(i, k)$, hence $d(i, k) > d(i, j)$. □

Item (i) in Proposition 2 shows choices in a MUM reveal a complete and transitive ranking over the options represented by the utility parameter u. Item (ii) shows how every violation of SST is explained by the differentiation parameter d. To illustrate, consider the choice data from Example 4. By Proposition 2, the analyst concludes that any MUM that generates this data must satisfy $u(B) > u(A) > u(C)$ and $d(A, C) < d(B, C)$. Likewise, every MUM that generates the data in Example 3 must satisfy $u(1) > u(\ell) = u(h) > u(2)$, $d(2, \ell) < d(2, h)$ and $d(1, \ell) < d(1, h)$.

It is worth noting that the inequalities $d(A, C) < d(B, C)$, $d(2, \ell) < d(2, h)$ and $d(1, \ell) < d(1, h)$ revealed by choice data agree with the inequalities an analyst would obtain by applying the standard Euclidean distance, angle distance, or Manhattan distance to the vectors of measurable attributes in Example 3 and Example 4. That is, options that are revealed to be 'close' in the subjective parameter d are in fact 'close' in the space of observable attributes. The best way to map the abstract utility and differentiation parameters to the observable attributes must be determined empirically in any given application.[†] For an illustration, consider the example of angular distance used in Hausman and Wise (1978). In Example 3, options 1 and ℓ in form a small angle with respect to the origin, so that $d(1, \ell)$ is small, while options 1 and h form a wider angle with respect to the origin, so that $d(1, h)$ is large. Hence, a decision maker may ascribe to h and ℓ the same utility values, and yet have an easier time comparing 1 to ℓ than to h. In Example 4, options A and C form a smaller angle with respect to the origin than options B and C. Even if options A and B are close in value, the frogs find option C much easier to compare to A than to B. As these examples show, a MUM can address both situations in which the ease of comparison involves dominance (Example 3) and non-dominance (Example 4) in the attribute space.

[†]See Apesteguia and Ballester (2018) for some important issues that may arise when the analyst maps the abstract parameters of a random choice model to the attribute space.

4 Moderate expected utility model

We continue to let Z be a finite set of objects and we extend the domain of choice alternatives to the set of all lotteries over Z, denoted by Δ. We identify Δ with the $n-1$ dimensional simplex $\{x \in [0,1]^n : x_1 + \cdots + x_n = 1\}$. The function $U : \Delta \to \mathbb{R}$ is an *expected utility function* if it is linear. A choice rule $\rho : \Delta^2 \to [0,1]$ is a *moderate expected utility model* (MEM) if there exist an expected utility function U, a norm $\|\cdot\|$ induced by an inner product, and a strictly increasing and continuous transformation F, such that, for any $x \neq y$ in Δ,

$$\rho(x, y) = F\left(\frac{U(x) - U(y)}{\|x - y\|}\right). \tag{2}$$

The MEM linearly extends the MUM formula to a subset of $\mathbb{R}^n$. This type of linear extension is commonly used in empirical industrial organization (Hausman and Wise, 1978; Berry, Levinsohn and Pakes, 1995; Nevo, 2000):

Example 5. *The random coefficients model of Hausman and Wise (1978) assumes tastes are linear in a vector of attributes $V(x) = \beta_1 x_1 + \cdots + \beta_n x_n$ and the vector of coefficients $\beta = (\beta_1, \ldots, \beta_n)$ is random with a joint Gaussian distribution $\beta \sim \mathcal{N}\left(\bar{\beta}, \Sigma\right)$. In our lottery setting, each attribute x_i measures the probability of obtaining prize i. The probability that x is preferred to y in this model is given by*

$$\rho(x, y) = \mathbb{P}\{V(x) > V(y)\} = \Phi\left(\frac{\bar{\beta}' x - \bar{\beta}' y}{\sqrt{(x - y)' \Sigma (x - y)}}\right)$$

which is clearly a special case of the MEM formula (2).

Every MEM satisfies MST+. This can be shown by repeating the argument for a MUM in the proof of Theorem 1. Compared to the MUM, however, the MEM is defined in the richer domain of lotteries contained in a linear vector space; it imposes linearity on the utility function U and continuity of the transformation F; and it requires the distance metric to be a norm induced by an inner product. These assumptions carry a few additional testable implications beyond MST+.

First, it is immediate from the formula that every MEM is *continuous* at every point in the domain except along the diagonal $\{(x, x) \in \Delta^2 : x \in \Delta\}$. Second, the assumptions of linear utility and norm distance imply that every MEM is *linear*, that is, for any lotteries $x, y, z \in \Delta$ and $0 < \alpha < 1$ we have $\rho(x, y) = \rho(\alpha x + (1 - \alpha) z, \alpha y + (1 - \alpha) z)$.

Finally, the assumption that the norm comes from an inner product implies that every MEM satisfies a weak form of *independence*: whenever $\rho(x, y) = 1/2$, $\rho(x, z) > \rho(y, z) > 1/2$, and $1/2 < \alpha < 1$, we have $\rho(\alpha x + (1 - \alpha)y, z) > \rho(\alpha y + (1 - \alpha)x, z)$.

Continuity and linearity are familiar postulates from the random choice literature (see, for example, Gul and Pesendorfer, 2006), while independence deserves some discussion. Suppose an individual is equally likely to choose a trip to Paris (P) and a trip to Rome (R) so that $\rho(P, R) = 1/2$ as in Example 1. Moreover, suppose that Paris and Rome both beat London (L) in a binary comparison, with Paris beating London more often than Rome; that is, $\rho(P, L) > \rho(R, L) > 1/2$. Our independence postulate says that if the mixture lottery $\alpha P + (1 - \alpha)R$ gives more weight to Paris than to Rome ($\alpha > 1/2$) then it must be chosen over London with higher probability than the opposite mixture $\alpha R + (1 - \alpha)P$. Intuitively, in this example, Paris and Rome have the same value, but the decision maker is more decisive when comparing Paris and London than when comparing Rome and London. The postulate requires the decision maker to remain more decisive when we substitute lottery $\alpha P + (1 - \alpha)R$ which favors Paris for Paris, and lottery $\alpha R + (1 - \alpha)P$ which favors Rome for Rome. This intuitive property holds because the norm in the MEM comes from an inner product. Examples easily show the postulate may be violated for norms that are not generated by an inner product.

Our next Theorem shows that these postulates, in addition to MST+, are necessary and sufficient for a choice rule to be a MEM.

Theorem 2. *ρ is a MEM iff it satisfies continuity, linearity, independence, and MST+.*

We prove Theorem 2 in the Appendix. Necessity is straightforward. To prove sufficiency, we first show that ρ has a unique linear extension to the $n - 1$ dimensional hyperplane H that contains Δ. Transitivity, linearity and continuity allow us to invoke a standard result to obtain the expected utility function U on H. The indifference sets $I(y) := \{x \in H : U(x) = U(y)\}$ are then parallel hyperplanes of dimension $n - 2$. To construct the norm, we fix one indifference set I, and one lottery $\bar{y}$ with $U(x) > U(\bar{y})$ for all $x \in I$. The bulk of the work is showing that the contour sets $\{x \in I : \rho(x, \bar{y}) \geq \alpha\}$ must be compact, convex, dilations of one another, and centrally symmetric around the point $\hat{x}$ that maximizes $x \mapsto \rho(x, \bar{y})$ on I. Then, we take one such contour set to be the unit ball that defines the norm in the $n - 2$ dimensional subspace parallel to I. We use the independence postulate and a characterization of inner product spaces to show that there exists an inner product which generates the norm. We then extend this inner product in one more dimension to obtain the MEM representation.

To determine the uniqueness of the MEM parameters, we first establish in the Lemma below a useful property of the MEM: the direction of maximum choice probability is always orthogonal to the indifference plane.

Lemma 2 (Orthogonality)**.** *Let* $(U, \|\cdot\|, F)$ *be a MEM representation for* ρ *with* $\bar{p} = \max_{x,y} \rho(x, y) > 1/2$*. Let* $\langle \cdot, \cdot \rangle$ *denote the inner product that generates the norm* $\|\cdot\|$*. Then for every* x, y, w, z *with* $\rho(x, y) = 1/2$ *and* $\rho(w, z) = \bar{p}$ *we have* $\langle x - y, w - z \rangle = 0$*.*

With this useful property in hand we now show that the parameters of the MEM are unique up to one shifting and three scaling factors:

Proposition 3 (Uniqueness)**.** *Let* $(U_1, \|\cdot\|_1, F_1)$ *be a MEM representation for* ρ *with* $\bar{p} = \max_{x,y} \rho(x, y) > 1/2$*. Then* $(U_2, \|\cdot\|_2, F_2)$ *is another MEM representation for* ρ *if and only if there exist* $A \in \mathbb{R}$ *and* $B, C, D > 0$ *such that:*

(i) $U_2(x) = A + BU_1(x)$ *for all* x*;*

(ii) $\|x - y\|_2 = C\|x - y\|_1$ *for all* x, y *with* $\rho(x, y) = 1/2$*;*

(iii) $\|x - y\|_2 = D\|x - y\|_1$ *for all* x, y *with* $\rho(x, y) = \bar{p}$*;*

(iv) $F_2(t) = F_1\left(\frac{TCt}{\sqrt{B^2T^2 + (C^2 - D^2)t^2}}\right)$ *for all* $t \in [-BT/D, BT/D]$*, where* $T := F_1^{-1}(\bar{p})$*.*

Item (i) in Proposition 3 says the only permissible transformations of utility in the MEM representation are affine transformations. This comes from the standard vNM expected utility representation result. Items (ii) and (iii) say the only permissible transformations of the norm are simple rescaling by a factor $C > 0$ along the indifference plane and by a factor $D > 0$ along the direction of maximum choice probability (which is always orthogonal to the indifference plane by Lemma 2). Item (iv) determines the only permissible transformations of F, providing an explicit formula for how to obtain F_2 from F_1 and the scaling factors $B, C, D > 0$. In particular, when $C = D$ the formula becomes a simple rescaling of the domain $F_2(t) = F_1(tC/B)$.

Proposition 3 implies that all the parameters of the MEM are uniquely pinned down with straightforward normalizations. First, we must normalize the shift and scale of the utility function. This is commonly done in applications, for example, by fixing the utility of the best prize to 1 and the utility of the worst prize to zero. Second, we must normalize the two scaling factors of the norm. Fix any two lotteries $x \neq y$ with $\rho(x, y) = 1/2$ and set $\|x - y\| = 1$; then fix any w, z with $\rho(w, z)$ equal to the maximum choice probability and let $\|w - z\| = 1$. Proposition 3 then implies that no further transformations are permissible; hence, all the parameters of the MEM are uniquely pinned down.

For estimating the parameters of a MEM, it is convenient to express the inner product that generates the norm in matrix form:

$$\rho(x, y) = F\left(\frac{u'(x-y)}{\sqrt{(x-y)'M(x-y)}}\right)$$

Since the domain of the inner product is the $n-1$ dimensional subspace $\{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$, the $n \times n$ matrix M can always be written with zeroes on the last row and the last column:

$$M = \begin{bmatrix} \tilde{M} & \mathbf{0} \\ \mathbf{0'} & 0 \end{bmatrix}, \tag{3}$$

where $\tilde{M}$ is a symmetric, positive definite matrix of dimension $n-1$ by $n-1$. With restrictions imposed on u and $\tilde{M}$ to implement the normalizations described in the previous paragraph, the parameters $u, \tilde{M}$ and F are point-identified.

Finally, suppose the analyst specializes the MEM by fixing a particular transformation F. Then, Proposition 3 implies the only permissible transformations are affine transformations of utility and rescaling of the norm with $B = C = D > 0$. In particular, we obtain the following corollary to Proposition 3:

Corollary. *The random coefficients model of Example 5 is point-identified.*

Proof. fix $F = \Phi$ to be the cdf of the standard Gaussian distribution, write the covariance matrix as in (3), normalize the utility function, and apply Proposition 3. □

Notably, the parameters of the model are identified just from binary comparison data. Moreover, while we assumed the choice options are lotteries, the result immediately extends to more general options described by vectors in $\mathbb{R}^n$. To identify the parameters, it is enough to observe binary choices restricted to a simplex.

5 Related literature

Our results build on a long history of discrete choice modelling in economics and in psychology. Marschak (1960) presented the MST postulate and attributed its formulation to Chipman (1958, 1960) and Georgescu-Roegen (1958). Chipman (1958, 1960) formulated restrictions on the distribution of tastes in a population of standard rational consumers that lead to MST. Georgescu-Roegen (1958) obtained the slightly stronger MST+ postulate as a testable implication of his 'threshold model' (Georgescu-Roegen, 1936). As far as we are able to determine, Corollary 6 in Georgescu-Roegen (1958, p. 161) is the first and only time that MST+ appeared in the literature before. Georgescu-Roegen (1958, p. 160) also anticipated L. J. Savage's Example 1, explaining why it is natural to expect SST to be violated in a classical demand setting.

Halff (1976) proposed the MUM formula (1), and its psychological motivation can be traced back to Fechner (1859) and Thurstone (1927). The idea that differentiation can be represented by a "distance" between the options has been proposed in Tversky and Russo (1969), Domencich and McFadden (1975), and Hausman and Wise (1978). A related literature highlights the non-metric nature of similarity judgements (Tversky, 1977) and explores the role of similarity in intransitive choices (Rubinstein, 1988).

5.1 Famous examples of MUM

Many models proposed in the discrete choice literature to address empirical violations of strong transitivity turn out to be a MUM. While these models may appear to take very different forms, they all represent the differentiation between the options with a distance metric. The following examples are perhaps the most familiar.

Example 6. *The covariance probit model (Thurstone, 1927) is a MUM. The model is described by a Gaussian vector $(X_1, \ldots, X_n)$, each coordinate X_i corresponding to an option $i \in Z$, such that $\rho(i,j) = \mathbb{P}\{X_i > X_j\}$. Note that*

$$\rho(i,j) = \mathbb{P}\left\{\frac{X_i - X_j - \mathbb{E}[X_i - X_j]}{\sqrt{Var(X_i - X_j)}} > \frac{\mathbb{E}[X_i - X_j]}{\sqrt{Var(X_i - X_j)}}\right\} = \Phi\left(\frac{\mathbb{E}[X_i - X_j]}{\sqrt{Var(X_i - X_j)}}\right)$$

which is a special case of the MUM formula (1) with utility $u(i) = \mathbb{E}[X_i]$, distance metric $d(i,j) = \sqrt{Var(X_i - X_j)}$ (we allow correlation but rule out perfectly correlated variables), and $F = \Phi$ the standard Gaussian cdf. The 'standard' probit model is a special case of this model with independent variables, and therefore also a MUM.

Example 7. *The nested logit model (McFadden, 1978) is a MUM. The set of alternatives is partitioned into K disjoints nests $B_1 \cup \cdots \cup B_K = Z$. The utility of each option i is a random variable $U(i) = u(i) + \varepsilon_i$ where $u(i) \in \mathbb{R}$ is the deterministic part of utility, and* *the random taste shocks ε_i have the joint cumulative distribution*

$$G(\varepsilon_1, \ldots, \varepsilon_n) = \exp\left\{ -\sum_{k=1}^{K} \left[\sum_{j \in B_k} \exp\left(-\varepsilon_j / \lambda_k\right) \right]^{\lambda_k} \right\}$$

with parameters $\lambda_1, \ldots, \lambda_K \in (0, 1]$. To see the nested logit is a MUM, note the probability that option i is preferred to option j is

$$\rho(i, j) = \mathbb{P}\{U(i) > U(j)\} = \left[1 + \exp\left(\frac{u(j) - u(i)}{d(i, j)} \right) \right]^{-1}$$

where $d(i, j) = \lambda_k$ whenever i and j belong to nest B_k, and $d(i, j) = 1$ when i and j belong to different nests. Writing the nested logit as a MUM allows for a simpler interpretation of the λ_k parameters: they measure the distance between alternatives within nest k. When all $\lambda_k = 1$ we obtain the standard logit model (Luce, 1959; McFadden, 1974) as a special case. Hence, standard logit is also a MUM.

Example 8. *The elimination-by-aspects model (Restle, 1961; Tversky, 1972a,b) is a MUM. The model assumes that each option i has a set of aspects $A(i)$. There is a measure m defined over the set of aspects such that*

$$\rho(i, j) = \frac{m\left[A(i)\right] - m\left[A(j)\right]}{m\left[A(i) \setminus A(j)\right] + m\left[A(j) \setminus A(i)\right]}.$$

This formula is a special case of MUM where utility $u(i) = m[A(i)]$ is the measure of the set of option i's aspects, distance $d(i, j) = m\left[A(i) \setminus A(j)\right] + m\left[A(j) \setminus A(i)\right]$ is the measure of the set of aspects that do not overlap between the i and j, and with $F(t) = 1/2 + t/2$.

Further examples of MUM include the ideal point model of Coombs et al. (1961), the random coefficients model of Hausman and Wise (1978), and the Bayesian learning model of Natenzon (2019).

5.2 Behavioral characterizations for restrictions of MUM

The MUM characterized in Theorem 1 generalizes several nested models of stochastic binary choice in the literature, which we represent in order or generality in Figure 1. The most restrictive model, at the very bottom in Figure 1, is the binary *Logit model*

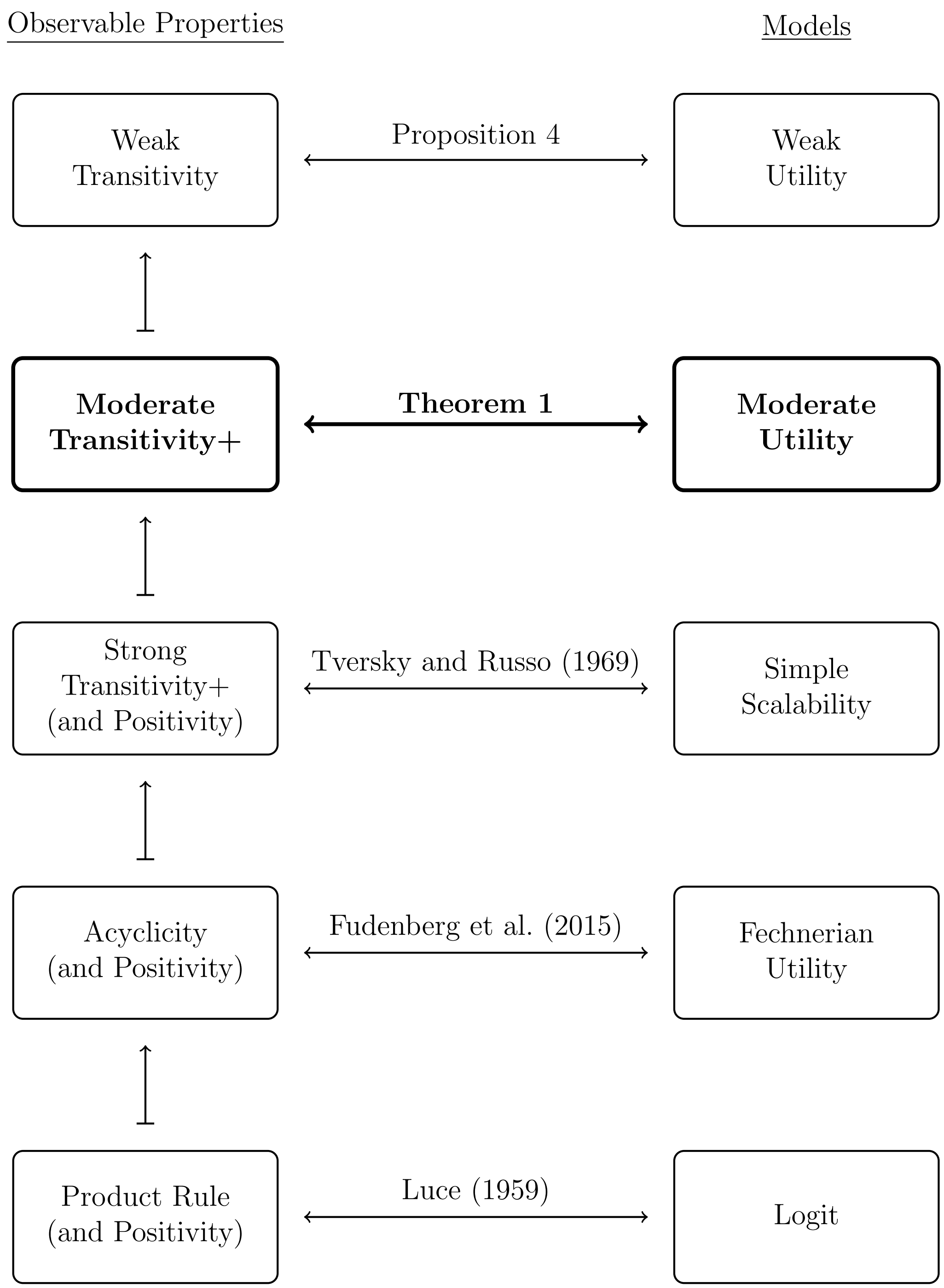


Figure 1: Relationship between models and postulates on choice probabilities for binary stochastic choice over a finite set of options. A double arrow ($\leftrightarrow$) indicates equivalence while an arrow ($\mapsto$) indicates implication in the direction of the arrow and failure of implication in the opposite direction.

in which choice probabilities are given by

$$\rho(i,j) = \frac{e^{u(i)}}{e^{u(i)} + e^{u(j)}} = \frac{1}{1 + e^{-[u(i)-u(j)]}} \tag{4}$$

for some utility function $u : Z \to \mathbb{R}$. Luce (1959) showed formula (4) is equivalent to the *product rule*

(PR) $\quad \rho(i,j)\rho(j,k)\rho(k,i) = \rho(i,k)\rho(k,j)\rho(j,i)$

which can be interpreted as saying that the probability of observing a choice cycle in the direction $i \succ j \succ k \succ i$ is always equal to the probability of observing a choice cycle in the opposite direction. Luce (1959) obtains this equivalence under the mild assumption of *positivity*, which requires that $\rho(i,j) > 0$ for all i, j.

Formula (4) is a special case of the *Fechnerian utility model* from psychophysics (Fechner, 1859; Debreu, 1958; Davidson and Marschak, 1959) where

$$\rho(i,j) = F(u(i) - u(j)) \tag{5}$$

for some utility function $u : Z \to \mathbb{R}$ and a strictly increasing $F : \mathbb{R} \to (0,1)$. Fudenberg et al. (2015) show that (5) is equivalent, under the assumption of positivity, to the postulate of *acyclicity*. This postulate rules out cycles of the form $\rho(w^i, x^i) \geq \rho(y^i, z^i)$ for all $i = 1, \ldots, n$ with at least one strict inequality, whenever $\{w^i, x^i\} = \{y^{f(i)}, z^{f(i)}\}$ and $w^i = y^{g(i)}$ for some permutations $f, g : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Formula (5) is a special case of *simple scalability* (Krantz, 1964) which requires

$$\rho(i,j) = F(u(i), u(j)) \tag{6}$$

for some utility function u and a real valued function F which is strictly increasing in the first argument and strictly decreasing in the second. Tversky and Russo (1969) showed that simple scalability is equivalent to positivity and a slightly stronger version of SST:

(SST+) $\quad \min\{\rho(i,j), \rho(j,k)\} \geq (>)\, 1/2 \Rightarrow \rho(i,k) \geq (>) \max\{\rho(i,j), \rho(j,k)\}$

which, compared to the original SST postulate, imposes the additional requirement that a strict inequality in the hypothesis entails a strict inequality in the conclusion.

A quick comparison of the formulas shows that (4) $\Rightarrow$ (5) $\Rightarrow$ (6). To see that (6) is a special case MUM, note that SST+ immediately implies MST+, and the result follows from Theorem 1. The failure of the reverse implications is also easily seen by examples.

5.3 A generalization of MUM equivalent to WST

Above, we imposed restrictions on the parameters of the MUM to obtain several special cases in the literature. Conversely, we now relax the triangle inequality property in the distance metric of the MUM to obtain a more general model, and we show this model is equivalent to the weak transitivity postulate.

A *semimetric* on Z is a function $s : Z^2 \to \mathbb{R}_+$ satisfying $s(i, j) = 0$ if and only if $i = j$, and $s(i, j) = s(j, i)$ for all i, j. A semimetric does not need to satisfy the triangle inequality. A choice rule ρ on a finite set Z is a *weak utility model (WUM)* if there is a utility function $u : Z \to \mathbb{R}$, a semimetric $s : Z^2 \to \mathbb{R}_+$ and a strictly increasing function F, such that for all $i \neq j$,

$$\rho(i, j) = F\left(\frac{u(i) - u(j)}{s(i, j)}\right) \tag{WUM}$$

where F satisfies $F(t) = 1 - F(-t)$ for all t.

Proposition 4. *A choice rule ρ on a finite set Z is a WUM iff it satisfies WST.*

We prove this result in the Appendix. Proposition 4 appears at the top of Figure 1. Combining Theorem 1 and Proposition 4, we conclude that the triangle inequality property of the distance metric in the MUM is exactly responsible for the restriction from WST to MST+ in observable choice behavior.

5.4 Relation to random utility models

A choice rule ρ on a finite Z is a *random utility model* (RUM) if there exists a probability measure μ over the strict orderings on Z such that $\rho(i, j)$ equals the probability under μ of the event in which i beats j. Block and Marschak (1959) and Falmagne (1978) characterize the set of RUMs in an abstract setting of choice options when choice data for all finite menus is available. A review of the literature that tackles the characterization of binary choice RUMs is provided by Fishburn (1992). The MUM and RUM families have a non-empty intersection which includes Examples 6 and 7 above. Next, we show that neither MUM nor RUM nest each other.

Example 9. *We modify an example given in de Souza (1983) to obtain a choice rule that is a MUM but not a RUM. Let* $Z = \{1, 2, 3, 4, 5, 6\}$*, let* $0 < \varepsilon < 3/46$ *and let the* *choice rule ρ on Z be given by*

$$\begin{aligned}\rho(4,5) = \rho(4,6) = \rho(2,5) = \rho(2,3) = \rho(1,6) = \rho(1,3) &= 1-\varepsilon \\ \rho(2,6) = \rho(1,5) &= \frac{1}{2}+\varepsilon \\ \rho(2,4) = \rho(1,4) = \rho(3,5) = \rho(3,6) &= \frac{1}{2}+\frac{\varepsilon}{2} \\ \rho(3,4) = \rho(1,2) = \rho(5,6) &= \frac{1}{2}+\frac{\varepsilon}{3}\end{aligned}$$

It is straightforward to verify that ρ satisfies MST+. Now suppose ρ is a RUM generated by the probability μ on the set of strict orderings over Z. Since $\rho(2,3) = \rho(4,6) = 1-\varepsilon$, the probability of the event $\{2 \succ 3\} \cap \{4 \succ 6\}$ is larger or equal to $1-2\varepsilon$. By transitivity, the event $\{2 \succ 3\} \cap \{3 \succ 4\} \cap \{4 \succ 6\}$ is contained in the event $\{2 \succ 6\}$. Hence the event $\{3 \succ 4\} \cap \{6 \succ 2\}$ has at most probability 2ε. By the same reasoning, $\{3 \succ 4\} \cap \{5 \succ 1\}$ has at most probability 2ε. And likewise $\{6 \succ 2\} \cap \{5 \succ 1\}$ has at most probability 2ε. Since μ is a probability, this implies $\rho(3,4) + \rho(5,1) + \rho(6,2) \leq 1 + 6\varepsilon$. But instead we have $\rho(3,4) + \rho(5,1) + \rho(6,2) = 3/2 - 5\varepsilon/3 > 1 + 6\varepsilon$ and therefore ρ cannot be a RUM.

A converse example based on the well-known Condorcet paradox shows that RUM models can violate MST+. Let μ assign equal probability to three strict orderings $i \succ j \succ k$, $j \succ k \succ i$ and $k \succ i \succ j$ over the options i, j and k. Then the binary choice rule ρ generated by μ has $\rho(i,j) = \rho(j,k) = \rho(k,i) = 2/3$ which violates WST, and therefore also violates MST+. Similarly, some recent models proposed in the random choice literature including the random consideration set rule (Manzini and Mariotti, 2014), the attribute rule (Gul et al., 2014), the single-crossing random utility rule (Apesteguia et al., 2017), the deliberately stochastic choice rule (Cerreia-Vioglio et al., 2019) and the focus-then-compare procedure (Ravid and Steverson, 2019) can be easily verified to violate WST and therefore, their binary choice restrictions are not nested by MUM.

In the richer domains of lotteries, Gul and Pesendorfer (2006) introduce the random expected utility model: they characterize the choice behavior of an agent who is an expected utility maximizer with a stochastic Bernoulli index. In our Theorem 2, we impose the same linearity and continuity assumptions employed by Gul and Pesendorfer (2006), specialized to the binary choice domain. Let REM denote any binary choice rule that is obtained as the restriction of a random expected utility model to the domain of binary menus. The relationship between our MEM and REM mirrors the relationship between MUM and RUM: neither model nests the other.

First, it is straightforward to construct a REM that violates WST in the same spirit as the Condorcet paradox example above. By Theorem 2, this behavior cannot be accounted for by a MEM. On the other hand, consider the MUM in Example 9. The proof of Theorem 1 provides a constructive proof to obtain a MUM representation (u, d, F) for the ρ in Example 9. As an exercise, it is straightforward to extend this MUM to a MEM on Δ. This MEM is obviously not a REM, for otherwise, its restriction to Z would be a RUM.

6 Multinomial choice

To conclude the paper, we show that, in addition to helping explain violations of strong transitivity in binary choice, differentiation also plays a key role addressing several non-binary choice phenomena that have attracted attention in the recent random choice literature. Throughout this section, we extend the definition of a choice rule and let $\rho(i, M)$ denote the probability that i is the preferred alternative when the finite set of options M is available.

An advantage of our approach is that we did not take a stance on how behavior in pairwise comparisons relates to behavior in comparisons with more than two options. This makes our results applicable to model and measure differentiation in wider variety of empirical settings. To illustrate this advantage, consider the basic postulate of *regularity*:

(R) $\quad i \in M \subset M'$ implies $\rho(i, M) \geq \rho(i, M')$

This postulate links behavior across choice problems of different sizes. It says that the market share of an existing option cannot increase when new options are introduced. By focusing on binary comparisons, we did not take a stance on regularity. Thus, our characterization and uniqueness results apply both to models of heterogeneous tastes in a population of rational consumers (which satisfy regularity) as well as to models of stochastic individual choice (which can violate regularity). We start by examining the role of differentiation in aggregate demand.

6.1 Differentiation and substitutability in consumer demand

Consider the automobile demand example of the introduction. The choice rule ρ describes the market shares of automobile purchases by a population of standard rational consumers with heterogeneous tastes. The distribution of tastes in the population is described by a probability measure $\mathbb{P}$ over strict preferences:

$$\rho(i, M) = \mathbb{P}\{i \succ j \text{ for all } i \neq j \in M\} \tag{7}$$

which implies ρ satisfies the regularity postulate (R).

Extensions of MUM to multinomial choice taking the form (7) abound in the literature. These include the famous Nested Logit (McFadden, 1978) and Covariance Probit (Hausman and Wise, 1978) from Examples 6–7. The main issue addressed by these models is allowing for flexible patterns of substitutability among the options.

For example, let c_1 and c_2 be two coupe automobiles and let m be a minivan. Suppose $\rho(c_1, c_2) = \rho(c_1, m)$, that is, the fraction of the population that ranks the fist coupe above the second coupe is equal to the fraction that ranks the first coupe above the minivan. In the classic logit model, this last equality would automatically imply the unrealistic consequence that c_2 and m have the same cross-price elasticity with respect to the first coupe. This is a well known limitation of the classic logit model (Nevo, 2000).

The differentiation parameter d in the MUM formula, however, allows us to accommodate realistic patterns of substitution. The first coupe is naturally closer in relevant characteristics to the second coupe than to the minivan: $d(c_1, c_2) < d(c_1, m)$. If a price decrease for c_1 increases its valuation $u(c_1)$ but not its level of differentiation from the other options, then by the MUM formula

$$\frac{\partial \rho(c_1, c_2)}{\partial u(c_1)} = F'\left(\frac{u(c_1) - u(c_2)}{d(c_1, c_2)}\right) \cdot \frac{1}{d(c_1, c_2)} > F'\left(\frac{u(c_1) - u(m)}{d(c_1, m)}\right) \cdot \frac{1}{d(c_1, m)} = \frac{\partial \rho(c_1, m)}{\partial u(c_1)}$$

and hence the demand for the second coupe can be more sensitive to price changes in the first coupe than the demand for the minivan.

6.2 Differentiation and violations of regularity

The MUM and MEM formulas can model the stochastic choices made by a single individual, reflecting the behavioral limitations of the decision maker (Thurstone, 1927; Luce, 1959; Gul et al., 2014; Matejka and McKay, 2014). In this interpretation, the differentiation metric d reflects the comparability of the options, driving choice probabilities for options that are harder to compare closer to one-half.

The experimental designs in Examples 3 and 4 varied the degree of differentiation between the options, making some pairs of options easier to compare than others. In addition to the binary choice data shown in Examples 3 and 4, both experiments also included treatments with three available options, and found significant violations of the regularity postulate (R). In Example 3, the addition of the inferior decoy 2 to the menu $\{\ell, h\}$ increased the choice probability of option ℓ (Soltani et al., 2012). In Example 4, the addition of option C to the menu $\{A, B\}$ increased the choice probability of option A (Lea and Ryan, 2015). The first type of regularity violation is known in the literature as the *attraction effect* (Huber et al., 1982), while the second type of regularity violation is known as the *compromise effect* (Simonson, 1989).

Differentiation plays a systematic role in these violations of regularity. In both cases, the added option boosts the probability of the existing option that is closer in characteristics and easier to compare. Natenzon (2019) shows how this behavior may arise from the optimal response of an imperfectly informed agent to the ease of comparison between the options. The attraction and compromise effects are then explained and predicted with a multinomial choice model in which binary comparisons are given by a MUM. The differentiation parameter d is the key variable that determines ease of comparison and drives the violations of regularity. Our characterization and uniqueness results can therefore be used to explain and predict violations of regularity in individual choice.

6.3 Differentiation and dynamic choice

Consider a static choice setting in which the differentiation parameter d in the MUM is approximately constant. For example, let the alternatives be ten different meal choices labeled 1 to 10. The meal options can be very different but present the same degree of substitutability between them. Hence, the classic logit model and other strongly transitive models provide reasonable approximations to meal choice problems.

Gul et al. (2014) point out that, even in the setting above, dynamic choice problems generate substantial variation in the degree of substitutability of the options, given the natural overlap that occurs between continuation problems. For example, let a restaurant be described as a finite menu of meals. In the first period, the decision maker chooses a restaurant. In the second period, the decision maker chooses a meal from that restaurant. Even if all meal options show the same degree of substitutability, the same is not true for restaurants: for example, a consumer that abandons restaurant $\{1, 2, 3, 4, 5, 6, 7\}$ due to a cost increase is more likely to substitute it with restaurant $\{2, 3, 4, 5, 6, 7, 8\}$ than with restaurant $\{9, 10\}$. The overlap in meals between the first two restaurants makes them more substitutable. Even when the differentiation parameter d plays no role in describing meal choice, it matters for restaurant choice. Hence, our discrete choice foundation for modeling and measuring differentiation as a distance metric might be particularly relevant in dynamic choice settings.

A Proofs

Proof of Proposition 1

Let $Z = \{1, 2, \ldots, n\}$ be the finite set of alternatives. Consider the set of choice rules ρ on Z which satisfy WST with $\rho(i, j) \geq 1/2$ whenever $i \leq j$ and for which the set $\{\rho(i, j) \in [0, 1] : i \neq j\}$ has maximum cardinality with $n(n-1)$ elements. Each such ρ induces a strict ordering $\succ_\rho$ of the $n(n+1)/2$ pairs $P_n := \{(i, j) : 1 \leq i < j \leq n\}$ given by $(i, j) \succ_\rho (k, \ell)$ if and only if $\rho(i, j) > \rho(k, \ell)$. This set of choice rules ρ induces $\#WST(n) = [n(n-1)/2]!$ different strict orderings $\succ_\rho$ on P_n.

MST and MST+ allow the same number of different strict orderings over P_n which we denote $\#MST(n)$. Now consider the addition of alternative $n + 1$ to the set Z.

Lemma 3. $\#MST(n + 1) \leq [n(n-1)/2 + 1]^n \, \#MST(n)$

Proof. Take a single strict ordering over P_n compatible with MST. There are multiple ways to extend this strict ordering to incorporate the new pairs $(1, n+1)$, $(2, n+1)$, ..., $(n, n+1)$ and obtain a strict ordering over P_{n+1} that is still compatible with MST. Since the original ordering has $n(n-1)/2$ pairs, there are $n(n-1)/2 + 1$ different positions to include $(n, n+1)$. In this way we obtain $n(n-1)/2 + 1$ different strict orderings, all of which respect MST. The total number of strict orderings over $P_n \cup \{(n, n+1)\}$ that satisfy MST is therefore $[n(n-1)/2+1] \, \#MST(n)$. Now we take one such strict ordering and extend it to incorporate a second pair $(n-1, n+1)$. This pair can in principle be added into $n(n-1)/2+2$ different positions, but placing it in the very last position would violate MST, since MST requires $\rho(n-1, n+1) > \min\{\rho(n-1, n), \rho(n, n+1)\}$. The total number of strict orderings over $P_n \cup \{(n, n+1), (n-1, n+1)\}$ which satisfy MST must therefore be smaller or equal to $[n(n-1)/2 + 1]^2 \, \#MST(n)$. A simple inductive argument completes the proof. $\square$

Lemma 4. $\lim_{n\to\infty} \left[\prod_{k=1}^{n} \frac{n(n-1)/2+k}{n(n-1)/2+1}\right] = e$

Proof. The result can be shown by verifying that, for each n,

$$\left(1+\frac{1}{n}\right)^{n-1} \leq \left[\prod_{k=1}^{n} \frac{n(n-1)/2+k}{n(n-1)/2+1}\right] \leq \left(1+\frac{1}{n}\right)^{n}$$

and taking the limit as $n \to \infty$. We leave the details to the reader. $\square$

Lemma 3 implies that

$$\begin{aligned}\frac{\#MST(n+1)}{\#WST(n+1)} &\leq \frac{\#MST(n)}{\#WST(n)} \frac{[n(n-1)/2]!}{[n(n+1)/2]!}[n(n-1)/2+1]^n \\ &= \frac{\#MST(n)}{\#WST(n)} \left[\prod_{k=1}^{n} \frac{n(n-1)/2+1}{n(n-1)/2+k}\right]\end{aligned}$$

and by Lemma 4 the last expression in brackets goes to $1/e$ when n goes to infinity, where $e \approx 2.718$ is the base of the natural logarithm. Hence for all n sufficiently large the ratio $\#MST(n+1)/\#WST(n+1)$ is less than half of the ratio $\#MST(n)/\#WST(n)$, which completes the proof.

Finally, we prove the additional claim, stated after Proposition 1, that

$$\lim_{n\to\infty} \#SST(n)/\#MST(n) = 0.$$

The choice probability $\rho(1,n)$ must be the highest choice probability in every ρ that satisfies SST. For each strict ordering of choice probabilities satisfying SST, there exist at least $n-2$ strict orderings which violate SST but satisfy MST: for each $k = 2, 3, \ldots, n-1$ change the value of $\rho(1,n)$ to be equal to $\max\{\rho(1,k), \rho(k,n)\} - \varepsilon$ for $\varepsilon > 0$ sufficiently small. It is immediate to see that each resulting ranking violates SST. To see that MST still holds, note that every inequality required by SST holds except those involving $\rho(1,n)$. In addition, SST implies that for each $k, j = 2, \ldots, n-1$, $\max\{\rho(1,k), \rho(k,n\}) > \min\{\rho(1,j), \rho(j,n\})$ hence for ε small we have $\rho(1,n) > \min\{\rho(1,j), \rho(j,n)\}$. Thus, $\#SST(n)/\#MST(n) \leq 1/(n-1) \to 0$ when $n \to \infty$. $\square$

Proof of Theorem 1

Necessity is shown in the main text. For sufficiency, suppose ρ satisfies MST+. In particular, ρ satisfies WST, and hence, by letting $x \succcurlyeq y$ if and only if $\rho(x,y) \geq 1/2$, we obtain a complete and transitive relation $\succcurlyeq$ over the finite set of options Z. The relation

$\succcurlyeq$ induced by ρ divides the n alternatives in Z into $k \leq n$ indifference classes. Therefore, there exists a utility function $u : Z \to \{1, \dots, k\}$ that is onto and represents $\succcurlyeq$, that is, $u(x) \geq u(y)$ if and only if $x \succcurlyeq y$ if and only if $\rho(x, y) \geq 1/2$.

Let $Y := \{\{x, y\} \subset Z : \rho(x, y) \neq 1/2\}$, and let m be the cardinality of the set $\{|\rho(x, y) - 1/2| : \{x, y\} \in Y\}$. Partition the set Y into m disjoint sets $Y_1 \cup Y_2 \cup \cdots \cup Y_m = Y$ such that for any two pairs $\{w, x\}$ and $\{y, z\}$ in Y we have $\{w, x\} \in Y_i$ and $\{y, z\} \in Y_j$ with $i \geq j$ if and only if $|\rho(w, x) - 1/2| \leq |\rho(y, z) - 1/2|$. Thus, the pairs in Y_1 have the highest value of $|\rho(x, y) - 1/2|$, while the pairs in Y_m have the lowest value of $|\rho(x, y) - 1/2|$ in Y.

The result is trivial when Z has $n \leq 2$ alternatives so suppose $n \geq 3$. Define a constant $C = (n-1)^{[n(n-1)/2+1]} > 0$ and define the sequence $D_1, D_2, \dots, D_m$ by:

$$D_1 = 0; D_j = (n-1)^{j-2} \text{ for } j = 2, \dots, m.$$

Let $d : Z \times Z \to [0, \infty)$ be defined as follows:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ C, & \text{if } x \neq y \text{ and } \rho(x, y) = 1/2 \\ (C/2 + D_j)\,|u(x) - u(y)|\,, & \text{if } \{x, y\} \in Y_j \end{cases} \tag{8}$$

From the definition (8) it is immediate that d satisfies (i) $d(x, y) \geq 0$; (ii) $d(x, y) = 0$ if and only if $x = y$; and (iii) $d(x, y) = d(y, x)$ for all $x, y \in Z$. To show that d is a metric, it remains to verify the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$. The inequality trivially holds when any two options among x, y, z are equal. Consider three distinct options $x, y, z \in Z$.

Case 1: $u(x) = u(y) = u(z)$. By the definition of u we have $\rho(x, y) = \rho(y, z) = \rho(x, z) = 1/2$. By the definition of d we have $d(x, z) = C < 2C = d(x, y) + d(y, z)$.

Case 2: $u(x) \neq u(y) = u(z)$. The definitions of u and d imply

$$\begin{aligned} d(x, y) + d(y, z) - d(x, z) &= (C/2 + D_i)\,|u(x) - u(y)| + C - (C/2 + D_j)\,|u(x) - u(z)| \\ &= (D_i - D_j)\,|u(x) - u(z)| + C \\ &\geq -(n-1)^{m-2}(n-1) + C \\ &= (n-1)^{[n(n-1)/2+1]} - (n-1)^{m-1} \\ &> 0 \end{aligned}$$

where the last inequality follows from the fact that we defined m to be the cardinality of $\{|\rho(x,y)-1/2| : \{x,y\} \in Y\}$ which is smaller or equal to $n(n-1)/2$.

Case 3: $u(y) \neq u(x) = u(z)$. The definitions of u and d imply

$$\begin{aligned} d(x,y)+d(y,z)-d(x,z) &= (C/2+D_i)\,|u(x)-u(y)| + (C/2+D_j)\,|u(y)-u(z)| - C \\ &= (C+D_i+D_j)\,|u(y)-u(z)| - C \\ &\geq 0. \end{aligned}$$

Case 4: $u(z) \neq u(x) = u(y)$. Result follows from the same argument as in Case 2.

Case 5: $u(x) > u(y) > u(z)$. By the definition of u we have $\{x,y\} \in Y_i$, $\{y,z\} \in Y_j$, and $\{x,z\} \in Y_\ell$, for some i,j,ℓ. The definition of d implies

$$\begin{aligned} d(x,y)+d(y,z)-d(x,z) &= (C/2+D_i)\,|u(x)-u(y)| + (C/2+D_j)\,|u(y)-u(z)| \\ &\quad - (C/2+D_\ell)\,|u(x)-u(y)+u(y)-u(z)| \\ &= (D_i-D_\ell)\,|u(x)-u(y)| + (D_j-D_\ell)\,|u(y)-u(z)| \end{aligned}$$

The definition of u implies $\rho(x,y) > 1/2$ and $\rho(y,z) > 1/2$. By MST+ we have either $\rho(x,y) = \rho(y,z) = \rho(x,z)$ or $\rho(x,z) > \min\{\rho(x,y),\rho(y,z)\}$. The first case implies $D_i = D_j = D_\ell$ above and therefore $d(x,y)+d(y,z)-d(x,z) = 0$. The second case implies $D_\ell < \max\{D_i,D_j\}$. If $D_\ell \leq \min\{D_i,D_j\}$ then both $(D_i - D_\ell)$ and $(D_j - D_\ell)$ above are positive and the desired inequality holds. It remains to show the inequality holds when $\min\{D_i,D_j\} < D_\ell < \max\{D_i,D_j\}$, which implies

$$\begin{aligned} d(x,y)+d(y,z)-d(x,z) &\geq (\max\{D_i,D_j\}-D_\ell)\,1 + (\min\{D_i,D_j\}-D_\ell)\,(n-2) \\ &\geq (n-1)^{\ell-1} - (n-1)^{\ell-2} + [0-(n-1)^{\ell-2}](n-2) \\ &= 0. \end{aligned}$$

Case 6: $u(x) > u(z) > u(y)$. By the definition of u we have $\{x,y\} \in Y_i$, $\{y,z\} \in Y_j$, and $\{x, z\} \in Y_\ell$, for some i, j, ℓ. The definition of d implies

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,[u(x) - u(z) + u(z) - u(y)] \\ &\quad + (C/2 + D_j)\,[u(z) - u(y)] - (C/2 + D_\ell)\,[u(x) - u(z)] \\ &= (D_i - D_\ell)\,[u(x) - u(z)] + (C + D_i + D_j)\,[u(z) - u(y)] \\ &\geq \left(0 - (n-1)^{m-2}\right)(n-2) + (C + 0 + 0)\,1 \\ &= -(n-1)^{m-1} + (n-1)^{m-2} + (n-1)^{n(n-1)/2+1} \\ &> 0. \end{aligned}$$

Case 7: $u(y) > u(x) > u(z)$. By the definition of u we have $\{x, y\} \in Y_i$, $\{y, z\} \in Y_j$, and $\{x, z\} \in Y_\ell$, for some i, j, ℓ. The definition of d implies

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,[u(y) - u(x)] \\ &\quad + (C/2 + D_j)\,[u(y) - u(x) + u(x) - u(z)] \\ &\quad - (C/2 + D_\ell)\,[u(x) - u(z)] \\ &= (C + D_i + D_j)\,[u(y) - u(x)] + (D_j - D_\ell)\,[u(x) - u(z)] \\ &> 0. \end{aligned}$$

Case 8: $u(y) > u(z) > u(x)$. Similarly to Case 7, we have

$$d(x,y) + d(y,z) - d(x,z) = (C + D_i + D_j)\,[u(y) - u(z)] + (D_i - D_\ell)\,[u(z) - u(x)] > 0.$$

Case 9: $u(z) > u(x) > u(y)$. Similarly to Cases 7 and 8, we have

$$d(x,y) + d(y,z) - d(x,z) = (C + D_i + D_j)\,[u(x) - u(y)] + (D_j - D_\ell)\,[u(z) - u(x)] > 0.$$

Case 10: $u(z) > u(y) > u(x)$. Since $d(x,y) + d(y,z) \leq d(x,z)$ if and only if $d(y,x) + d(z,y) \leq d(z,x)$, the inequality follows from the same argument as in Case 5.

By Cases 1 to 10 above, d satisfies the triangle inequality and is therefore a metric. Now we show u and d constructed above provide an ordinal representation for ρ, that is,

$$\rho(w,x) \geq \rho(y,z) \text{ if and only if } \frac{u(w) - u(x)}{d(w,x)} \geq \frac{u(y) - u(z)}{d(y,z)}. \tag{9}$$

First, $\rho(w,x) \geq \rho(y,z) > 1/2$ if and only if $\rho(w,x) > 1/2$, $\rho(y,z) > 1/2$, and $|\rho(w,x) - 1/2| \geq |\rho(y,z) - 1/2|$; if and only if $u(w) > u(x)$, $u(y) > u(z)$, $d(w,x) = (C/2 + D_i)[u(w) - u(x)]$, $d(y,z) = (C/2 + D_j)[u(y) - u(z)]$, and $i \leq j$; if and only if

$$\frac{u(w) - u(x)}{d(w,x)} = \frac{1}{C/2 + D_i} \geq \frac{1}{C/2 + D_j} = \frac{u(y) - u(z)}{d(y,z)} > 0.$$

Second, $1/2 > \rho(w,x) \geq \rho(y,z)$ if and only if $\rho(z,y) \geq \rho(x,w) > 1/2$ and the desired inequality follows from the step above. Finally, $\rho(w,x) \geq 1/2 \geq \rho(y,z)$ if and only if $u(w) - u(x) \geq 0 \geq u(y) - u(z)$ if and only if

$$\frac{u(w) - u(x)}{d(w,x)} \geq 0 \geq \frac{u(y) - u(z)}{d(y,z)}$$

hence the ordinal representation (9) holds. Finding a strictly increasing F such that the cardinal representation (1) holds is then straightforward and left to the reader. □

Proof of Theorem 2

To show necessity, let $U : \Delta \to \mathbb{R}$ be linear, let $\|\cdot\|$ be a norm on the subspace $\{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$, generated by an inner product $\|x\| = \sqrt{\langle x, x \rangle}$, and let F be a strictly increasing, continuous transformation such that the MEM representation (2) holds.

First, ρ must be continuous outside the diagonal since (i) U is linear; (ii) $\|\cdot\|$ is a norm hence $\|x - y\| > 0$ for $x \neq y$; and (iii) F is continuous. Second, ρ satisfies

$$\begin{aligned} \rho(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z) &= F\left(\frac{U(\alpha x + (1-\alpha)z) - U(\alpha y + (1-\alpha)z)}{\|\alpha x + (1-\alpha)z - [\alpha y + (1-\alpha)z]\|}\right) \\ &= F\left(\frac{\alpha[U(x) - U(y)]}{\alpha\|x - y\|}\right) \\ &= \rho(x,y) \end{aligned}$$

whenever $0 < \alpha < 1$ and $x \neq y$, and the equality holds trivially when $x = y$. Finally, we show that ρ must satisfy independence. Suppose $\rho(x,y) = 1/2$ and $\rho(x,z) > \rho(y,z) > 1/2$, and let $1 > \alpha > 1/2$. By (2) we have $U(x) = U(y) > U(z)$ and $\|x - z\| < \|y - z\|$.

Then,

$$
\begin{aligned}
\|\alpha x+(1-\alpha)y-z\|^2 &= \alpha^2\|x-z\|^2+2\alpha(1-\alpha)\,\langle x-z,y-z\rangle+(1-\alpha)^2\|y-z\|^2 \\
&< \alpha^2\|y-z\|^2+2\alpha(1-\alpha)\,\langle x-z,y-z\rangle+(1-\alpha)^2\|x-z\|^2 \\
&= \|\alpha y+(1-\alpha)x-z\|^2
\end{aligned}
$$

and by (2) we have $\rho(\alpha x+(1-\alpha)y,z)>\rho(\alpha y+(1-\alpha)x,z)$ as desired.

To show sufficiency, let the choice rule ρ on Δ satisfy linearity, continuity (outside the diagonal), independence, and MST+. The result is trivial for constant ρ, so we now consider the case in which ρ is not constant. For the remainder of the proof, we choose and fix two lotteries $\bar{x},\bar{y}\in\Delta$ with $\rho(\bar{x},\bar{y})>1/2$. First, we show that ρ has a unique linear extension to the $n-1$ dimensional hyperplane H that contains Δ.

Lemma 5. *ρ has a unique linear extension to $H=\{(x_1,\dots,x_n)\in\mathbb{R}^n : x_1+\cdots+x_n=1\}$.*

Proof. Let ρ' and ρ'' be two linear extensions of ρ and let $x,y\in\mathbb{R}^n$ with $x_1+\cdots+x_n=y_1+\cdots+y_n=1$. Let $z=(1/n,\dots,1/n)\in\Delta$. Take $0<\alpha<1$ sufficiently small such that $0<\alpha x_i+(1-\alpha)/n<1$ and $0<\alpha y_i+(1-\alpha)/n<1$ for each i. Then $\alpha x+(1-\alpha)z\in\Delta$, $\alpha y+(1-\alpha)z\in\Delta$ and, by linearity,

$$
\begin{aligned}
\rho'(x,y) &= \rho'(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)z) \\
&= \rho(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)z) \\
&= \rho''(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)z) \\
&= \rho''(x,y).
\end{aligned}
$$

□

From this point on, we identify ρ with its unique linear extension. Define the relation $\succcurlyeq\subset H\times H$ by $x\succcurlyeq y$ if and only if $\rho(x,y)\geq 1/2$. Since ρ satisfies MST+, this $\succcurlyeq$ is complete and transitive. By linearity and continuity, $\succcurlyeq$ satisfies all the vNM axioms and admits an expected utility representation. Let $U:H\to\mathbb{R}$ be a linear function representing $\succcurlyeq$.

For each lottery x, let $I(x):=\{y\in H:\rho(x,y)=1/2\}$ denote the set of lotteries that are stochastically indifferent to x. Note that $I(x)$ is an affine subspace of dimension $n-2$. By linearity, ρ is entirely determined by the values of the mapping $x\mapsto\rho(x,\bar{y})$ for $x\in I(\bar{x})$, where $\bar{x},\bar{y}$ are the two lotteries with $\rho(\bar{x},\bar{y})>1/2$ that we fixed above. For each $1/2<p\leq 1$, define the upper contour sets $B(p):=\{x\in I(\bar{x}):\rho(x,\bar{y})\geq p\}$.

Lemma 6. *$B(p)$ is convex for all $1/2 < p \leq 1$.*

Proof. Let $x, x' \in B(p)$ and let $0 < \alpha < 1$. Since $I(\bar{x})$ is an affine subspace, $\alpha x + (1-\alpha)x' \in I(\bar{x})$. Linearity implies $\rho(\alpha x + (1-\alpha)x', \alpha\bar{y} + (1-\alpha)x') = \rho(x, \bar{y}) \geq p$. Linearity also implies $\rho(\alpha\bar{y} + (1-\alpha)x', \bar{y}) = \rho(x', \bar{y}) \geq p$. Then, MST+ implies $\rho(\alpha x + (1-\alpha)x', \bar{y}) \geq p$. □

Lemma 7. *$B(p)$ is compact for all $1/2 < p \leq 1$.*

Proof. $B(p)$ is closed by continuity. Let $|\cdot|$ denote the standard Euclidean metric, not necessarily equal to the metric we are going to construct for the representation. If $B(p)$ were not bounded, there would exist a sequence $x(k)$ in $B(p)$ with $|x(k) - \bar{y}| \geq k$ for all $k \in \mathbb{N}$. For each k, by linearity $\rho(\bar{y} + (x(k) - \bar{y})/|x(k) - \bar{y}|, \bar{y}) = \rho(x(k), \bar{y}) \geq p$. By Bolzano-Weierstrass the sequence $\bar{y} + (x(k) - \bar{y})/|x(k) - \bar{y}|$ would have a subsequence converging to some $z \neq \bar{y}$. By the linearity of U we would have $U(z) = U(\bar{y})$ and $\rho(z, \bar{y}) = 1/2$, contradicting continuity. Hence $B(p)$ must also be bounded. □

Lemma 8. *The mapping $x \mapsto \rho(x, \bar{y})$ has a unique maximizer $\hat{x}$ on $I(\bar{x})$.*

Proof. Since $\rho(\bar{x}, \bar{y}) > 1/2$ we have $B(p) \neq \emptyset$ for some $p > 1/2$. Since ρ is continuous outside the diagonal, the mapping $x \mapsto \rho(x, \bar{y})$ is continuous on $I(\bar{x})$. $B(p)$ is compact by Lemma 7, hence the maximum $\rho(\hat{x}, \bar{y}) = \bar{p}$ is attained at some $\hat{x} \in B(p)$. Hence $B(\bar{p})$ is not empty, and by the previous lemmas it is compact and convex. Since ρ satisfies independence, $B(\bar{p})$ must be a singleton. Otherwise, by Lemmas 6 and 7 there would exist a nontrivial segment $[\hat{x}, \hat{x}']$ contained in $B(\bar{p})$ with $\hat{x}'$ on the boundary of $B(\bar{p})$, so that the point $x'' = \hat{x}' + (1/2)(\hat{x}' - \hat{x})$ lies outside $B(\bar{p})$, that is $\rho(x'', \bar{y}) < \bar{p} = \rho(\hat{x}, \bar{y})$. But then the point $(2/3)\hat{x} + (1/3)x'' = (1/2)\hat{x} + (1/2)\hat{x}' \in B(\bar{p})$ and the point $(1/3)\hat{x} + (2/3)x'' = \hat{x}' \in B(\bar{p})$, with $\rho((1/3)\hat{x} + (2/3)x'', \bar{y}) = \bar{p} = \rho((2/3)\hat{x} + (1/3)x'', \bar{y})$ contradicting independence. □

For the rest of the proof, we denote by $\hat{x}$ the unique maximizer of $x \mapsto \rho(x, \bar{y})$ on $I(\bar{x})$.

Lemma 9. *$x \in I(\bar{x})$ and $\rho(x, \bar{y}) = p$ implies $\rho(2\hat{x} - x, \bar{y}) = p$.*

Proof. The statement trivially holds if $x = \hat{x}$, so suppose $x \neq \hat{x}$. First note $2\hat{x} - x = \hat{x} + (\hat{x} - x) \in I(\bar{x})$. If $\rho(\hat{x} + (\hat{x} - x), \bar{y}) < p$, by continuity there is a sufficiently small $\varepsilon > 0$ such that $\rho(\hat{x} + (1-\varepsilon)(\hat{x} - x), \bar{y}) < p$. Taking $\alpha = 1/(2-\varepsilon)$ we have $\hat{x} = \alpha(\hat{x} + (1-\varepsilon)(\hat{x} - x)) + (1-\alpha)x$. By Lemma 8 we have $\rho(\alpha(\hat{x} + (1-\varepsilon)(\hat{x} - x)) +$

$(1-\alpha)x, \bar{y}) > \rho((1-\alpha)(\hat{x} + (1-\varepsilon)(\hat{x}-x)) + \alpha x, \bar{y})$ contradicting independence. Hence $\rho(2\hat{x} - x, \bar{y}) \geq p$. An entirely analogous argument shows that $\rho(2\hat{x} - x, \bar{y}) \leq p$. □

Recall that $\hat{x}$ is the unique maximizer $\rho(\hat{x}, \bar{y}) = \bar{p}$ on $I(\bar{x})$. Let $B = B(p) - \hat{x}$ for some fixed $p \in (1/2, \bar{p})$. We first define an auxiliary norm $\|\cdot\|_B$ on the $n-2$ dimensional subspace $I(\bar{x}) - \hat{x}$ using B as the unit ball.

Lemma 10. *$\|x\|_B := \inf\{\lambda \geq 0 : x \in \lambda B\}$ is a norm on $I(\bar{x}) - \hat{x}$.*

Proof. The *Minkowski functional* $\|\cdot\|_B$ defined above is a norm when B is a symmetric, convex set such that each line through zero meets B in a non-trivial, closed, bounded segment (Thompson, 1996). By definition $\|x\|_B \geq 0$ for all x. Moreover, if $\|x\|_B = 0$ then $x \in \lambda B$ for all $\lambda > 0$ and therefore $x = 0$. Now for each $\alpha \geq 0$ we have $x \in \lambda B$ if and only if $\alpha x \in \alpha\lambda B$ and therefore $\alpha\|x\|_B = \|\alpha x\|_B$. Lemma 9 implies $x \in \lambda B$ if and only if $-x \in \lambda B$ and therefore $\|x\|_B = \|-x\|_B$. To verify the triangle inequality, note that B is closed by Lemma 7, and therefore $x/\|x\|_B \in B$ for all $x \neq 0$. B is also convex by Lemma 6, and therefore

$$\frac{x + x'}{\|x\|_B + \|x'\|_B} = \left(\frac{\|x\|_B}{\|x\|_B + \|x'\|_B}\right)\frac{x}{\|x\|_B} + \left(\frac{\|x'\|_B}{\|x\|_B + \|x'\|_B}\right)\frac{x'}{\|x'\|_B} \in B.$$

Thus,

$$\left\|\frac{x + x'}{\|x\|_B + \|x'\|_B}\right\|_B \leq 1$$

and the triangle inequality $\|x + x'\|_B \leq \|x\|_B + \|x'\|_B$ holds. □

Lemma 11. *If $\bar{p} \geq p \geq q > 1/2$ then $B(p) = \hat{x} + \lambda\,[B(q) - \hat{x}]$ for some $0 \leq \lambda \leq 1$.*

Proof. MST+ implies that, for any $x \neq \hat{x}$ in $B(p)$, the function $t \mapsto \rho(t\hat{x} + (1-t)x, \bar{y})$ is strictly increasing for $0 \leq t \leq 1$. It suffices to show that if $\rho(x^1, \bar{y}) = \rho(x^2, \bar{y})$ for $x^1, x^2 \in I(\bar{x})$ and $0 < \alpha < 1$, then $\rho(\alpha x^1 + (1-\alpha)\hat{x}, \bar{y}) = \rho(\alpha x^2 + (1-\alpha)\hat{x}, \bar{y})$. To see that equality must hold, suppose instead that $\rho(\alpha x^1 + (1-\alpha)\hat{x}, \bar{y}) < \rho(\alpha x^2 + (1-\alpha)\hat{x}, \bar{y})$. Continuity implies $\rho(\beta x^2 + (1-\beta)\hat{x}, \bar{y}) = \rho(\alpha x^1 + (1-\alpha)\hat{x}, \bar{y})$ for some $0 < \alpha < \beta < 1$.

Letting

$$
\begin{aligned}
z^1 &= x^1 + \frac{\beta(1-\alpha)}{\beta-\alpha}(x^2 - x^1) \\
z^2 &= x^1 + x^2 - z^1 \\
z^3 &= 2\hat{x} - z^1 \\
z^4 &= \alpha x^1 + \beta x^2 + (2-\alpha-\beta)\hat{x} - z^1
\end{aligned}
$$

we have that the line segment $[z^1, z^2]$ contains the line segment $[x^1, x^2]$; the line segment $[z^1, z^4]$ contains the line segment $[\alpha x^1 + (1-\alpha)\hat{x}, \beta x^2 + (1-\beta)\hat{x}]$ and

$$
\begin{aligned}
z^1/2 + z^2/2 &= x^1/2 + x^2/2 \\
z^1/2 + z^3/2 &= \hat{x} \\
z^1/2 + z^4/2 &= (\beta x^2 + (1-\beta)\hat{x})/2 + (\alpha x^1 + (1-\alpha)\hat{x})/2
\end{aligned}
$$

By Lemma 9 we have $\rho(z^3, \bar{y}) = \rho(z^1, \bar{y})$. We must also have $\rho(z^2, \bar{y}) = \rho(z^1, \bar{y})$, for otherwise $\rho(z^2, \bar{y}) \neq \rho(z^1, \bar{y})$ and $\rho(x^1, \bar{y}) = \rho(x^2, \bar{y})$ would contradict independence. And again by independence we have $\rho(z^4, \bar{y}) = \rho(z^1, \bar{y})$. Now since $0 < \alpha < \beta < 1$ we have

$$
0 < \frac{\alpha\beta(2-\alpha-\beta)}{\alpha(1-\alpha)+\beta(1-\beta)} < 1 < \frac{\alpha(1-\beta)+\beta(1-\alpha)}{\alpha(1-\alpha)+\beta(1-\beta)}.
$$

Letting

$$
z^5 = \left(\frac{\alpha\beta(2-\alpha-\beta)}{\alpha(1-\alpha)+\beta(1-\beta)}\right) z^2 + \left(1 - \frac{\alpha\beta(2-\alpha-\beta)}{\alpha(1-\alpha)+\beta(1-\beta)}\right) z^3
$$

we have z^5 belongs to the segment $[z^2, z^3]$ and by Lemma 6 it must be $\rho(z^5, \bar{y}) \geq \rho(z^4, \bar{y})$. On the other hand, it is straightforward to verify the equality

$$
z^5 - \hat{x} = \left[\frac{\alpha(1-\beta)+\beta(1-\alpha)}{\alpha(1-\alpha)+\beta(1-\beta)}\right](z^4 - \hat{x})
$$

so z^4 lies in the interior of the segment $[z^5, \hat{x}]$. But then the mapping $t \mapsto \rho(t\hat{x} + (1-t)z^5, \bar{y})$ is not strictly increasing for $0 \leq t \leq 1$, contradicting MST+. $\square$

Lemma 12. *$\|\cdot\|_B$ is Euclidean, i.e., $\|x\|_B = \sqrt{\langle x, x\rangle_B}$ where $\langle \cdot, \cdot\rangle_B$ is an inner product.*

Proof. We use a characterization of inner product spaces by Gurari and Sozonov (1970), who showed that a normed linear space is an inner product space if and only if

$$\|x\| = \|y\| = 1 \text{ and } 0 \leq \alpha \leq 1 \quad \text{imply} \quad \left\| \frac{1}{2}x + \frac{1}{2}y \right\| \leq \|\alpha x + (1-\alpha)y\|. \tag{10}$$

If $\|x\|_B = \|y\|_B = 1$ then x, y are on the boundary of B, hence $\rho(x + \hat{x}, \bar{y}) = \rho(y + \hat{x}, \bar{y}) = p > 1/2$ and $\rho(x + \hat{x}, y + \hat{x}) = 1/2$. A violation of condition (10) would entail $\|(1/2)x + (1/2)y\|_B > \|\alpha x + (1-\alpha)y\|_B$ where, without loss of generality $1/2 < \alpha < 1$. Let

$$\begin{aligned}
x^\alpha &= \alpha(x + \hat{x}) + (1-\alpha)(y + \hat{x}) \\
x^{1/2} &= (1/2)(x + \hat{x}) + (1/2)(y + \hat{x}) \\
x' &= (\alpha - 1/2)(x + \hat{x}) + (3/2 - \alpha)(y + \hat{x}) \\
x'' &= (y + \hat{x}) + (1/2)(y - x)
\end{aligned}$$

By Lemma 11 the sets $B\left(\rho(x^\alpha, \bar{y})\right) - \hat{x}$ and $B\left(\rho(x^{1/2}, \bar{y})\right) - \hat{x}$ are dilations of $B(p) - \hat{x}$, hence $\rho(x^\alpha, \bar{y}) > \rho(x^{1/2}, \bar{y})$. By construction, the segment $[x^\alpha, x'']$ contains the segment $[x', y + \hat{x}]$ and both have the midpoint $(\alpha/2 - 1/4)(x + \hat{x}) + (5/4 - \alpha/2)(y + \hat{x})$. Likewise, the segment $[x + \hat{x}, x']$ contains the segment $[x^\alpha, x^{1/2}]$ and both have the midpoint $(1/4 + \alpha/2)(x + \hat{x}) + (3/4 - \alpha/2)(y + \hat{x})$. By Lemma 6 $\rho(x', \bar{y}) \geq \rho(y + \hat{x}, \bar{y})$. This last inequality must in fact be strict, for otherwise by independence we would have $\rho(x'', \bar{y}) = \rho(x^\alpha, \bar{y}) > \rho(y + \hat{x}, \bar{y})$ violating Lemma 6. Thus $\rho(x', \bar{y}) > \rho(y + \hat{x}, \bar{y}) = \rho(x + \hat{x}, \bar{y})$. But then $\rho(x', \bar{y}) > \rho(x + \hat{x}, \bar{y})$ and independence imply $\rho(x^{1/2}, \bar{y}) > \rho(x^\alpha, \bar{y})$, a contradiction. Hence $\|\cdot\|_B$ satisfies (10). □

Now we extend the inner product $\langle \cdot, \cdot \rangle_B$ on the $n-2$ dimensional subspace $I(\bar{x}) - \hat{x}$ obtained in the last Lemma to an inner product $\langle \cdot, \cdot \rangle$ on the $n-1$ dimensional subspace $H - \hat{x}$. Let $v_1, \ldots, v_{n-2}$ be an orthonormal base for the subspace $I(\bar{x}) - \hat{x}$ endowed with $\langle \cdot, \cdot \rangle_B$. Let $v_{n-1} := \hat{x} - \bar{y}$ and for every $1 \leq i, j \leq n-1$ let $\langle v_i, v_j \rangle = 0$ if $i \neq j$ and $\langle v_i, v_j \rangle = 1$ if $i = j$. We let the norm be induced by this inner product $\|x\| := \sqrt{\langle x, x \rangle}$ for all $x \in H - \hat{x}$. Before showing an ordinal representation of ρ with U and $\|\cdot\|$ holds, we must show the upper contour sets $B(p) = \{x \in I(\bar{x}) : \rho(x, \bar{y}) \geq p\}$ are dilations of one another.

Lemma 13. *U and $\|\cdot\|$ provide an ordinal representation of ρ, that is, for any $w \neq x$*

and $y \neq z$ we have

$$\rho(w,x) \geq \rho(y,z) \iff \frac{U(w)-U(x)}{\|w-x\|} \geq \frac{U(y)-U(z)}{\|y-z\|}.$$

Proof. First, suppose $\rho(w,x) \geq \rho(y,z) > 1/2$. Then $w \succ x$, $y \succ z$ and since U represents $\succcurlyeq$ we have $U(w) > U(x)$ and $U(y) > U(z)$. Let

$$w' = \bar{y} + \frac{U(\hat{x})-U(\bar{y})}{U(w)-U(x)}(w-x)$$
$$y' = \bar{y} + \frac{U(\hat{x})-U(\bar{y})}{U(y)-U(z)}(y-z)$$

and note that $w', y' \in H$. Since U is linear, $U(w') = U(y') = U(\bar{x})$ and hence $w', y' \in I(\bar{x})$. By the linearity of ρ, $\rho(w',\bar{y}) = \rho(w,x) \geq \rho(y,z) = \rho(y',\bar{y})$. By Lemma 11 the sets $B(\rho(w',\bar{y}))$ and $B(\rho(y',\bar{y}))$ are dilations of one another and hence $\|w'-\hat{x}\|_B \leq \|y'-\hat{x}\|_B$. By construction, $\hat{x}-\bar{y}$ is orthogonal to $I(\bar{x})-\hat{x}$, and therefore

$$\|w'-\bar{y}\|^2 = \|w'-\hat{x}\|^2 + \|\hat{x}-\bar{y}\|^2 \leq \|y'-\hat{x}\|^2 + \|\hat{x}-\bar{y}\|^2 = \|y'-\bar{y}\|^2.$$

Thus,

$$\left\|\frac{U(\hat{x})-U(\bar{y})}{U(w)-U(x)}(w-x)\right\| = \|w'-\bar{y}\| \leq \|y'-\bar{y}\| = \left\|\frac{U(\hat{x})-U(\bar{y})}{U(y)-U(z)}(y-z)\right\|$$

which implies

$$\frac{U(w)-U(x)}{\|w-x\|} \geq \frac{U(y)-U(z)}{\|y-z\|}.$$

Next, suppose $\rho(w,x) \geq 1/2 \geq \rho(y,z)$ with $w \neq x$ and $y \neq z$. Then $U(w) \geq U(x)$ and $U(z) \geq U(y)$ which implies

$$\frac{U(w)-U(x)}{\|w-x\|} \geq 0 \geq \frac{U(y)-U(z)}{\|y-z\|}.$$

Finally, suppose $1/2 > \rho(w,x) \geq \rho(y,z)$. Then $\rho(z,y) \geq \rho(x,w) > 1/2$ and the desired inequality follows from the first step.

Reversing the argument above to show the converse is straightforward and left to the reader. □

Lemma 14. *The image of ρ is an interval $[1-\bar{p},\bar{p}]$.*

Proof. Linearity implies ρ is entirely determined by the values of the mapping $x \mapsto \rho(x, \bar{y})$ for $x \in I(\bar{x})$. Hence, ρ achieves its maximum at $\bar{p} = \rho(\hat{x}, \bar{y})$. Linearity also implies ρ is entirely determined by the values of the mapping $x \mapsto \rho(x, \bar{y})$ for x in a unit sphere around $\bar{y}$. The continuity of ρ outside the diagonal implies $x \mapsto \rho(x, \bar{y})$ is continuous on the unit sphere around $\bar{y}$. The result then easily follows from the intermediate value theorem. $\square$

To construct F, we first define an auxiliary function $f : [1-\bar{p}, \bar{p}] \to \mathbb{R}$. Let $f(1/2) = 0$. For each $t \neq 1/2$, let $f(t) = [U(x) - U(y)]/\|x - y\|$ for any x, y such that $\rho(x, y) = t$. By Lemma 13 and Lemma 14, the function f is well defined. To see that the image of f must be a compact interval in $\mathbb{R}$, take any lottery $x \neq \hat{x}$ with $U(x) = U(\hat{x})$. Then we have $U(\hat{x} + t(x - \hat{x})) - U(\bar{y}) = U(\hat{x}) - U(\bar{y})$ for all $t > 0$ and $\|\hat{x} + t(x - \hat{x}) - \bar{y}\| \geq t\|x - \hat{x}\| - \|\hat{x} - \bar{y}\|$ which goes to infinity when t goes to infinity. Hence $[U(\hat{x} + t(x - \hat{x})) - U(\bar{y})]/\|\hat{x} + t(x - \hat{x}) - \bar{y}\|$ goes to zero when t goes to infinity. Thus the image of f is the interval $[-T, T]$, where $T = [U(\hat{x}) - U(\bar{y})]/\|\hat{x} - \bar{y}\|$. By Lemma 13 f is strictly increasing and has an inverse. Repeating the argument in the proof of Lemma 14 shows f is continuous. Leting $F = f^{-1}$ be the continuous inverse of f, it follows that $(U, \|\cdot\|, F)$ is a MEM representation of ρ. $\square$

Proof of Lemma 2 and Proposition 3

Fix $\hat{x}, \bar{y}$ exactly as in the proof of Theorem 2, that is, $\rho(\hat{x}, \bar{y}) = \max_{x,y} \rho(x, y) > 1/2$. Let $(U, \|\cdot\|, F)$ be a MEM representation of ρ as in (2), and let $\langle \cdot, \cdot \rangle$ be the inner product that induces the norm. To prove Lemma 2, by linearity, we can equivalently prove the following Lemma:

Lemma 15. $\langle x - \hat{x}, \hat{x} - \bar{y} \rangle = 0$ *for all* x *with* $\rho(x, \hat{x}) = 1/2$.

Proof. When $x = \hat{x}$ the statement is obviously true. Suppose $x \neq \hat{x}$. By Lemma 9 $\rho(x, \bar{y}) = \rho(2\hat{x} - x, \bar{y})$. By the representation (2) it must be $\|x - \bar{y}\| = \|2\hat{x} - x - \bar{y}\|$. Hence

$$\begin{aligned}
\|x - \hat{x}\|^2 + 2\langle x - \hat{x}, \hat{x} - \bar{y} \rangle + \|\hat{x} - \bar{y}\|^2 &= \langle x - \bar{y}, x - \bar{y} \rangle \\
&= \langle 2\hat{x} - x - \bar{y}, 2\hat{x} - x - \bar{y} \rangle \\
&= \|x - \hat{x}\|^2 + 2\langle x - \hat{x}, \bar{y} - \hat{x} \rangle + \|\hat{x} - \bar{y}\|^2
\end{aligned}$$

which implies $4\langle x - \hat{x}, \hat{x} - \bar{y} \rangle = 0$ as desired. $\square$

Lemma 16. *$\rho(x,\hat{x}) = \rho(x',\hat{x}) = 1/2$ and $\rho(x,\bar{y}) = \rho(x',\bar{y})$ implies $\|x-\hat{x}\| = \|x'-\hat{x}\|$.*

Proof. By the representation (2) we must have $\|x-\bar{y}\| = \|x'-\bar{y}\|$. By Lemma 15, $\langle x-\hat{x}, \hat{x}-\bar{y}\rangle = \langle x'-\hat{x}, \hat{x}-\bar{y}\rangle = 0$. Thus,

$$\|x-\hat{x}\|^2 + \|\hat{x}-\bar{y}\|^2 = \|x-\bar{y}\|^2 = \|x'-\bar{y}\|^2 = \|x'-\hat{x}\|^2 + \|\hat{x}-\bar{y}\|^2$$

and therefore $\|x-\hat{x}\| = \|x'-\hat{x}\|$ as desired. $\square$

To prove necessity, suppose $(U_1, \|\cdot\|_1, F_1)$ and $(U_2, \|\cdot\|_2, F_2)$ are two MEM representations of the same choice rule ρ. By the definition of MEM, both norms $\|x\|_1 = \sqrt{\langle x,x\rangle_1}$ and $\|x\|_2 = \sqrt{\langle x,x\rangle_2}$ are induced by inner products. The two norms and their respective inner products are defined on the linear subspace $\ker(\mathbf{1}) := \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$. Denote by H the $n-1$ dimensional hyperplane that contains Δ, and identify ρ with its linear extension to H, which is unique by Lemma 3.

To prove (i), note that by the definition of MEM, the utility functions $U_1, U_2 : \Delta \to \mathbb{R}$ are linear and represent the preference $\succcurlyeq$ given by $x \succcurlyeq y$ if and only if $\rho(x,y) \geq 1/2$. By the uniqueness result for vNM expected utility we have $U_2 = A + BU_1$ for some $A \in \mathbb{R}$ and $B > 0$, as desired.

To prove (ii), we fix a lottery $x' \neq \hat{x}$ with $\rho(x',\hat{x}) = 1/2$ and let $C = \|x'-\hat{x}\|_2/\|x'-\hat{x}\|_1 > 0$. Then, we let $D = \|\hat{x}-\bar{y}\|_2/\|\hat{x}-\bar{y}\|_1 > 0$. Now consider any x, y, w, z with $\rho(x,y) = 1/2$ and $\rho(w,z) = \bar{p}$. By linearity we may assume, without loss of generality, that $\hat{x} + (x-y) \in \Delta$. By the representation, $U_1(x) = U_1(y)$ and $\rho(\hat{x}+(x-y), \hat{x}) = 1/2$. Lemma 15 then implies

$$\langle x-y, \hat{x}-\bar{y}\rangle_1 = \langle (\hat{x}+x-y)-\hat{x}, \hat{x}-\bar{y}\rangle_1 = 0 = \langle (\hat{x}+x-y)-\hat{x}, \hat{x}-\bar{y}\rangle_2 = \langle x-y, \hat{x}-\bar{y}\rangle_2 .$$

By Lemma 8 and linearity, $\rho(w,z) = \bar{p} = \rho(\hat{x},\bar{y})$ implies $w-z = \alpha(\hat{x}-\bar{y})$ for some $\alpha > 0$. Hence $\langle x-y, w-z\rangle_2 = \langle x-y, w-z\rangle_1 = 0$ as desired. By Lemma 8 and Lemma 11 we have $\rho(\hat{x}+\beta(x-y), \hat{x}) = \rho(x',\hat{x})$ for some $\beta > 0$. Lemma 16 and the definition of C imply

$$\|\hat{x}+\beta(x-y)-\hat{x}\|_2 = \|x'-\hat{x}\|_2 = C\|x'-\hat{x}\|_1 = \|\hat{x}+\beta(x-y)-\hat{x}\|_2$$

hence $\beta\|x-y\|_2 = \beta C\|x-y\|_1$ and since $\beta > 0$ we have $\|x-y\|_2 = C\|x-y\|_1$ as desired. Finally, the equality $w-z = \alpha(\hat{x}-\bar{y})$ also implies $\|w-z\|_2 = D\|w-z\|_1$ as desired.

To prove (iii), for each x with $U_2(x) = U_2(\hat{x})$, define $t(x) = [U_2(x) - U_2(\bar{y})]/\|x - \bar{y}\|_2 > 0$. Lemma 15 and items (i) and (ii) above imply for each such x

$$\|x - \bar{y}\|_2^2 = \|x - \hat{x}\|_2^2 + \|\hat{x} - \bar{y}\|_2^2 = C^2\|x - \hat{x}\|_1^2 + D^2\|\hat{x} - \bar{y}\|_1^2.$$

Let $T := F_1^{-1}(\max_{x,y} \rho(x, y)) = F_1^{-1}(\bar{p}) = [U_1(\hat{x}) - U_1(\bar{y})]/\|\hat{x} - \bar{y}\|_1$. Substituting and rearranging we obtain for each x with $U_2(x) = U_2(\hat{x})$,

$$\frac{\|x - \hat{x}\|_1^2}{\|\hat{x} - \bar{y}\|_1^2} = \frac{(BT)^2 - D^2 t(x)^2}{C^2 t(x)^2}.$$

By linearity, for each $0 < t \leq BT/D$ we have $t = t(x)$ for some x with $U_2(x) = U_2(\hat{x})$, thus

$$\begin{aligned}
F_2(t) = F_2(t(x)) &= \rho(x, \bar{y}) \\
&= F_1\left(\frac{U(\hat{x}) - U(\bar{y})}{\sqrt{\|x - \hat{x}\|_1^2 + \|\hat{x} - \bar{y}\|_1^2}}\right) \\
&= F_1\left(T\Big/\sqrt{\frac{\|x - \hat{x}\|_1^2}{\|\hat{x} - \bar{y}\|_1^2} + 1}\right) \\
&= F_1\left(T\Big/\sqrt{\frac{(BT)^2 - D^2 t(x)^2}{C^2 t(x)^2} + 1}\right) \\
&= F_1\left(\frac{TCt}{\sqrt{B^2T^2 + (C^2 - D^2)t^2}}\right).
\end{aligned}$$

The results follows since $F_2(t) = 1 - F_2(-t)$ for all t. Sufficiency is straightforward. $\square$

Proof of Proposition 4

Necessity is straightforward. To show sufficiency, suppose the binary choice rule ρ on a finite set Z satisfies WST. As in the proof of Theorem 1, WST implies there is a utility function $u : Z \to \{1, \ldots, m\}$ representing the complete and transitive binary relation given by $i \succcurlyeq j$ if and only if $\rho(i, j) \geq 1/2$. Fix k, ℓ with $\rho(k, \ell) = \max_{i,j \in Z} \rho(i, j)$. Define $s(k, \ell) = 1$ and define $F : [u(\ell) - u(k), u(k) - u(\ell)] \to \mathbb{R}$ by

$$F(t) = \frac{1}{2} + t\left(\frac{\rho(k, \ell) - 1/2}{u(k) - u(\ell)}\right).$$

Finally, for each $i, j \in Z$ define

$$s(i,j) = \frac{u(i) - u(j)}{F^{-1}(\rho(i,j))}.$$

It is easy to verify that the WUM with the parameters u, s, F above represents ρ. $\square$

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